Extended Bell and Stirling numbers from hypergeometric exponentiation

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Abstract

Exponentiating the hypergeometric series ${}_0F_L(1,1,\ldots,1;z)$, $L=0,1,2,\ldots$, furnishes a recursion relation for the members of integer sequences $b_L(n)$, $n=0,1,2,\ldots$. For $L>0$, the $b_L(n)$'s are certain generalizations of conventional Bell numbers, $b_0(n)$. The corresponding associated Stirling numbers of the second kind are also generated and investigated. For $L=1$ one can give a combinatorial interpretation of the numbers $b_1(n)$, and of some Stirling numbers associated with them. We also consider the $L\geq 1$ analogues of Bell numbers for restricted partitions.

The conventional Bell numbers [1] $b_0(n)$, $n=0,1,2,\ldots$, have a well known exponential generating function

$$B_0(z) \equiv e^{(e^z-1)} = \sum_{n=0}^{\infty} b_0(n)\frac{z^n}{n!}, \tag{1}$$

which can be derived by interpreting $b_0(n)$ as the number of partitions of a set of n distinct elements. In this note we obtain recursion relations for related sequences of positive integers, called $b_L(n)$, $L=0,1,2,\ldots$, obtained by exponentiating the hypergeometric series ${}_0F_L(1,1,\ldots,1;z)$ defined by [2]:

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$$ {}_0F_L(\underbrace{1,1,\ldots,1}_{L};z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{L+1}}, \tag{2} $$

(for which we shall use throughout the short notation ${}_0F_L(z)$) and which includes the special cases ${}_0F_0(z) \equiv e^z$ and ${}_0F_1(z) \equiv I_0(2\sqrt{z})$, where $I_0(x)$ is the modified Bessel function of the first kind. For $L > 1$, the functions ${}_0F_L(z)$ are related to the so-called hyper-Bessel functions [3], [4], [5], which have recently found application in quantum mechanics [6], [7]. Thus, we are interested in $b_L(n)$ given by

$$ e^{[\,{}_0F_L(z)-1]} = \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}}, \tag{3} $$

thereby defining a <u>hypergeometric</u> generating function for the numbers $b_L(n)$. From eq.(3) it follows formally that

$$ b_L(n) = (n!)^L \cdot \frac{d^n}{dz^n} \left(e^{[\,{}_0F_L(z)-1]} \right)\Big|_{z=0}. \tag{4} $$

For $L = 0$ the r.h.s of eq.(4) can be evaluated in closed form:

$$ b_0(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} = \left\{ \frac{1}{e^z} \left[\left(z\frac{d}{dz} \right)^n e^z \right] \right\}_{z=1}. \tag{5} $$

The first equality in (5) is the celebrated Dobiński formula [1], [8], [9]. The second equality in eq.(5) follows from observing that for a power series $R(z) = \sum_{k=0}^{\infty} A_k z^k$

$$ \left(z\frac{d}{dz} \right)^n R(z) = \sum_{k=0}^{\infty} A_k\, k^n\, z^k \tag{6} $$

holds, and by applying eq.(6) to the exponential series ($A_k = (k!)^{-1}$).

The reason for including the divisors $(n!)^{L+1}$ rather than $n!$ as in the usual exponential generating function, arises from the fact that only through eq.(3) are the numbers $b_L(n)$ actually integers. This can be seen from general formulas for exponentiation of a power series [8], which employ the (exponential) Bell polynomials, complicated and rather unwieldy objects. It cannot however be considered as a proof that the $b_L(n)$ are integers. At this stage we shall use eq.(3) with $b_L(n)$ real and apply to it an efficient method, exposed in [9], which will yield the recursion relation for the $b_L(n)$. (For the proof that the $b_L(n)$ are integers, see below eq.(11)). To this end we first obtain a result for the multiplication of two power-series of the type (3). Suppose that we have to multiply $f(x) = \sum_{n=0}^{\infty} a_L(n) \frac{x^n}{(n!)^{L+1}}$ and $g(x) = \sum_{n=0}^{\infty} c_L(n) \frac{x^n}{(n!)^{L+1}}$. We get $f(x) \cdot g(x) = \sum_{n=0}^{\infty} d_L(n) \frac{x^n}{(n!)^{L+1}}$, where

$$ d_L(n) = (n!)^{L+1} \sum_{r+s=n}^{\infty} \frac{a_L(r) c_L(s)}{(r!)^{L+1} (s!)^{L+1}} = \sum_{r=0}^{n} \binom{n}{r}^{L+1} a_L(r)\, c_L(n-r). \tag{7} $$

Substitute eq.(2) into eq.(3) and take the logarithm of both sides of eq.(3):

$$\sum_{n=1}^{\infty} \frac{z^n}{(n!)^{L+1}} = \ln\left(\sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}}\right). \tag{8}$$

Now differentiate both sides of eq.(8) and multiply by z. It produces

$$\left(\sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}}\right)\left(\sum_{n=0}^{\infty} n \frac{z^n}{(n!)^{L+1}}\right) = \sum_{n=0}^{\infty} n\, b_L(n) \frac{z^n}{(n!)^{L+1}}, \tag{9}$$

which with eq.(7) yields the desired recurrence relation

$$b_L(n+1) = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k}^{L+1} (n+1-k)\, b_L(k), \qquad n = 0, 1, \ldots \tag{10}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{n+1}{k}^{L} b_L(k), \tag{11}$$

$$b_L(0) = 1. \tag{12}$$

Since eq.(11) involves only positive integers, it follows that the $b_L(n)$ are indeed positive integers. For $L = 0$ one gets the known recurrence relation for the Bell numbers [9]:

$$b_0(n+1) = \sum_{k=0}^{n} \binom{n}{k} b_0(k). \tag{13}$$

We have used eq.(11) to calculate some of the $b_L(n)$'s, listed in Table I, for $L = 0, 1, \ldots, 6$. Eq.(11), for n fixed, gives closed form expressions for the $b_L(n)$ directly as a function of L (columns in Table I): $b_L(2) = 1 + 2^L$, $b_L(3) = 1 + 3 \cdot 3^L + (3!)^L$, $b_L(4) = 1 + 4 \cdot 4^L + 3 \cdot 6^L + 6 \cdot 12^L + (4!)^L$, etc.

The sets of $b_L(n)$ have been checked against the most complete source of integer sequences available [10]. Apart from the case $L = 0$ (conventional Bell numbers) only the first non-trivial sequence $L = 1$ is listed: it turns out that this sequence $b_1(n)$, listed under the heading A023998 in [10], can be given a combinatorial interpretation as the number of block permutations on a set of n objects, which are uniform, i.e. corresponding blocks have the same size [12].

Eq.(1) can be generalized by including an additional variable x, which will result in "smearing out" the conventional Bell numbers $b_0(n)$ with a set of integers $S_0(n,k)$, such that for $k > n$, $S_0(n,k) = 0$, and $S_0(0,0) = 1$, $S_0(n,0) = 0$. In particular,

$$B_0(z,x) \equiv e^{x(e^z - 1)} = \sum_{n=0}^{\infty} \left[\sum_{k=1}^{n} S_0(n,k)\, x^k\right] \frac{z^n}{n!}, \tag{14}$$

which leads to the (exponential) generating function of $S_0(n,l)$, the conventional Stirling numbers of the second kind, (see [1], [8]), in the form

$$\frac{(e^z-1)^l}{l!} = \sum_{n=l}^{\infty} \frac{S_0(n,l)}{n!} z^n, \tag{15}$$

and defines the so-called exponential or Touchard polynomials $l_n^{(0)}(x)$ as

$$l_n^{(0)}(x) = \sum_{k=1}^{n} S_0(n,k) x^k. \tag{16}$$

They satisfy

$$l_n^{(0)}(1) = b_0(n), \tag{17}$$

justifying the term "smearing out" used above.

The appearance of integers in eq.(3) suggests a natural extension with an additional variable x:

$$B_L(z,x) \equiv e^{x[\,{}_0F_L(z)-1]} = \sum_{n=0}^{\infty} \left[\sum_{k=1}^{n} S_L(n,k)\, x^k \right] \frac{z^n}{(n!)^{L+1}}, \tag{18}$$

where we include the right divisors $(n!)^{L+1}$ in the r.h.s of (18).

This in turn defines "hypergeometric"polynomials of type L and order n through

$$l_n^{(L)}(x) = \sum_{k=1}^{n} S_L(n,k) x^k, \tag{19}$$

which satisfy

$$l_n^{(L)}(1) = b_L(n), \tag{20}$$

with the $b_L(n)$ of eq.(10). Thus, the polynomials of eq.(19) "smear out" the $b_L(n)$ with the generalized Stirling numbers of the second kind, of type L, denoted by $S_L(n,k)$ (with $S_L(n,k)=0$, if $k>n$, $S_L(n,0)=0$ if $n>0$ and $S_L(0,0)=1$), which have, from eq.(18) the "hypergeometric"generating function

$$\frac{(\,{}_0F_L(z)-1)^l}{l!} = \sum_{n=l}^{\infty} \frac{S_L(n,l)}{(n!)^{L+1}}\, z^n, \qquad L=0,1,2,\ldots. \tag{21}$$

Eq.(21) can be used to derive a recursion relation for the numbers $S_L(n,k)$, in the same manner as eq.(3) yielded eq.(12). Thus we take the logarithm of both sides of eq.(21), differentiate with respect to z, multiply by z and obtain:

$$\left(\sum_{n=0}^{\infty} \frac{S_L(n,l-1)}{(n!)^{L+1}}\, z^n \right) \left(\sum_{n=0}^{\infty} \frac{n}{(n!)^{L+1}}\, z^n \right) = \sum_{n=0}^{\infty} \frac{n\, S_L(n,l)}{(n!)^{L+1}}\, z^n, \tag{22}$$

which, with the help of eq.(7), produces the required recursion relation

$$S_L(n+1,l) = \sum_{k=l-1}^{n} \binom{n}{k} \binom{n+1}{k}^L S_L(k,l-1), \tag{23}$$

$$S_L(0,0) = 1, \qquad S_L(n,0) = 0, \tag{24}$$

which for $L = 0$ is the recursion relation for the conventional Stirling numbers of the second kind [1], [8], and in eq.(23) the appropriate summation range has been inserted. Since the recursions of eq.(23) and eq.(24) involve only integers we conclude that $S_L(n,l)$ are positive integers.

We have calculated some of the numbers $S_L(n,l)$ using eq.(21) and have listed them in Tables II and III, for $L = 1$ and $L = 2$ respectively. Observe that $S_1(n,2) = \binom{2n+1}{n+1} - 1$ and $S_L(n,n) = (n!)^L$, $L = 1,2$. Also, by fixing n and l, the individual values of $S_L(n,l)$ have been calculated as a function of L with the help of eq.(23), see Table IV, from which we observe

$$S_L(n,n) = (n!)^L, \qquad L = 1,2,\ldots. \tag{25}$$

which is the lowest diagonal in Table IV. We now demonstrate that the repetitive use of eq.(23) permits one to establish closed-form expressions for any supra-diagonal of order p, i.e. the sequence $S_L(n+p,n)$, for $p = 1,2,3,\ldots$, if one knows the expression for all $S_L(n+k,n)$ with $k < p$. We shall illustrate it here for $p = 1,2$. To this end fix $l = n$ on both sides of eq.(23). It becomes, upon using eq.(25), and defining $\alpha_L(n) \equiv S_L(n+1,n)$, a linear recursion relation

$$\alpha_L(n) = \frac{n[(n+1)!]^L}{2^L} + (n+1)^L \alpha_L(n-1), \qquad \alpha_L(0) = 0, \tag{26}$$

with the solution

$$\alpha_L(n) = S_L(n+1,n) = \frac{n(n+1)}{2}\left[\frac{(n+1)!}{2}\right]^L \tag{27}$$

$$= \left[\frac{(n+1)!}{2}\right]^L S_0(n+1,n), \tag{28}$$

which gives the second lowest diagonal in Table IV. Observe that for any L, $S_L(n+1,n)$ is proportional to $S_0(n+1,n) = n(n+1)/2$. The sequence $S_1(n+1,n) = 1,\ 9,\ 72,\ 600,\ 5\,400, 8\,564\,480,\ \ldots$ is of particular interest: it represents the sum of inversion numbers of all permutations on n letters [10]. For more information about this and related sequences see the entry A001809 in [10]. The $S_L(n+1,n)$ for $L > 1$ do not appear to have a simple combinatorial interpretation. A recurrence equation for $\beta_L(n) \equiv S_L(n+2,n)$ is obtained upon substituting eq.(25) and eq.(27) into eq.(23):

$$\beta_L(n) = \frac{n(n+1)}{2!}\left[\frac{(n+2)!}{2!}\right]^L\left(\frac{n-1}{2^L}+\frac{1}{3^L}\right)+(n+2)^L\beta_L(n-1), \qquad \beta_L(0)=0. \tag{29}$$

It has the solution

$$S_L(n+2,n) = \frac{n(n+1)(n+2)}{3\cdot 2^3}\left[\frac{(n+2)!}{2}\right]^L\left(\frac{3}{2^L}(n-1)+\frac{4}{3^L}\right) \tag{30}$$

which is a closed form expression for the second lowest diagonal in Table IV. Clearly, eq.(30) for $L=0$ gives the combinatorial form for the series of conventional Stirling numbers

$$S_0(n+2,n) = \frac{n(n+1)(n+2)(3n+1)}{4!}. \tag{31}$$

In a similar way we obtain

$$\begin{aligned} S_L(n+3,n) &= \frac{n(n+1)(n+2)(n+3)}{3\cdot 2^4}\left[\frac{(n+3)!}{3}\right]^L \\ &\times\left(n^2\left(\frac{3}{8}\right)^L+n\left(\frac{1}{4^{L-1}}-\frac{3^{L+1}}{8^L}\right)+\frac{2+2\cdot 3^L}{8^L}-\frac{1}{4^{L-1}}\right) \end{aligned} \tag{32}$$

which for $L=0$ reduces to

$$S_0(n+3,n) = \frac{1}{48}n^2(n+1)^2(n+2)(n+3). \tag{33}$$

Combined with the standard definition [8], [9]

$$S_0(n,l) = \frac{(-1)^l}{l!}\sum_{k=1}^{l}(-1)^k\binom{l}{k}\,k^n. \tag{34}$$

eqs.(28), (31) and (33) give compact expressions for the summation form of $S_0(n+p,n)$. Further, from eq.(34), use of eq.(6) gives the following generating formula

$$S_0(n,l) = \frac{(-1)^l}{l!}\left[\left(z\frac{d}{dz}\right)^n\left(\sum_{k=1}^{l}(-1)^k\binom{l}{k}\,z^k\right)\right]_{z=1} \tag{35}$$

$$= \frac{(-1)^l}{l!}\left[\left(z\frac{d}{dz}\right)^n\left[(1-z)^l-1\right]\right]_{z=1}, \qquad n\geq l. \tag{36}$$

The formula (1) can be generalized by putting restrictions on the type of resulting partitions. The generating function for the number of partitions of a set of n distinct elements without singleton blocks $b_0(1,n)$ is [8], [14], [15],

$$B_0(1,z) = e^{e^z-1-z} = \sum_{n=0}^{\infty} b_0(1,n)\frac{z^n}{n!}, \tag{37}$$

or more generally, without singleton, doubleton ..., $p-$blocks ($p=0,1,\ldots$) is [15]

$$B_0(p,z) = e^{e^z - \sum_{k=0}^{p} \frac{z^k}{k!}} = \sum_{n=0}^{\infty} b_0(p,n) \frac{z^n}{n!}, \tag{38}$$

with the corresponding associated Stirling numbers defined by analogy with eq.(14) and eq.(22). The numbers $b_0(1,n)$, $b_0(2,n)$, $b_0(3,n)$, $b_0(4,n)$ can be read off from the sequences A000296, A006505, A057837 and A057814 in [10], respectively. For more properties of these numbers see [11].

We carry over this type of extension to eq.(3) and define $b_L(p,n)$ through

$$B_L(p,z) \equiv e^{\,{}_0F_L(z) - \sum_{k=0}^{p} \frac{z^k}{(k!)^{L+1}}} = \sum_{n=0}^{\infty} b_L(p,n) \frac{z^n}{(n!)^{L+1}}, \tag{39}$$

where $b_L(0,n) = b_L(n)$ from eq.(3). (We know of no combinatorial meaning of $b_L(p,n)$ for $L \geq 1$, $p > 0$). The $b_L(p,n)$ satisfy the following recursion relations:

$$b_L(p,n) = \sum_{k=0}^{n-p} \binom{n}{k} \binom{n+1}{k}^L b_L(p,k), \tag{40}$$

$$b_L(p,0) = 1, \tag{41}$$

$$b_L(p,1) = b_L(p,2) = \cdots = b_L(p,p) = 0, \tag{42}$$

$$b_L(p,p+1) = 1. \tag{43}$$

That the $b_L(p,n)$ are integers follows from eq.(40). Through eq.(39) additional families of integer Stirling-like numbers $S_{L,p}(n,k)$ can be readily defined and investigated.

The numbers $b_0(p,n)$ are collected in Table V, and Tables VI and VII contain the lowest values of $b_1(p,n)$ and $b_2(p,n)$, respectively.

Formula (1) can be used to express e in terms of $b_0(n)$ in various ways. Two such lowest order (in differentiation) forms are

$$e = 1 + \ln\left(\sum_{n=0}^{\infty} \frac{b_0(n)}{n!}\right) = \tag{44}$$

$$= \ln\left(\sum_{n=0}^{\infty} \frac{b_0(n+1)}{n!}\right). \tag{45}$$

In the very same way, eq.(3) can be used to express the values of ${}_0F_L(z)$ and its derivatives at $z = 1$ in terms of certain series of $b_L(n)$'s. For $L = 1$, the analogues of eq.(44) and eq.(45) are

$$I_0(2) = 1 + \ln\left(\sum_{n=0}^{\infty} \frac{b_1(n)}{(n!)^2}\right), \tag{46}$$

$$I_0(2) + \ln(I_1(2)) = 1 + \ln\left(\sum_{n=0}^{\infty} \frac{b_1(n+1)}{(n+1)(n!)^2}\right) \tag{47}$$

and for $L = 2$ the corresponding formulas are

$$ {}_0F_2(1,1;1) = 1 + \ln\left(\sum_{n=0}^{\infty} \frac{b_2(n)}{(n!)^3}\right), \tag{48} $$

$$ {}_0F_2(1,1;1) + \ln\left({}_0F_2(2,2;1)\right) = 1 + \ln\left(\sum_{n=0}^{\infty} \frac{b_2(n+1)}{(n+1)^2(n!)^3}\right). \tag{49} $$

By fixing z_0 at values other than $z_0 = 1$, one can link the numerical values of certain combinations of ${}_0F_L(1,1,\ldots;z_0)$, ${}_0F_L(2,2,\ldots;z_0)$,... and their logarithms, with other series containing the $b_L(n)$'s.

The above considerations can be extended to the exponentiation of the more general hypergeometric functions of type ${}_0F_L(k_1,k_2,\ldots,k_L;z)$ where $k_1,k_2,\ldots,k_L$ are positive integers. We conjecture that for every set of k_n's a different set of integers will be generated through an appropriate adaptation of eq.(3). We quote one simple example of such a series. For

$$ {}_0F_2(1,2;z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)(n!)^3} \tag{50} $$

eq.(3) extends to

$$ e^{[\,{}_0F_2(1,2;z)-1]} = \sum_{n=0}^{\infty} f_2(n) \frac{z^n}{(n+1)(n!)^3} \tag{51} $$

where

$$ f_2(n) = (n+1)(n!)^2 \left[\frac{d^n}{dz^n} e^{[\,{}_0F_2(1,2;z)-1]}\right]_{z=0} \tag{52} $$

turn out to be integers: $f_2(n)$, $n = 0,1,\ldots,8$ are: 1, 1, 4, 37, 641, 18 276, 789 377, 48 681 011, etc. The analogue of equations (23) and (44) is:

$$ {}_0F_2(1,2;1) = 1 + \ln\left(\sum_{n=0}^{\infty} \frac{f_2(n)}{(n+1)(n!)^3}\right). \tag{53} $$

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TABLES

TABLE I. Table of $b_L(n)$; $L, n = 0, 1, \dots, 6$.

L	$b_L(0)$	$b_L(1)$	$b_L(2)$	$b_L(3)$	$b_L(4)$	$b_L(5)$	$b_L(6)$
0	1	1	2	5	15	52	203
1	1	1	3	16	131	1 496	22 482
2	1	1	5	64	1 613	69 026	4 566 992
3	1	1	9	298	25 097	4 383 626	1 394 519 922
4	1	1	17	1 540	461 105	350 813 126	573 843 627 152
5	1	1	33	8 506	9 483 041	33 056 715 626	293 327 384 637 282
6	1	1	65	48 844	209 175 233	3 464 129 078 126	173 566 857 025 139 312

TABLE II. Table of $S_L(n, l)$; $L = 1$, $l, n = 1, 2, \dots, 8$.

l	$S_1(1,l)$	$S_1(2,l)$	$S_1(3,l)$	$S_1(4,l)$	$S_1(5,l)$	$S_1(6,l)$	$S_1(7,l)$	$S_1(8,l)$
1	1	1	1	1	1	1	1	1
2		2	9	34	125	461	1 715	6 434
3			6	72	650	5 400	43 757	353 192
4				24	600	10 500	161 700	2 361 016
5					120	5 400	161 700	4 116 000
6						720	52 920	2 493 120
7							5 040	564 480
8								40 320

TABLE III. Table of $S_L(n, l)$; $L = 2$, $l, n = 1, 2, \dots, 8$.

l	$S_2(1,l)$	$S_2(2,l)$	$S_2(3,l)$	$S_2(4,l)$	$S_2(5,l)$	$S_2(6,l)$	$S_2(7,l)$	$S_2(8,l)$
1	1	1	1	1	1	1	1	1
2		4	27	172	1 125	7 591	52 479	369 580
3			36	864	17 500	351 000	7 197 169	151 633 440
4				576	36 000	1 746 000	80 262 000	3 691 514 176
5					14 400	1 944 000	191 394 000	17 188 416 000
6						518 400	133 358 400	23 866 214 400
7							25 401 600	11 379 916 800
8								1 625 702 400

TABLE IV. Table of $S_L(n,l)$; $l,n=1,2,\ldots,6$.

l	$S_L(1,l)$	$S_L(2,l)$	$S_L(3,l)$	$S_L(4,l)$	$S_L(5,l)$	$S_L(6,l)$
1	1	1	1	1	1	1
2		$(2!)^L$	$3\cdot 3^L$	$4\cdot 4^L+3\cdot 6^L$	$5\cdot 5^L+10\cdot 10^L$	$6\cdot 6^L+15\cdot 15^L+10\cdot 20^L$
3			$(3!)^L$	$6\cdot 12^L$	$10\cdot 20^L+15\cdot 30^L$	$15\cdot 30^L+60\cdot 60^L+15\cdot 90^L$
4				$(4!)^L$	$10\cdot 60^L$	$20\cdot 120^L+45\cdot 180^L$
5					$(5!)^L$	$15\cdot 360^L$
6						$(6!)^L$

TABLE V. Table of $b_0(p,n)$; $p=0,1,2,3$; $n=0,\ldots,10$.

n	$b_0(0,n)$	$b_0(1,n)$	$b_0(2,n)$	$b_0(3,n)$
0	1	1	1	1
1	1	0	0	0
2	2	1	0	0
3	5	1	1	0
4	15	4	1	1
5	52	11	1	1
6	203	41	11	1
7	877	162	36	1
8	4 140	715	92	36
9	21 147	3 425	491	127
10	115 975	17 722	2 557	337

TABLE VI. Table of $b_1(p,n)$; $p=0,1,2$; $n=0,\dots,9$.

n	$b_1(0,n)$	$b_1(1,n)$	$b_1(2,n)$
0	1	1	1
1	1	0	0
2	3	1	0
3	16	1	1
4	131	19	1
5	1 496	101	1
6	22 482	1 776	201
7	426 833	23 717	1 226
8	9 934 563	515 971	5 587
9	277 006 192	11 893 597	493 333

TABLE VII. Table of $b_2(p,n)$; $p=0,1,2$; $n=0,\dots,8$.

n	$b_2(0,n)$	$b_2(1,n)$	$b_2(2,n)$
0	1	1	1
1	1	0	0
2	5	1	0
3	64	1	1
4	1 613	109	1
5	69 026	1 001	1
6	4 566 992	128 876	4 001
7	437 665 649	4 682 637	42 876
8	57 903 766 800	792 013 069	347 117

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