Coherent States from Combinatorial Sequences [†]

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Abstract

We construct coherent states using sequences of combinatorial numbers such as various binomial and trinomial numbers, and Bell and Catalan numbers. We show that these states satisfy the condition of the *resolution of unity* in a natural way. In each case the positive weight functions are given as solutions of associated Stieltjes or Hausdorff moment problems, where the moments are the combinatorial numbers.

We describe the construction of coherent states which are generalizations of the standard coherent states defined for complex z by[1]

$$|z\rangle_0 = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{1}$$

where $|n\rangle$ are eigenfunctions of a Hermitian operator H (usually the system Hamiltonian) with

$$H|n\rangle = \varepsilon_n |n\rangle \quad \langle n|n'\rangle = \delta_{n,n'}, \quad n = 0, 1, \ldots. \tag{2}$$

The generalization consists in defining new states

$$|z\rangle_c = \mathcal{N}_c{}^{-\frac{1}{2}}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{c(n)}} |n\rangle, \tag{3}$$

where

$$\mathcal{N}_c(x) = \sum_{n=0}^{\infty} \frac{x^n}{c(n)}, \quad x \leq R. \tag{4}$$

In Eq.(3) and Eq.(4), $\mathcal{N}_c(x)$ ($x \equiv |z|^2$) is the normalization of $|z\rangle$, and $R \leq \infty$ is its radius of convergence. We shall assume that in Eq.(3) the positive numbers $c(n)$ for $n = 0, 1, \ldots,$ arise from sequences of combinatorial numbers; that is, integers which count the characteristic properties of an ensemble of n objects

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($c(0) = 1$ by convention). In this note we shall refrain from providing the precise combinatorial interpretation of the sequences used, but shall rather limit ourselves to giving their formulae. For more details of the $c(n)$ we refer the interested reader to the classical literature [2, 3, 4].

In general, the states $|z\rangle_c$ are not orthogonal ($\langle z|z'\rangle_c \neq 0$); the resolution of unity condition is given by the following sum of weighted non-orthogonal projectors

$$\iint d^2z |z\rangle_c W_c(|z|^2)\ {}_c\langle z| = I = \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{5}$$

where in Eq.(5) the integration is restricted to the part of the complex plane where the normalization converges. Equations (4) and (5) may be succinctly written as

$$\int_0^R x^n \tilde{W}_c(x) dx = c(n), \qquad n = 0, 1, \ldots, \tag{6}$$

with $\tilde{W}_c(x) = \pi W_c(x)/\mathcal{N}_c(x) > 0$. For $R = \infty$, Eq.(6) is the Stieltjes, and for $R < \infty$, the Hausdorff moment problem, with moments $c(n)$.

The motivation for choosing the combinatorial sequences $\{c(n)\}$ as input moments to Eq.(3) is that for many such sequences solutions can be explicitly constructed by considering Eq.(6) as a Mellin transform; see [5, 6, 7, 8] for related instances. For example, in those cases where the $c(n)$ may be expressed solely in terms of gamma functions, the use of properties of Meijer's G function is instrumental in obtaining solutions of Eq.(6). In what follows we shall list a number of sequences $\{c(n)\}$ and simply quote the solutions of Eq.(6), specifying R in each case:

1. $c(n) = (2n)!$; $\tilde{W}_1(x) = \frac{1}{2} e^{-\sqrt{x}}/\sqrt{x}$; $R = \infty$.

2. $c(n) = (2n)!/n!$; $\tilde{W}_2(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x}{4}}/\sqrt{x}$; $R = \infty$.

3. $c(n) = \left(\begin{array}{c} 2n \\ n \end{array}\right)$ (so-called central binomial coefficients) ;
   $\tilde{W}_3(x) = \frac{1}{\pi}[x(4-x)]^{-\frac{1}{2}}$; $R = 4$.

4. $c(n) = \left(\begin{array}{c} 2n \\ n \end{array}\right)/(n+1)$ (so-called Catalan numbers) ;
   $\tilde{W}_4(x) = \frac{1}{\pi}[(4-x)/x]^{\frac{1}{2}}$; $R = 4$.

5. $c(n) = (2n)!/(n+1)!$; $\tilde{W}_5(x) = -\frac{1}{2} + \frac{1}{\sqrt{\pi x}} e^{-\frac{x}{4}} + \frac{1}{2}\mathrm{erf}(\frac{\sqrt{x}}{2})$; $R = \infty$;
   $\mathrm{erf}(y)$ is the error function.

6. $c(n) = (2n)!/(n+1)$; $\tilde{W}_6(x) = \frac{1}{\sqrt{x}} e^{-\sqrt{x}} + \mathrm{Ei}(-\sqrt{x})$; $R = \infty$;
   $\mathrm{Ei}(y)$ is the exponential integral.

7. $c(n) = (3n)!/n!$; $\tilde{W}_7(x) = \frac{1}{3\pi\sqrt{x}} K_{\frac{1}{3}}(2\sqrt{\frac{x}{27}})$; $R = \infty$;
   $K_\nu(y)$ is the modified Bessel function of the second kind.

8. $c(n) = (3n)!/(2n)!$; $\tilde{W}_8(x) = \frac{\sqrt{3}}{27\pi}\exp(-\frac{2x}{27})[K_{\frac{1}{3}}(\frac{2x}{27}) + K_{\frac{2}{3}}(\frac{2x}{27})]$; $R = \infty$.

9. $c(n) = (3n)!/(n!)^3$; (so-called middle trinomial coefficients);
$\tilde{W}_9(x) = \alpha x^{-\frac{2}{3}}{}_2F_1(\frac{1}{3}, \frac{1}{3}; \frac{2}{3}; \frac{x}{27}) + \beta x^{-\frac{1}{3}}{}_2F_1(\frac{2}{3}, \frac{2}{3}; \frac{4}{3}; \frac{x}{27})$;
where $\alpha = \frac{1}{3}[\Gamma(\frac{2}{3})]^{-3}$ and $\beta = -\frac{\sqrt{3}}{8\pi^3}[\Gamma(\frac{2}{3})]^3$, $R = 27$.

10. $c(n) = \binom{3n}{n}/(2n+1)$;
$\tilde{W}_{10}(x) = \frac{\sqrt{3}\,2^{\frac{2}{3}}}{12\pi}\frac{[2^{\frac{1}{3}}(27+3\sqrt{81-12x})^{\frac{2}{3}} - 6x^{\frac{1}{3}}]}{x^{\frac{2}{3}}(27+3\sqrt{81-12x})^{\frac{1}{3}}}$; $R = \frac{27}{4}$.

The above weight functions, which may be seen to be positive by inspection, constitute only a small selection of examples of this type. Moreover, with the $c(n)$ of Examples 1–10, the normalization $\mathcal{N}_c(x)$ of Eq.(4) can always be expressed in terms of known hypergeometric functions. This renders calculations of many expectation values in the state $|z\rangle_c$ entirely analytic.

We now extend our considerations to include the important sequence of Bell numbers, $c(n) = B(n)$, which count the number of partitions of a set of n objects. The Bell numbers are given by the celebrated Dobinski formula[2]

$$B(n) = 1/e \sum_{k=1}^{\infty} \frac{k^n}{k!}, \quad n = 0, 1, \ldots, \tag{7}$$

from which it readily follows that the solution of

$$\int_0^{\infty} x^n \tilde{W}_B(x) dx = B(n)$$

is

$$\tilde{W}_B(x) = 1/e \sum_{k=1}^{\infty} \delta(x-k)/k!, \tag{8}$$

which is a sum of weighted Dirac delta function spikes situated on the positive integers. Since $x = |z|^2$, the weight is positive on concentric equidistant circles around the origin of the complex plane. The particular form of Eq.(8) permits one to write down the weight function whose moments are $c(n)B(n)$, where $c(n)$ is any moment from Examples 1–10. For instance, if $c(n)$ are the Catalan numbers, the nth moment of

$$\tilde{W}_{CB}(x) = 1/(2\pi e) \sum_{k=1}^{\infty} \frac{1}{kk!}\sqrt{\frac{4k-x}{x}} H(4-\frac{x}{k}) \tag{9}$$

equals $(\binom{2n}{n}/(n+1))\ B(n)$; $H(y)$ is the Heaviside function. The powerful convolution property of the Mellin transform[6, 7, 8] enables one to obtain Eq.(9). It has been shown elsewhere[9, 10] that by a different form of parametrization (replacing z by two independent real variables) the states described above may be applied to specific physical systems. A Hamiltonian with energy levels ε_n can be associated with the sequence $\{c(n)\}$ as follows:

$$\varepsilon_0 = 0, \quad \varepsilon_n = c(n)/c(n-1), \;\; n > 0. \tag{10}$$

The wealth of the preceding examples allows for different forms of ε_n, differing from the standard $\varepsilon_n = n$, associated with the standard coherent state characterized by the $|z\rangle_0$ of Eq.(1). These examples are associated with models having discrete non-linear spectra, and may well provide good approximations to some real nonlinear systems.

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