Combinatorics of Rooted Trees and Hopf Algebras

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Abstract

We begin by considering the graded vector space with a basis consisting of rooted trees, with grading given by the count of non-root vertices. We define two linear operators on this vector space, the growth and pruning operators, which respectively raise and lower grading; their commutator is the operator that multiplies a rooted tree by its number of vertices, and each operator naturally associates a multiplicity to each pair of rooted trees. By using symmetry groups of trees we define an inner product with respect to which the growth and pruning operators are adjoint, and obtain several results about the associated multiplicities.

Now the symmetric algebra on the vector space of rooted trees (after a degree shift) can be endowed with a coproduct to make a Hopf algebra; this was defined by Kreimer in connection with renormalization. We extend the growth and pruning operators, as well as the inner product mentioned above, to Kreimer's Hopf algebra. On the other hand, the vector space of rooted trees itself can be given a noncommutative multiplication: with an appropriate coproduct, this leads to the Hopf algebra of Grossman and Larson. We show that the inner product on rooted trees leads to an isomorphism of the Grossman-Larson Hopf algebra with the graded dual of Kreimer's Hopf algebra, correcting an earlier result of Panaite.

1 Introduction

In recent work on renormalization of quantum field theory, D. Kreimer and his collaborators [1, 3, 13, 14, 15, 16] introduce a Hopf algebra (here denoted $\mathcal{H}_K$) whose generators are rooted trees. Various other Hopf algebras based on rooted trees have appeared in the literature, in particular that of R. Grossman and R. G. Larson [10], which F. Panaite [18] connected to $\mathcal{H}_K$. The proof of the principal result of [18] actually contains an error due to the confusion of two kinds of multiplicities associated to triples of rooted trees. In this paper we show how to correct Panaite's result, while clarifying the combinatorial significance of these multiplicities.

Kreimer's Hopf algebra $\mathcal{H}_K$ admits a derivation called the growth operator, which is important in describing the relation of this algebra to another Hopf algebra studied earlier by A. Connes and H. Moscovici [4]. We introduce a complementary derivation called the pruning operator. In fact, we find it easiest to start (in §2) in the vector space $k\{\mathcal{T}\}$ of rooted trees rather than in $\mathcal{H}_K$. There we have growth and pruning operators (denoted $\mathfrak{N}$ and $\mathfrak{P}$ respectively), and for each pair of rooted trees t, t' with $|t| \leq |t'|$ (where $|t|$ is the number of vertices of t) there are natural multiplicities $n(t; t')$ and $m(t; t')$ associated with $\mathfrak{N}$ and $\mathfrak{P}$ respectively. A comparison of these multiplicities using symmetry groups of rooted trees leads to the definition of an inner product with respect to which $\mathfrak{N}$ and $\mathfrak{P}$ are adjoint. The operators $\mathfrak{N}$ and $\mathfrak{P}$ are very similar to the adjoint operators that appear in R. Stanley's theory of differential posets [21, 22], and can be described in terms of S. Fomin's somewhat more general theory of dual graded graphs [6, 7, 8]. The techniques of Stanley and Fomin can be used to obtain various results about n and m.

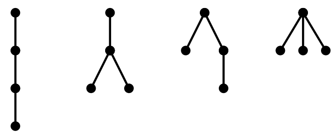
In §3 we extend the growth and pruning operators (as well as the inner product) to the Hopf algebra $\mathcal{H}_K$. In this setting the growth and pruning operators are not quite adjoint, but the deviation from adjointness is easily described (Proposition 3.3). We also describe the duals of the growth and pruning operators. In §4 we extend the multiplicities n and m to multiplicities $n(t_1, t_2; t_3)$ and $m(t_1, t_2; t_3)$ associated to triples of rooted trees t_1, t_2, t_3 with $|t_1| + |t_2| = |t_3|$. We then give an explicit isomorphism of the Grossman-Larson Hopf algebra onto the graded dual of $\mathcal{H}_K$ (Proposition 4.4), providing a corrected version of Panaite's result. We also show how the isomorphism gives another description of the duals of the growth and pruning operators.

In addition to the references cited above, two recent articles treat aspects of $\mathcal{H}_K$ not covered here: see [2] for connections between Kreimer's Hopf algebra and earlier work on Runge-Kutta methods, and [5] for an analysis of the primitives of $\mathcal{H}_K$. The author thanks the referee for bringing them to his attention.

2 The Vector Space of Rooted Trees

A rooted tree is a partially ordered set (whose elements are called vertices) with a unique greatest element (the root vertex), such that, for any vertex v, the vertices exceeding v in the partial order form a chain. If v exceeds w in the partial order, we call w a descendant of v and v an ancestor of w. If v covers w in the partial order (i.e., v is an ancestor of w and there are no vertices between v and w in the order), we call w a child of v and v the parent of w.

We can visualize a rooted tree as a directed graph by putting an edge from each vertex to each of its children: the root is the only vertex with no incoming edge. We call a vertex terminal if it has no outgoing edges (i.e., no children). The condition that the set of ancestors of any vertex forms a chain insures this graph has no cycles. For a finite rooted tree t, we denote by $|t|$ the number of vertices of t: let $\mathcal{T}$ be the set of finite rooted trees, and $\mathcal{T}_n = \{t \in \mathcal{T} : |t| = n+1\}$. For example, $\mathcal{T}_0 = \{\bullet\}$, where $\bullet$ is the tree consisting of only the root vertex, and below are the four elements of $\mathcal{T}_3$, with the root placed at the top:



We can define a partial order $\preceq$ on the set $\mathcal{T}$ itself by setting $t \preceq t'$ if t can be obtained from t' by removing some non-root vertices and edges; of course $t \preceq t'$ implies $|t| \leq |t'|$. Evidently t' covers t for this order exactly in the case when t can be obtained by removing from t' a single terminal vertex and the edge into it. In this case we write $t \lhd t'$.

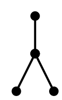
If $t \lhd t'$, we can get from t' to t by removing a (terminal) edge, and from t to t' by adding an edge (and accompanying terminal vertex). This leads to the definitions of two numbers associated with the pair (t, t'):

$$n(t;t') = \text{the number of vertices of } t \text{ to which a new edge can be added to obtain } t',$$

and

$$m(t;t') = \text{the number of edges of } t' \text{ which when removed leave } t.$$

That these two numbers are not always equal can be seen from the example of

$t =$  , $t' =$ 

where $n(t;t') = 1$ and $m(t;t') = 2$.

The relation between the numbers $n(t;t')$ and $m(t;t')$ can be clarified by introducing symmetry groups of trees. For a rooted tree t, let $V(t)$ be its set of vertices: then for each $v \in V(t)$, there is a rooted tree t_v consisting of v and its descendants with the order inherited from t. We call this the subtree of t with v as root. For $v \in V(t)$, let $SG(t, v)$ be the group of permutations of identical branches out of v, i.e., if $\{v_1, v_2, \ldots, v_k\}$ are the children of v, then $SG(t, v)$ is the group generated by the permutations that exchange t_{v_i} with t_{v_j} when they are isomorphic rooted trees. The symmetry group of t is the direct product

$$SG(t) = \prod_{v \in V(t)} SG(t, v).$$

For a given $v \in V(t)$, let $\operatorname{Fix}(v, t) \leq SG(t)$ be the subgroup of $SG(t)$ that fixes v; note that $\operatorname{Fix}(w, t) \leq \operatorname{Fix}(v, t)$ whenever w is a descendant of v.

Now suppose $t \lhd t'$, and let $v \in V(t)$ be such that, when a new edge and terminal vertex w are added to t at v, the result is t'. If $\operatorname{Orb}(v, t)$ is the orbit of v under $SG(t)$, then evidently

$$n(t;t') = |\operatorname{Orb}(v, t)| = |SG(t)/\operatorname{Fix}(v, t)| = \frac{|SG(t)|}{|\operatorname{Fix}(v, t)|}.$$

On the other hand, if $\operatorname{Orb}(w, t')$ is the orbit of $w \in V(t')$ under $SG(t')$, then

$$m(t;t') = |\operatorname{Orb}(w, t')| = |SG(t')/\operatorname{Fix}(w, t')| = \frac{|SG(t')|}{|\operatorname{Fix}(w, t')|}.$$

But there is an evident identification of $\operatorname{Fix}(v, t)$ with $\operatorname{Fix}(w, t')$, so we have the following result.

Proposition 2.1. *If $t \lhd t'$, then $|SG(t)|m(t;t') = n(t;t')|SG(t')|$.*

Let
$$k\{\mathcal{T}\} = \bigoplus_{n\geq 0} k\{\mathcal{T}_n\}$$
be the graded vector space (over a field k of characteristic 0) with basis consisting of rooted trees: we put the rooted tree t in grade $|t|-1$. We define two linear operators on $k\{\mathcal{T}\}$ as follows. For $n \geq 0$, the growth operator $\mathfrak{N} : k\{\mathcal{T}_n\} \to k\{\mathcal{T}_{n+1}\}$ is defined by
$$\mathfrak{N}(t) = \sum_{t \lhd t'} n(t;t')t', \tag{1}$$
and for $n \geq 1$ the pruning operator $\mathfrak{P} : k\{\mathcal{T}_n\} \to k\{\mathcal{T}_{n-1}\}$ is given by
$$\mathfrak{P}(t) = \sum_{t' \lhd t} m(t';t)t'; \tag{2}$$
we set $\mathfrak{P}(\bullet) = 0$. Then $\mathfrak{P}$ and $\mathfrak{N}$ satisfy the following commutation relation.

Proposition 2.2. *As operators on $k\{\mathcal{T}\}$, $[\mathfrak{P}, \mathfrak{N}] = \mathfrak{D}$, where $\mathfrak{D}$ is the operator given by $\mathfrak{D}(t) = |t|t$.*

Proof. It suffices to show $\mathfrak{P}\mathfrak{N}(t) - \mathfrak{N}\mathfrak{P}(t) = |t|t$ for any rooted tree t. Let $V(t) = \{v_1, \dots, v_n, v_{n+1}, \dots, v_{|t|}\}$ be the vertices of t, with v_i terminal for $1 \leq i \leq n$. Then
$$\mathfrak{N}(t) = \sum_{i=1}^{|t|} t_i \quad \text{and} \quad \mathfrak{P}(t) = \sum_{i=1}^{n} t^{(i)},$$
where t_i is the tree obtained from t by adding a new edge and terminal vertex to t at v_i, and $t^{(i)}$ comes from t by removing the edge that ends in v_i. Then
$$\mathfrak{P}\mathfrak{N}(t) = \sum_{i=1}^{|t|} \mathfrak{P}(t_i) = \sum_{i=1}^{|t|} \left(t + \sum_{1\leq j\leq n, j\neq i} (t_i)^{(j)} \right) = |t|t + \sum_{i=1}^{|t|} \sum_{1\leq j\leq n, j\neq i} (t_i)^{(j)}$$
and
$$\mathfrak{N}\mathfrak{P}(t) = \sum_{j=1}^{n} \mathfrak{N}(t^{(j)}) = \sum_{i=1}^{|t|} \sum_{1\leq j\leq n, j\neq i} (t^{(j)})_i.$$
Since $(t_i)^{(j)} = (t^{(j)})_i$ for $i \neq j$, the conclusion follows. □

Now we can endow $k\{\mathcal{T}\}$ with an inner product by setting
$$(t, t') = |SG(t)|\delta_{t,t'}$$
for any rooted trees t, t'.

Proposition 2.3. *The operators $\mathfrak{N}$ and $\mathfrak{P}$ are adjoint with respect to the inner product $(\cdot,\cdot)$.*

Proof. It suffices to show
$$(\mathfrak{N}(t), t') = (t, \mathfrak{P}(t'))$$
when $t \lhd t'$ (otherwise both sides are zero). In this case, we have
$$(\mathfrak{N}(t), t') = n(t;t')(t',t') = n(t;t')|SG(t')|$$
from equation (1) and
$$(t, \mathfrak{P}(t')) = m(t;t')(t,t) = m(t;t')|SG(t)|$$
from equation (2); but then the result follows by Proposition 2.1. □

Putting the last two results together gives the following.

Proposition 2.4. *For rooted trees t_1 and t_2,*
$$(\mathfrak{N}(t_1), \mathfrak{N}(t_2)) - (\mathfrak{P}(t_1), \mathfrak{P}(t_2)) = \begin{cases} 0, & \text{if } t_1 \neq t_2, \\ |t||SG(t)|, & \text{if } t_1 = t_2 = t. \end{cases}$$

Proof. This follows from the calculation
$$(\mathfrak{N}(t_1), \mathfrak{N}(t_2)) - (\mathfrak{P}(t_1), \mathfrak{P}(t_2)) = (t_1, (\mathfrak{P}\mathfrak{N} - \mathfrak{N}\mathfrak{P})(t_2)) = (t_1, \mathfrak{D}(t_2)) = |t_2|(t_1, t_2).$$
□

Remark. The second alternative of this result can be written
$$\sum_{t \lhd t'} n(t;t')^2 |SG(t')| - \sum_{t'' \lhd t} m(t'';t)^2 |SG(t'')| = |t||SG(t)|,$$
or, dividing by $|SG(t)|$,
$$\sum_{t \lhd t'} n(t;t')m(t;t') - \sum_{t'' \lhd t} n(t'';t)m(t'';t) = |t|$$
for any rooted tree t.

We can extend the definitions of $m(t;t')$ and $n(t;t')$ to any pair of rooted trees t, t' with $|t'| - |t| = k \geq 0$ by setting
$$\mathfrak{N}^k(t) = \sum_{|t'|=|t|+k} n(t;t')t' \tag{3}$$
and
$$\mathfrak{P}^k(t') = \sum_{|t'|=|t|+k} m(t;t')t. \tag{4}$$
With these definitions, we have the following result.

Proposition 2.5. *Let t, t' be rooted trees with $|t| \le |t'|$. Then*
1. $n(t;t')|SG(t')| = |SG(t)|m(t;t')$.
2. If $|t| \le k \le |t'|$*, then*

$$n(t;t') = \sum_{|t''|=k} n(t;t'')n(t'';t'),$$

and similarly for n replaced by m.
3. $n(t;t')$ *and* $m(t;t')$ *are nonzero if and only if* $t \preceq t'$.

Proof. The first part follows immediately from equations (3) and (4):

$$n(t;t')|SG(t')| = (t', \mathfrak{N}^{|t'|-|t|}(t)) = (\mathfrak{P}^{|t'|-|t|}(t'), t) = m(t;t')|SG(t)|.$$

For the second part, we have for $|t| \le k \le |t'|$,

$$\begin{aligned}
n(t;t') &= \frac{(\mathfrak{N}^{|t'|-|t|}(t), t')}{|SG(t')|} \\
&= \frac{(\mathfrak{N}^{k-|t|}(t), \mathfrak{P}^{|t'|-k}(t'))}{|SG(t')|} \\
&= \sum_{|t''|=k} \frac{(\mathfrak{N}^{k-|t|}(t), m(t'';t')t'')}{|SG(t')|} \\
&= \sum_{|t''|=k} \frac{(\mathfrak{N}^{k-|t|}(t), t'')}{|SG(t'')|} \frac{|SG(t'')|}{|SG(t')|} m(t'';t') \\
&= \sum_{|t''|=k} n(t;t'')n(t'';t').
\end{aligned}$$

(For the corresponding equation with m replacing n, reverse the roles of $\mathfrak{N}$ and $\mathfrak{P}$.) Finally, the third part is evident for $|t'| - |t| = 1$ and can be proved by induction on $|t'| - |t|$ using the second part. ☐

Remark. The second and third parts say that $\mathfrak{T}$ is a weighted-relation poset, in the terminology of [11], for either of the weights $n(t;t')$ or $m(t;t')$. In fact, $\mathfrak{T}$ with weights $n(t;t')$ is discussed as Example 7 in [11]. In the terminology of [7], $\mathfrak{T}$ with multiplicities $n(t;t')$ and $\mathfrak{T}$ with multiplicities $m(t;t')$ are a pair of graded graphs that are $\mathbf{r}$-dual for the sequence $\mathbf{r} = (0, 1, 2, \dots)$.

If $t \preceq t'$, we can think of $n(t;t')$ as counting the ways of building up t' from t by adding new edges and terminal vertices, and $m(t;t')$ as counting ways of getting from t' to t by removing terminal edges. In particular, since $\bullet \preceq t$ for every rooted tree t, we can think of $n(\bullet;t)$ as the number of ways to build up t, and $m(\bullet;t)$ as the number of ways to tear it down. A more precise formulation can be given using the idea of labellings of trees: a labelling of a rooted tree t is a bijection $f : V(t) \to \{0, 1, \dots, |t|\}$ such that $f(v) > f(w)$ whenever v is a descendant of w (necessarily f sends the root vertex to 0). We call labellings f and g equivalent if $f\phi = g$ for some $\phi \in SG(t)$.

Proposition 2.6. *Let t be a rooted tree. Then t has $m(\bullet;t)$ labellings and $n(\bullet;t)$ labellings mod equivalence.*

Proof. First note that $|SG(t)|n(\bullet;t) = m(\bullet;t)$ by the first part of Proposition 2.5 since $\bullet$ has trivial symmetry group. It follows from the discussion of [11, Ex. 7] that $n(\bullet;t)$ counts labellings mod equivalence, and the statement about $m(\bullet;t)$ follows since each equivalence class of labellings has $|SG(t)|$ elements. □

Remark. The "Connes-Moscovici weight" [1, 15] or "tree multiplicity" [2] of t is $n(\bullet;t)$. Cf. [20, Sect. 22] and [12, Ex. 5.1.4-20], where a hook-length formula for the number of labellings of t is given: this is $m(\bullet;t)$.

In [22] Stanley defined the notion of a sequentially differential poset (generalizing his definition of a differential poset in [21]). A sequentially differential poset P is a locally finite graded poset with a single element $\hat{0}$ in grade 0, so that the linear operators

$$U(p) = \sum_{p \lhd p'} p' \quad \text{and} \quad D(p) = \sum_{p' \lhd p} p'$$

on $k\{P\}$ satisfy the identity $(DU - UD)(p) = r_j p$ for any $p \in P$ of rank j: here $r_0, r_1, \dots$ are nonnegative integers. The results of [22] can be applied to $\mathfrak{T}$ (with $r_j = j+1$), provided we replace U and D with $\mathfrak{N}$ and $\mathfrak{P}$ respectively, and suitably reinterpret the statements of theorems to incorporate multiplicities. For example, for $x \in P$ Stanley writes $e(x)$ for the number of saturated chains from $\hat{0}$ to x, but in the proofs $e(x)$ really appears as the inner product of x with $U^k\hat{0}$, where k is the rank of x: so for a rooted tree $x \in \mathfrak{T}_k$ we replace $e(x)$ by

$$(\mathfrak{N}^k \bullet, x) = n(\bullet;x)(x,x) = n(\bullet;x)|SG(x)| = m(\bullet;x).$$

Let $w = w_1 w_2 \cdots w_r$ be a word in $\mathfrak{N}$ and $\mathfrak{P}$, and let $x \in \mathfrak{T}_k$. Clearly $(w\bullet, x) = 0$ unless (a) for each $1 \le i \le r$, the number of $\mathfrak{P}$'s in $w_i w_{i+1} \cdots w_r$ does not exceed the number of $\mathfrak{N}$'s; and (b) the number of $\mathfrak{N}$'s minus the number of $\mathfrak{P}$'s in w is k. In this case we call w a valid x-word, and we have the following result.

Proposition 2.7. *Let $x \in \mathfrak{T}_k$, $w = w_1 \cdots w_r$ a valid x-word. Let $S = \{i : w_i = \mathfrak{P}\}$. For each $i \in S$, let $a_i = |\{j : j \ge i, w_j = \mathfrak{P}\}|$, $b_i = |\{j : j > i, w_j = \mathfrak{N}\}|$, and $c_i = b_i - a_i$. Then*

$$(w\bullet, x) = m(\bullet;x) \prod_{i \in S} \binom{c_i + 2}{2}.$$

Proof. Replace U, D by $\mathfrak{N}, \mathfrak{P}$ in Theorem 2.3 of [22]. □

This result has the following corollary (cf. Theorem 1.5.2 of [7]).

Proposition 2.8. *For any rooted tree $x \in \mathfrak{T}_k$ and nonnegative integer a,*

$$\sum_{|t|=k+a+1} m(x;t)n(\bullet;t) = n(\bullet;x) \prod_{i=2}^{a+1} \binom{k+i}{2}.$$

Proof. In Proposition 2.7 set $w = \mathfrak{P}^a \mathfrak{N}^{a+k}$ to get

$$(\mathfrak{P}^a \mathfrak{N}^{a+k} \bullet, x) = m(\bullet; x) \prod_{i=1}^{a} \binom{k+a-i+2}{2} = m(\bullet; x) \prod_{i=2}^{a+1} \binom{k+i}{2}.$$

Now the left-hand side can be expanded as

$$(\mathfrak{N}^{a+k} \bullet, \mathfrak{N}^a(x)) = \sum_{|t|=k+a+1} n(x;t)(\mathfrak{N}^{a+k}\bullet, t) = \sum_{|t|=k+a+1} n(x;t) m(\bullet; t)$$
$$= \sum_{|t|=k+a+1} m(x;t)|SG(x)| n(\bullet; t),$$

where we have the first part of Proposition 2.5 in the last step. Hence

$$\sum_{|t|=k+a+1} m(x;t)|SG(x)|n(\bullet;t) = m(\bullet;x) \prod_{i=2}^{a+1} \binom{k+i}{2},$$

and dividing by $|SG(x)|$ gives the conclusion. □

Remark. In the case $x = \bullet$, this result becomes

$$\sum_{|t|=a+1} m(\bullet;t) n(\bullet;t) = \sum_{|t|=a+1} n(\bullet;t)^2 |SG(t)| = \prod_{i=2}^{a+1} \binom{i}{2}.$$

Cf. Corollary 1.5.4 of [7].

In [11, Ex. 7] it is shown that $\sum_{|t|=k+1} n(\bullet;t) = k!$. Further sum formulas involving $n(\bullet;t)$ appear in [15, Sect. 5] and [2, Sect. 5]. A result of [22] gives a formula for $\sum_{|t|=k+1} m(\bullet;t)$. To state it we will need some definitions. Let $\mathrm{Inv}(k)$ be the set of involutions in the group Σ_k of permutations of $\{1, 2, \ldots, k\}$. For $\sigma \in \Sigma_k$, call i a weak excedance of σ if $\sigma(i) \geq i$; let $\mathrm{Wex}(\sigma)$ be the set of weak excedances of σ. For $\sigma \in \Sigma_k$ and $i \in \{1, \ldots, k\}$, let $\eta(\sigma, i)$ be the number of integers j such that $j < i$ and $\sigma(j) < \sigma(i)$. Then we have the following result.

Proposition 2.9. *With the definitions above,*

$$\sum_{|t|=k+1} m(\bullet;t) = \sum_{\sigma \in \mathrm{Inv}(k)} \prod_{i \in \mathrm{Wex}(\sigma)} (\eta(\sigma, i) + 1).$$

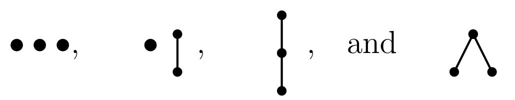
Proof. In the proof of Theorem 2.1 of [22], replace U, D with $\mathfrak{N}, \mathfrak{P}$: in the conclusion, this replaces $\alpha(0 \to k) = \sum_{\mathrm{rank}\, x = k} e(x)$ with $\sum_{|t|=k+1} m(\bullet;t)$. □

For example, a sum over the four involutions 123, 213, 132, and 321 of Σ_3 gives

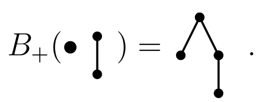
$$\sum_{|t|=4} m(\bullet;t) = 1 \cdot 2 \cdot 3 + 1 \cdot 3 + 1 \cdot 2 + 1 \cdot 1 = 12.$$

3 Kreimer's Hopf Algebra

In this section we discuss the Hopf algebra $\mathcal{H}_K$ defined by D. Kreimer and his collaborators [1, 3, 13, 14, 15, 16] in connection with renormalization. As an algebra $\mathcal{H}_K$ is generated by rooted trees; so as a vector space $\mathcal{H}_K$ is generated by monomials in rooted trees, i.e.,"forests" of rooted trees. For a rooted tree t, we give the corresponding generator degree $|t|$ in $\mathcal{H}_K$; in degree 0, $\mathcal{H}_K$ is generated by the unit element 1. For example, the degree-3 part of $\mathcal{H}_K$ is generated as a vector space by the four elements



There is a linear map $B_+ : \mathcal{H}_K \to k\{\mathcal{T}\}$ which takes a forest to a single tree with a new root vertex connected to all the roots of the forest: e.g.,



The map B_+ takes the degree-n part of $\mathcal{H}_K$ onto $k\{\mathcal{T}_n\}$: if we set $B_+(1) = \bullet$, then B_+ is a vector space isomorphism. We write B_- for the inverse of B_+. On the other hand, except for the degree shift, $\mathcal{H}_K$ is just the symmetric algebra on $k\{\mathcal{T}\}$. Thus, if $T_n = \dim k\{\mathcal{T}_n\} = |\mathcal{T}_n|$, we have

$$\sum_{n\geq 0} T_n x^n = \prod_{n\geq 1} \frac{1}{(1-x^n)^{T_{n-1}}} \tag{5}$$

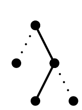
from which we can compute recursively $T_0 = 1$, $T_1 = 1$, $T_2 = 2$, $T_3 = 4$, $T_4 = 9$, etc. (see [19] for more information).

To define the bialgebra structure on $\mathcal{H}_K$, we let the counit send all elements of positive degree to 0, and the unit element 1 in degree 0 to $1 \in k$. The comultiplication Δ has $\Delta(1) = 1 \otimes 1$,

$$\Delta(t) = t \otimes 1 + (\mathrm{id} \otimes B_+)\Delta(B_-(t)) \tag{6}$$

for a rooted tree t, and $\Delta(t_1 t_2 \cdots t_n) = \Delta(t_1)\Delta(t_2)\cdots\Delta(t_n)$ for monomials $t_1 t_2 \cdots t_n$.

Equation (6) gives a recursive definition of the coproduct, but there is also a nonrecursive definition in terms of cuts. A cut of a rooted tree t is a set of edges of t. A cut is elementary if its cardinality is 1. When the elements of a cut C of t are removed, what remains is a collection of rooted trees: the one containing the root is denoted $R^C(t)$, and the remaining rooted trees form a monomial denoted $P^C(t)$. For example, if t is the tree



and C consists of the dotted edges, then

$$R^C(t) = \quad$$  $$\quad \text{and} \quad P^C(t) = \bullet\,\bullet\,.$$

The order of a cut C of t is the largest number of edges in C between the root of t and any of its terminal vertices: a cut of order at most 1 is called admissible. The empty cut $\emptyset$ is the only cut of order 0 (note that $R^{\emptyset}(t) = t$ and $P^{\emptyset}(t) = 1$). The following formula for $\Delta(t)$ is proved in [3].

Proposition 3.1. *For a rooted tree t, $\Delta(t)$ can be written*

$$\Delta(t) = t \otimes 1 + \sum_{C \text{ admissible cut of } t} P^C(t) \otimes R^C(t).$$

We can define growth and pruning operators N and P on $\mathcal{H}_K$ as follows. The growth operator N is simply $\mathfrak{N}$ extended as a derivation, i.e.,

$$N(t_1 t_2 \cdots t_n) = \sum_{i=1}^{n} t_1 \cdots \mathfrak{N}(t_i) \cdots t_n.$$

We also define P as a derivation, but set $P(t) = \mathfrak{P}(t)$ only for $|t| \geq 2$; we put $P(\bullet) = 1$. If $D : \mathcal{H}_K \to \mathcal{H}_K$ is the extension of $\mathfrak{D}$ as a derivation (i.e., the linear map that multiplies a monomial by its degree), then the identity

$$[P, N] = D \tag{7}$$

holds. To prove equation (7), note that both sides are derivations, so it suffices to prove it for rooted trees t; but in that case (7) follows from Proposition 2.2. The map B_+ interacts with the growth and pruning operators as follows.

Proposition 3.2. *For monomials u of $\mathcal{H}_K$,*
1. $B_+ P(u) = \mathfrak{P} B_+(u)$,
2. $B_+ N(u) = \mathfrak{N} B_+(u) - B_+(\bullet u)$.

Proof. Suppose $u = t_1 \cdots t_k$ with each $|t_i| \geq 1$. Then applying B_+ to

$$P(u) = P(t_1) t_2 \cdots t_k + t_1 P(t_2) t_3 \cdots t_k + \cdots + t_1 \cdots t_{n-1} P(t_n)$$

gives a sum of rooted trees that includes all those obtained by removing terminal edges of $B_+(u)$, and the cases with $t_i = \bullet$ (hence $P(t_i) = 1$) work correctly: these are exactly those cases where an edge coming out of the root of $B_+(u)$ is terminal. If $u = 1$, then $\mathfrak{P} B_+(1) = \mathfrak{P}(\bullet) = 0 = B_+ P(1)$. So in any case $B_+ P(u)$ coincides with $\mathfrak{P} B_+(u)$.

Now for $u = t_1 \cdots t_k$, B_+ applied to

$$N(u) = N(t_1) t_2 \cdots t_k + t_1 N(t_2) t_3 \cdots t_k + \cdots + t_1 \cdots t_{n-1} N(t_n)$$

will include all those trees obtained by adding a new edge to each vertex of $B_+(u)$ except one–the "new" root vertex. Thus, $B_+ N(u)$ is missing the term obtained by adding a new edge to the root of $B_+(u)$, namely $B_+(\bullet u)$. On the other hand, if $u = 1$ we have $B_+ N(1) = 0 = \mathfrak{N}(\bullet) - B_+(\bullet)$. $\square$

We can extend the inner product of the previous section to $\mathcal{H}_K$ by setting

$$(u_1, u_2) = (B_+(u_1), B_+(u_2))$$

for monomials u_1, u_2; there is no ambiguity since $(B_+(t), B_+(t')) = (t, t')$ for rooted trees t, t'. With this definition, we can state the adjointness relation between P and N.

Proposition 3.3. *On $\mathcal{H}_K$, the adjoint of P with respect to the inner product above is $N + M_\bullet$, where $M_\bullet$ is the operator that sends u to $\bullet u$; equivalently, the adjoint of N is $P - \frac{\partial}{\partial \bullet}$.*

Proof. Let u_1, u_2 be monomials of $\mathcal{H}_K$. Then

$$(u_1, P(u_2)) = (B_+(u_1), B_+P(u_2)) = (B_+(u_1), \mathfrak{P}B_+(u_2)) = (\mathfrak{N}B_+(u_1), B_+(u_2)),$$

from which the first statement follows using the second part of Proposition 3.2. For the second statement, note that $\frac{\partial}{\partial \bullet}$ is adjoint to $M_\bullet$. □

We now compute the characteristic polynomial of the restriction PN_k of PN to the degree-k part of $\mathcal{H}_K$. Let $\mathrm{Ch}(L, \lambda) = \det(\lambda I - L)$ for a linear transformation L.

Proposition 3.4. *For $k \geq 1$,*

$$\mathrm{Ch}(PN_k, \lambda) = \left(\lambda - \binom{k+1}{2}\right) \prod_{r=0}^{k-1} \left(\lambda - \sum_{j=0}^{r} (k-j)\right)^{T_{k-r} - T_{k-r-1}},$$

where as above T_i is the dimension of the degree-i part of $\mathcal{H}_K$.

Proof. We follow the argument of [21, Theorem 4.1]. Evidently PN_1 is the identity, so the result holds for $k = 1$; assume it inductively for $k \geq 1$. From elementary linear algebra

$$\mathrm{Ch}(NP_{k+1}, \lambda) = \lambda^{T_{k+1} - T_k} \mathrm{Ch}(PN_k, \lambda),$$

while from equation (7) we have

$$\mathrm{Ch}(PN_{k+1}, \lambda) = \mathrm{Ch}(NP_{k+1}, \lambda - (k+1))$$

since $D_{k+1} = (k+1)I$. The induction step then follows. □

The preceding result implies that N_k is injective for all $k \geq 1$, and that P_k is surjective for $k \geq 2$; of course P_1 is also surjective. In addition, the maximal eigenvalue of PN_k is $\binom{k+1}{2}$. In fact, the element

$$f_k = N^{k-1}(\bullet) = \sum_{|t|=k} n(\bullet; t) t$$

is a corresponding eigenvector. To see this, note that

$$PN_k(f_k) = P(f_{k+1}) = \sum_{|t'|=k} t' \sum_{|t|=k+1} n(\bullet;t)m(t';t) = \sum_{|t'|=k} n(\bullet;t')\binom{k+1}{2}t' = \binom{k+1}{2}f_k,$$

where we have used Proposition 2.8. The f_k are the "naturally grown forests" of [3] (where they are denoted δ_k).

The following result, which describes how N behaves with respect to the coproduct, is essentially [3, Prop. 6]. We give the proof since it can be stated concisely and illustrates the use of Proposition 3.1.

Proposition 3.5. *$\Delta N = (N \otimes \mathrm{id} + \mathrm{id} \otimes N + M_\bullet \otimes D)\Delta$.*

Proof. Since both sides are derivations, it suffices to show that

$$\Delta N(t) = (N \otimes \mathrm{id} + \mathrm{id} \otimes N + M_\bullet \otimes D)\Delta(t)$$

for any rooted tree t. As in the proof of Proposition 2.2, write $N(t) = \sum_i t_i$, where each t_i is the result of adding an edge to t. Then

$$\begin{aligned}\Delta N(t) &= \sum_i t_i \otimes 1 + \sum_i \sum_{C_i \text{ admissible cut of } t_i} P^{C_i}(t_i) \otimes R^{C_i}(t_i) \\ &= N(t) \otimes 1 + \sum_i \sum_{C_i \text{ admissible cut of } t_i} P^{C_i}(t_i) \otimes R^{C_i}(t_i).\end{aligned}$$

Now each cut C_i of t_i either includes the "new" edge or it doesn't. Suppose first that C_i does not include the new edge. Then C_i corresponds to a cut C of t and either $P^{C_i}(t_i) \otimes R^{C_i}(t_i)$ is a term in $P^C(t) \otimes NR^C(t)$ (if the new edge is in the component of the root) or a term in $NP^C(t) \otimes R^C(t)$ (if it isn't). Together with the leading term $N(t) \otimes 1$, these give all the terms of $(N \otimes \mathrm{id} + \mathrm{id} \otimes N)\Delta(t)$.

Now suppose that C_i includes the new edge of t_i. If C is the cut of t given by C_i minus the new edge, then the new edge must have been attached to a vertex of $R^C(t)$ (by the definition of admissibility), and so

$$P^{C_i}(t_i) \otimes R^{C_i}(t_i) = \bullet P^C(t) \otimes R^C(t).$$

Since (for each admissible cut C of t) there are $|R^C(t)|$ vertices to which the new edge could be attached, terms of this form contribute $(M_\bullet \otimes D)\Delta(t)$. □

Remark. It follows from this result that the f_k, $k \geq 1$, generate a sub-Hopf-algebra of $\mathcal{H}_K$. This Hopf algebra is isomorphic to the graded dual of the universal enveloping algebra of $\mathcal{A}^1$, the Lie algebra of formal vector fields on $\mathbf{R}$ that vanish to order 2 at the origin (see [3]).

Since $\mathcal{H}_K$ is a locally finite commutative Hopf algebra, its graded dual $\mathcal{H}_K^{gr}$ is a locally finite cocommutative Hopf algebra, hence (by the results of [17]) the universal enveloping algebra of the Lie algebra $\mathcal{P}(\mathcal{H}_K^{gr})$, the primitives of $\mathcal{H}_K^{gr}$. Primitives of $\mathcal{H}_K^{gr}$ are dual to indecomposables of $\mathcal{H}_K$, and so are linear combinations of elements Z_t for rooted trees t, where $\langle Z_t, u\rangle = \delta_{t,u}$ for monomials $u \in \mathcal{H}_K$. The duals of N and P can be described as follows.

Proposition 3.6. *1. N^* is given by $N^*(Z_\bullet) = 0$,*

$$N^*(Z_t) = \sum_{|t'|=|t|-1} n(t';t)Z_{t'} \tag{8}$$

for $|t| \geq 2$, and

$$N^*(wv) = (N^*w)v + w(N^*v) + \frac{\partial w}{\partial Z_\bullet}|v|v \tag{9}$$

for $w, v \in \mathcal{H}_K^{gr}$.
2. $P^*(w) = Z_\bullet w$ for $w \in \mathcal{H}_K^{gr}$.

Proof. To prove the statements about $N^*(Z_t)$, note that $\langle N^*(Z_t), u\rangle = \langle Z_t, N(u)\rangle$ is zero unless u is a scalar multiple of t', for some $t' \lhd t$; but then equation (8) follows from equation (1). Equation (9) follows from Proposition 3.5 since the multiplication in $\mathcal{H}_K^{gr}$ is induced by Δ.

For the second part, let t be a rooted tree. If we write $P(t) = \sum_i t^{(i)}$ as in the proof of Proposition 2.2, then evidently

$$\bullet \otimes P(t) = \sum_i \bullet \otimes t^{(i)}$$

are (by Proposition 3.1) exactly those terms of $\Delta(t)$ of the form $\bullet \otimes t'$. Now let $u = t_1 t_2 \cdots t_n$ be a monomial of $\mathcal{H}_K$. Then

$$\begin{aligned}
\Delta(u) &= \prod_{i=1}^n \Delta(t_i) \\
&= \prod_{i=1}^n (1 \otimes t_i + \bullet \otimes P(t_i) + \cdots) \\
&= 1 \otimes t_1 t_2 \cdots t_n + \bullet \otimes (P(t_1)t_2 \cdots t_n + \cdots + t_1 \cdots t_{n-1}P(t_n)) + \cdots \\
&= 1 \otimes u + \bullet \otimes P(u) + \cdots
\end{aligned}$$

and thus

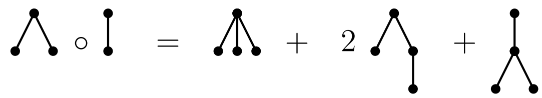
$$\langle Z_\bullet w, u\rangle = \langle Z_\bullet \otimes w, \Delta(u)\rangle = \langle w, P(u)\rangle = \langle P^*(w), u\rangle$$

for all $w \in \mathcal{H}_K^{gr}$ and monomials u of $\mathcal{H}_K$. □

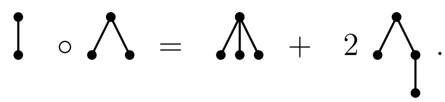
Remark. The Lie algebra $\mathcal{P}(\mathcal{H}_K^{gr})$ is in fact free; see [5].

4 The Grossman-Larson Hopf Algebra

We can define a noncommutative multiplication on the graded vector space $k\{\mathcal{T}\}$ as follows. Let t, t' be rooted trees, and suppose $B_-(t) = t_1 t_2 \cdots t_k$. There are $|t'|^k$ rooted trees obtainable by attaching each of the k rooted trees $t_1, t_2, \ldots, t_k$ to some vertex of t' (by a new edge): let $t \circ t' \in k\{\mathcal{T}\}$ be the sum of these trees (If $t = \bullet$, we define $t \circ t'$ to be t'). For example,



while



This product makes $k\{\mathcal{T}\}$ a graded algebra: note that for $t \in k\{\mathcal{T}_n\}$ and $t' \in k\{\mathcal{T}_m\}$, we have $t \circ t' \in k\{\mathcal{T}_{n+m}\}$. The element $\bullet \in \mathcal{T}_0$ is a two-sided identity. Note also that

$$B_+(\bullet) \circ t =$$  $$\circ\, t = \mathfrak{N}(t)$$

for any rooted tree t.

Now define a coproduct $\Delta : k\{\mathcal{T}\} \to k\{\mathcal{T}\} \otimes k\{\mathcal{T}\}$ by

$$\Delta(t) = \sum_{I \cup J = \{1,\ldots,k\}} B_+(t_I) \otimes B_+(t_J) \tag{10}$$

where $B_-(t) = t_1 \cdots t_k$ and the sum is over pairs (I, J) of (possibly empty) subsets I, J of $\{1, \ldots, k\}$ such that $I \cup J = \{1, \ldots, k\}$: t_I means the product of t_i for $i \in I$. The following result is proved in [10] and [9]: the main things to check are the associativity of the product $\circ$ (Lemma 2.6 of [10]) and the compatibility of the coproduct with $\circ$ (Lemma 2.8 of [10]).

Proposition 4.1. *The vector space $k\{\mathcal{T}\}$ with product $\circ$ and coproduct Δ is a graded Hopf algebra $\mathcal{H}_{GL}$.*

Since the coproduct Δ is cocommutative, by results of [17] it follows that $\mathcal{H}_{GL}$ is the universal enveloping algebra on its Lie algebra $\mathcal{P}(\mathcal{H}_{GL})$ of primitives. From equation (10), elements of the form $B_+(t)$, where t is a rooted tree, are primitive. We call such elements "primitive trees": they are those rooted trees whose root has exactly one child. If we let $\mathcal{PT}$ be the set of primitive trees (graded, like $\mathcal{T}$, by the number of non-root vertices), then we have following result (for another proof, see [10, Theorem 4.1]).

Proposition 4.2. *The vector space $k\{\mathcal{PT}\}$ generated by the primitive trees is $\mathcal{P}(\mathcal{H}_{GL})$.*

Proof. Since

$$B_+(t_1) \circ B_+(t_2) = B_+(t_1 t_2) + B_+(B_+(t_1) \circ t_2),$$

$k\{\mathcal{PT}\} \subseteq \mathcal{P}(\mathcal{H}_{GL})$ is a sub-Lie-algebra. Also, since B_+ is an isomorphism of $k\{\mathcal{T}_{n-1}\}$ onto $k\{\mathcal{PT}_n\}$, we have $\dim k\{\mathcal{PT}_n\} = T_{n-1}$. Then the Poincaré-Birkhoff-Witt theorem implies that the universal enveloping algebra of $k\{\mathcal{PT}\}$ has the same dimension in grade n as does the symmetric algebra on $k\{\mathcal{PT}\}$: but in view of equation (5), this is $T_n = \dim(\mathcal{H}_{GL})_n$. Hence $k\{\mathcal{PT}\} = \mathcal{P}(\mathcal{H}_{GL})$. □

Suppose t_1, t_2, t_3 are rooted trees so that $|t_1| + |t_2| = |t_3|$. If there is an elementary cut C of t_3 so that

$$P^C(t_3) = t_1 \quad \text{and} \quad R^C(t_3) = t_2, \tag{11}$$

let $m(t_1, t_2; t_3)$ be the number of distinct elementary cuts C of t_3 for which equations (11) hold: otherwise, set $m(t_1, t_2; t_3) = 0$. If (and only if) $m(t_1, t_2; t_3) \neq 0$, it is also true that t_3 can be obtained by attaching (via a new edge) the root vertex of t_1 to some vertex of t_2: let $n(t_1, t_2; t_3)$ be the number of vertices of t_2 for which this is true. Evidently

$$n(\bullet, t_2; t_3) = n(t_2; t_3) \quad \text{and} \quad m(\bullet, t_2; t_3) = m(t_2; t_3)$$

for trees $t_2 \lhd t_3$, so we have generalized the multiplicities of §2. (The reader is warned that $n(t_1, t_2; t_3)$ as used in [3] and [5] is our $m(t_1, t_2; t_3)$.) We now show how symmetry groups can be used to relate the two multiplicities.

Proposition 4.3. *For rooted trees t_1, t_2, t_3 with $|t_1| + |t_2| = |t_3|$,*

$$|SG(t_1)||SG(t_2)|m(t_1, t_2; t_3) = n(t_1, t_2; t_3)|SG(t_3)|.$$

Proof. For any rooted tree t and $v \in V(t)$, let $\operatorname{Fix}(t_v, t) \leq SG(t)$ be the subgroup of $SG(t)$ that holds t_v (the subtree of t with v as root) pointwise fixed. We can assume there is an elementary cut $C = \{e\}$ of t_3 so that equations (11) hold (otherwise both sides of the conclusion are zero). If e has source v and target w, then t_w is isomorphic to t_1. Also, if $\operatorname{Orb}(e, t_3)$ is the orbit of e under $SG(t_3)$, then

$$m(t_1, t_2; t_3) = |\operatorname{Orb}(e, t_3)| = |SG(t_3)/\operatorname{Fix}(t_v, t_3) \times SG(t_w)| = \frac{|SG(t_3)|}{|\operatorname{Fix}(t_v, t_3)||SG(t_1)|}.$$

On the other hand, since $R^C(t_3)$ is isomorphic to t_2,

$$n(t_1, t_2; t_3) = |\operatorname{Orb}(v, R^C(t_3))| = |SG(R^C(t_3))/\operatorname{Fix}(v, R^C(t_3))| = \frac{|SG(t_2)|}{|\operatorname{Fix}(v, R^C(t_3))|}.$$

Since there is an evident identification of $\operatorname{Fix}(t_v, t_3)$ with $\operatorname{Fix}(v, R^C(t_3))$, we have

$$\frac{|SG(t_1)|}{|SG(t_3)|} m(t_1, t_2; t_3) = \frac{n(t_1, t_2; t_3)}{|SG(t_2)|}$$

and the conclusion follows. □

We can now use the inner product on Kreimer's Hopf algebra $\mathcal{H}_K$ to define an isomorphism of $\mathcal{H}_{GL}$ onto the graded dual of $\mathcal{H}_K$.

Proposition 4.4. *There is an isomorphism* $\chi : \mathcal{H}_{GL} \to \mathcal{H}_K^{gr}$ *defined by*

$$\langle \chi(t), u \rangle = (B_-(t), u) = (t, B_+(u))$$

for any rooted tree t *and monomial* u *of* $\mathcal{H}_K$.

Proof. Since $\mathcal{H}_K$ is locally finite, it suffices to prove that χ is an injective homomorphism. We first show χ is a homomorphism, i.e., that

$$\langle \chi(t_1 \circ t_2), u \rangle = \langle \chi(t_1) \otimes \chi(t_2), \Delta(u) \rangle = \sum_u \langle \chi(t_1), u' \rangle \langle \chi(t_2), u'' \rangle = \sum_u (t_1, B_+(u'))(t_2, B_+(u''))$$

for any monomial u of $\mathcal{H}_K$ with coproduct

$$\Delta(u) = \sum_u u' \otimes u''. \tag{12}$$

In view of Proposition 4.2, $\mathcal{H}_{GL}$ is generated as an algebra by the primitive trees. So it suffices to show that

$$\langle \chi(B_+(t) \circ t_2), u \rangle = \sum_u (B_+(t), B_+(u'))(t_2, B_+(u'')) = \sum_u (t, u')(t_2, B_+(u'')). \tag{13}$$

Now from the definition of $n(t_1, t_2; t_3)$,

$$\langle \chi(B_+(t) \circ t_2), u \rangle = (B_+(t) \circ t_2, B_+(u)) = \sum_{|t_3| = |t| + |t_2|} n(t, t_2; t_3)(t_3, B_+(u)) = n(t, t_2; B_+(u))|SG(B_+(u))|.$$

On the other hand, if $\Delta(u)$ is given by equation (12), then

$$\Delta(B_+(u)) = B_+(u) \otimes 1 + \sum_u u' \otimes B_+(u'') \tag{14}$$

by equation (6). Now the only nonzero terms of

$$\sum_u (t, u')(t_2, B_+(u''))$$

are those with $u' = t$ and $t_2 = B_+(u'')$: and (comparing Proposition 3.1 with equation (14)) there are $m(t, t_2; B_+(u))$ such terms. Hence

$$\sum_u (t, u')(t_2, B_+(u'')) = m(t, t_2; B_+(u))|SG(t)||SG(t_2)|,$$

and equation (13) follows from Proposition 4.3: thus, χ is a homomorphism.

Now suppose $v = \sum_i a_i t_i \in \ker \chi$. Then

$$\langle \chi(v), u \rangle = \sum_i a_i (t_i, B_+(u)) = 0$$

for all monomials u of $\mathcal{H}_K$. But setting $u = B_-(t_i)$ implies that $a_i = 0$ for each i, so $v = 0$. □

Remark. In [18, Prop. 2.1] (and also in [9, Theorem 14.16]) it is wrongly asserted that the map sending $B_+(t)$ to Z_t induces an isomorphism of $\mathcal{H}_{GL}$ onto $\mathcal{H}_K^{gr}$: the error is due to a failure to distinguish the multiplicities $n(t_1, t_2; t_3)$ and $m(t_1, t_2; t_3)$, since Panaite confuses the coefficients in

$$[Z_{t_1}, Z_{t_2}] = \sum_{|t_3|=|t_1|+|t_2|} (m(t_1, t_2; t_3) - m(t_2, t_1; t_3)) Z_{t_3}$$

with those in

$$B_+(t_1) \circ B_+(t_2) - B_+(t_2) \circ B_+(t_1) = \sum_{|t_3|=|t_1|+|t_2|} (n(t_1, t_2; t_3) - n(t_2, t_1; t_3)) B_+(t_3).$$

In fact, since

$$\langle \chi(B_+(t)), u \rangle = (t, u) = |SG(t)| \delta_{t,u} = |SG(t)| \langle Z_t, u \rangle,$$

we have $\chi(B_+(t)) = |SG(t)| Z_t$.

We can use the isomorphism χ to express the duals of P and N as maps of $\mathcal{H}_{GL}$ (cf. Proposition 3.6 above).

Proposition 4.5. *1. The map $\chi^{-1} P^* \chi : \mathcal{H}_{GL} \to \mathcal{H}_{GL}$ is left multiplication by $B_+(\bullet)$, i.e., $\chi^{-1} P^* \chi(t) = \mathfrak{N}(t) = B_+(\bullet) \circ t$.*
2. For rooted trees t, $\chi^{-1} N^* \chi(t) = \mathfrak{P}(t) - B_+ \frac{\partial}{\partial \bullet} B_-(t)$.

Proof. Using Propositions 2.3, 3.2, and 3.3, we have for any rooted tree t and monomial u of $\mathcal{H}_K$,

$$\langle \chi(t), P(u) \rangle = (t, B_+ P(u)) = (t, \mathfrak{P} B_+(u)) = (\mathfrak{N}(t), B_+(u)) = \langle \chi(\mathfrak{N}(t)), u \rangle$$

and

$$\begin{aligned} \langle \chi(t), N(u) \rangle &= (t, B_+ N(u)) = (t, \mathfrak{N} B_+(u)) - (t, B_+(\bullet u)) = (\mathfrak{P}(t), B_+(u)) - (B_-(t), \bullet u) \\ &= (\mathfrak{P}(t), B_+(u)) - \left(\frac{\partial}{\partial \bullet} B_-(t), u \right) = \left\langle \chi \left(\mathfrak{P}(t) - B_+ \frac{\partial}{\partial \bullet} B_-(t) \right), u \right\rangle. \end{aligned}$$

□

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