SPINDLE CONFIGURATIONS OF SKEW LINES

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Abstract. We simplify slightly the exposition of some known invariants for configurations of skew lines and use them to define a natural partition of the lines in a skew configuration.

Finally, we describe an algorithm constructing a spindle in a given switching class, provided such a spindle exists.

1. Introduction

A configuration of n skew lines in $\mathbb{R}^3$ is an arrangement of n lines in general position (distinct lines are never coplanar).

Two configurations of skew lines C_1 and C_2 are *isotopic* if there exists an isotopy from C_1 to C_2 (a continuous deformation of configurations of skew lines starting at C_1 and ending at C_2).

The study and classification of configurations of skew lines (up to isotopy) was initiated by Viro [14] and continued for example in [1], [2], [3], [5], [8], [9], [10], [11] and [15].

A *spindle* (or *isotopy join*) is a particularly nice configuration of skew lines in which all lines of the configuration intersect an oriented extra line, called the *axis* of the spindle. The isotopy class of a spindle is completely described by its *spindle permutation* $\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ encoding the order in which a half-plane bounded by the axis A hits the lines during a half-turn around its boundary A (see Section 6 for the precise definition). A *spindle configuration* is a configuration of skew lines which is isotopic to a spindle.

Consider the *spindle-equivalence* relation on permutations of $\{1, \dots, n\}$ which is generated by transformations of the following three types:

(1) $\sigma \sim \mu$ if $\mu(i) = s + \sigma(i + t \pmod n) \pmod n$ for some integers $0 \le s, t < n$ (with integers read modulo n).

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(2) $\sigma \sim \mu$ if $\sigma(i) < k$ for $i < k$ and

$$\mu(i) = \begin{cases} k - \sigma(k-i) & i < k \\ \sigma(i) & i \geq k \end{cases}$$

for some integer $k \leq n+1$.

(3) $\sigma \sim \mu$ if $\sigma(i) < k$ for $i < k$ and

$$\mu(i) = \begin{cases} \sigma^{-1}(i) & i < k \\ \sigma(i) & i \geq k \end{cases}$$

for some integer $k \leq n+1$.

Conjecture 59 in [3] states that two spindle configurations are isotopic if and only if they are described by spindle-equivalent permutations.

This paper contains a slight improvement of Theorem 62 in [3]:

Theorem 1.1. *Equivalent spindle-permutations yield isotopic spindle configurations.*

Part of the information about a configuration of labeled and oriented skew lines can be encoded by the *crossing matrix* X (a symmetric matrix encoding the crossing data). Its *switching class* (or *two-graph*, see page 7 of [16]) is equivalent to the *homological equivalence class* of the underlying configuration $\mathcal{C}$ of unoriented unlabeled skew lines (see [2]). It is also equivalent to the description of the set of *linking numbers* or *linking coefficients* of $\mathcal{C}$ (see [2] or [15]).

An *invariant* is an application

$$\{\text{isotopy classes of configurations of skew lines}\} \longrightarrow \mathcal{R}$$

where $\mathcal{R}$ is some set, usually a ring or vector space. An invariant is *complete* if the above map is injective.

The set of known invariants for configurations of skew lines together with information about their completeness can be resumed as follows:

(1) Equivalence classes of skew pseudoline diagrams (see Section 2): Completeness unknown. A powerful combinatorial invariant somewhat tedious to describe and handle. Switching classes and Kauffman polynomials factorize through this invariant.

(2) Switching classes (or homological equivalence classes, see [2], or sets of linking numbers, see [15]). Not complete: There exist two configurations of 6 skew lines which are not isotopic (they have different Kauffman polynomials) but they define the same switching class (see Remark 3.4). This example is minimal in the sense that switching classes characterize isotopy classes of configurations of at most 5 skew lines.

An interesting feature of a switching class is the definition of its *Euler partition* - a natural partition of the corresponding lines into equivalence classes. The definition of this partition depends on the parity of the number n of lines. For odd n, the definition is fairly simple since odd switching classes are in bijection with so-called *Euler graphs* (see Proposition 5.1). For even n the definition is more delicate. Some features attached to these partitions will be studied in Section 5.

Slightly weaker (but more elementary to handle) than the switching class is the characteristic polynomial

$$P_X(t) = \sum_{i=0}^{n} \alpha_i t^i = \det(t\mathrm{I} - X)$$

of a switching class represented by a crossing matrix X. The description of P_X is of course equivalent to the description of the spectrum of X or of the traces of the first n powers of X. The coefficient α_{n-3} of the characteristic polynomial conveys the same information as *chirality*, a fairly weak invariant considered sometimes (see [3, Section 3]).

(3) Kauffman polynomials: Completeness unknown. A powerful invariant which is very difficult to compute if there are many lines (its complexity is $O(2^{\binom{n}{2}})$).

(4) Link invariants for links in $\mathcal{S}^3$ applied to the preimage $\pi^{-1}(C) \subset \mathcal{S}^3$ (called a *Temari model* by some authors, see for instance section 11 of [3]) of a configuration $C \subset \mathbb{R}^3 \subset \mathbb{RP}^3$ under the double covering $\pi : \mathcal{S}^3 \longrightarrow \mathbb{RP}^3$.

(5) Existence (and description) of a spindle structure. A generally very weak invariant since spindle structures are rare among configurations with many lines. Section 7 contains a somewhat curious (and rather weak) invariant of spindle configurations. In Section 8 we describe an algorithm which computes a spindle structure or proves non-existence of a structure representing a given switching class.

In this context, one should mention also the so-called *shellability order*, a kind of generalization of the notion of spindle structure, cf. [2] where it is used as a tool for classifications.

The above invariants can be used in order to distinguish isotopy classes of configurations of skew lines. Section 9 contains some computational data.

2. Skew pseudoline diagrams

Skew pseudoline diagrams are purely combinatorial objects and are among the main tools for studying configurations of skew lines.

Definition 2.1. *A* **pseudoline** *in* $\mathbb{R}^2$ *is a smooth simple curve representing a non-trivial cycle in* $\mathbb{RP}^2$*. An* **arrangement of** n **pseudolines** *in* $\mathbb{R}^2$ *is a set of* n *pseudolines with pairs of pseudolines intersecting transversally exactly once. The arrangement is* **generic** *if all intersections involve only two distinct pseudolines.*

Definition 2.2. A skew pseudoline diagram of n **pseudolines** *in* $\mathbb{R}^2$ *is a generic arrangement of* n *pseudolines with crossing data at intersections. The crossing data selects at each intersection the overcrossing pseudoline.*

One represents skew pseudoline diagrams graphically by drawing them with the conventions used for knots and links: under-crossing curves are slightly interrupted at the crossing.

Skew pseudoline diagrams are *equivalent* if and only if they are related by a finite sequence of the following two moves (see [3, Section 9]):

(1) Reidemeister-3 (or $*$-move), the most interesting of the three classical moves for knots and links (see Figure 1).

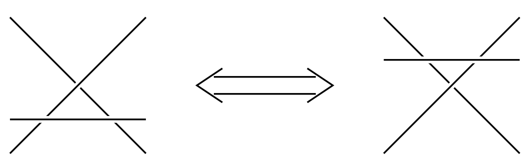


Figure 1. Local description of a Reidemeister-3 move

(2) Projective move (or $||$-move): pushing a crossing through infinity (see Figure 2).

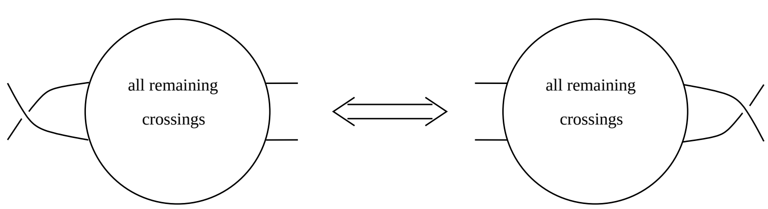


Figure 2. Projective move

Generic projections of isotopic configurations of skew lines are easily seen to yield equivalent skew-pseudoline diagrams (cf. for instance [3, Theorem 48]).

There exist equivalence classes of skew pseudoline diagrams which are not projections of configurations of skew lines: for instance alternating skew pseudoline diagrams with more than 3 pseudolines (see [7]). There are even generic arrangements of pseudolines which are not stretchable, i.e. cannot be realized as an arrangement of straight lines, see [4] for the smallest possible example having 9 lines.

The existence of non-isotopic configurations of skew lines inducing equivalent skew pseudoline diagrams is however unknown, cf. [3, Section 17, Problem 2]. This is a major difficulty for classifications. Indeed, classifying diagrams of skew pseudolines is purely combinatorial, but lifting such diagrams into configurations of skew lines (if such configurations exist) seems not obvious and it would of course be interesting to understand the number of possible (non-isotopic) liftings for instance in purely combinatorial terms.

3. Crossing matrices and switching classes

We assign signs to pairs of oriented under- or over-crossing lines. Figure 3 displays a positive and a negative crossing using the conventions of knot theory which we adopt in the sequel of this paper.

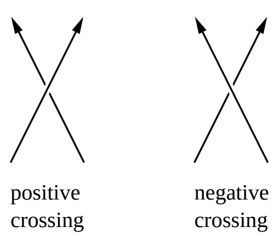


Figure 3. Positive and negative crossings

One can compute the sign between two oriented skew lines $L_A, L_B \subset \mathbb{R}^3$ algebraically as follows: choose ordered pairs of points (A_α, A_ω) on L_A (resp. (B_α, B_ω) on L_B) which induce the orientations. The sign of the crossing determined by L_A and L_B is then given by

$$\operatorname{sign}\left(\det \begin{pmatrix} A_\omega - A_\alpha \\ B_\alpha - A_\omega \\ B_\omega - B_\alpha \end{pmatrix}\right) \in \{\pm 1\}$$

where $\operatorname{sign}(x) = 1$ if $x > 0$ and $\operatorname{sign}(x) = -1$ if $x < 0$.

The signs of crossings are of course also well-defined between pairs of oriented pseudolines in skew pseudoline diagrams.

The *crossing matrix* of a diagram of n oriented and labeled skew pseudolines $L_1, \dots, L_n$ is the symmetric $n \times n$ matrix X where $x_{i,j}$ is the sign of the crossing between L_i and L_j for $i \neq j$ and $x_{i,i} = 0$.

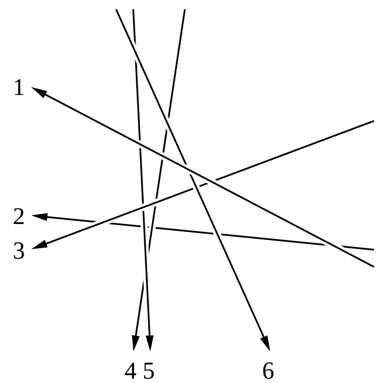


FIGURE 4. An example

The labeled and oriented configuration of six skew lines presented in Figure 4 gives rise to the crossing matrix

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

Two symmetric matrices X and Y are *switching-equivalent* if

$$Y = D\ P^t\ X\ P\ D$$

where P is a permutation matrix and D is a diagonal matrix with diagonal coefficients in $\{\pm 1\}$. Since $(PD)^{-1} = DP^t$, the diagonalizable matrices X and Y have identical characteristic polynomials and the same eigenvalues.

Proposition 3.1. *The switching class of a crossing matrix X of a skew pseudoline diagram D (and hence of a configuration of skew lines) is well-defined (i.e. independent of the labeling and the orientations of the lines in D).*

Proof. Changing the labeling of the lines in D conjugates X by a permutation matrix. Reversing the orientation of some lines amounts to conjugating X by a diagonal matrix with diagonal entries ± 1. $\square$

Remark 3.2. *The terminology "switching classes" (many authors use also "two-graphs") has its origin in the following combinatorial interpretation and definition of switching classes:*

Two finite simple graphs Γ_1 and Γ_2 with vertices $V = \{v_1, \dots, v_n\}$ are switching-equivalent if there exists a partition $I \cup J = \{1, \dots, n\}$ such that $\{v_i, v_j\}$ is an edge in Γ_2 if and only if either $\{v_i, v_j\}$ is an edge in Γ_1 and the labels i, j belong either both to I or both to J or there is no edge between v_i and v_j in Γ_1 and exactly one of the indices i, j is an element of I (see Figure 5 where $I = \{1, 2\}$ and $J = \{3, 4\}$).

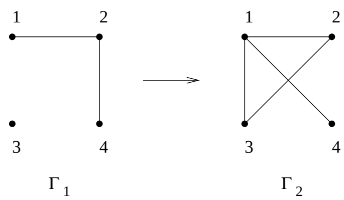


FIGURE 5. Γ_1 and Γ_2 are switching-equivalent (switch the vertices 1 and 2)

One says then that Γ_1 and Γ_2 are related by switching the vertices $\{v_i\}_{i \in I}$. The switching class of a graph Γ_1 defines a switching class of matrices by considering the symmetric matrix X with entries $x_{i,i} = 0$ and $x_{i,j} = \pm 1$ according to the existence (giving rise to an entry 1) of an edge between the two distinct vertices v_i and v_j. Conjugation by permutations is of course introduced in order to get rid of the labeling of the vertices in Γ_1. In the sequel we will also use the terminology "switching" in order to denote conjugation by a diagonal matrix with entries ± 1.

Remark 3.3. *The characteristic polynomial of a crossing matrix of a skew pseudoline diagram (or of a configuration of skew lines) is a weaker invariant than its switching class: One can see that the crossing matrices*

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

are in different switching classes (see Example 5.3), but have common characteristic polynomial

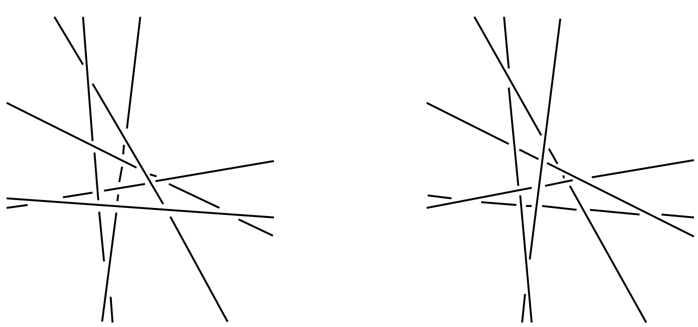
$$(t-3)(t-1)^2(t+1)(t+3)^2(t^2-2t-11)\ .$$

This example is minimal: suitable (i.e. zero entries on the diagonal and ± 1 *entries elsewhere) symmetric matrices of order less than* 8 *in distinct switching classes have distinct characteristic polynomials.*

Remark 3.4. *Configurations of at most* 5 *lines are (up to isotopy) completely characterized by the switching class of an associated crossing matrix.*

However, the two configurations of 6 *lines in Figure 6 have both crossing matrices in the switching class of*

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$



(a) (b)

Figure 6. Two different configurations with switching-equivalent crossing matrices

but are not isotopic. The Kauffman polynomial (see [3, Section 14 and Appendix]*) of the first configuration is*

$$5A^{12}B^3 + 10A^{11}B^4 - 10A^9B^6 + A^8B^7 + 16A^7B^8 + 10A^6B^9$$
$$-6A^5B^{10} - 5A^4B^{11} + 6A^3B^{12} + 6A^2B^{13} - B^{15}$$

whereas the second configuration has Kauffman polynomial

$$-A^{15} + 6A^{13}B^2 + 6A^{12}B^3 - 5A^{11}B^4 - 6A^{10}B^5$$
$$+10A^9B^6 + 16A^8B^7 + A^7B^8 - 10A^6B^9 + 10A^4B^{11} + 5A^3B^{12}$$

Each skew pseudoline diagram D with oriented and labeled pseudolines gives rise to a crossing matrix. Moreover, given an arbitrary pseudoline arrangement on n labeled and oriented lines and a suitable symmetric matrix X of order n, then the matrix X can be turned into the crossing matrix of D by choosing the appropriate (and uniquely defined) crossing data on D. This shows for instance that the number of equivalence classes of skew pseudoline diagrams having n pseudolines equals at least the number of switching classes of order n. This inequality is generally strict as shown by the example of Remark 3.4.

One may look at the mirror image of a configuration: Given a configuration C of skew lines with crossing matrix X, the mirror configuration $\overline{C}$ (obtained for instance by reflecting C through the $z = 0$ plane) has opposite crossing data. The pair of configurations of 6 lines in Figure 6 is an example of two configurations which are each others mirror. The crossing matrix $\overline{X}$ of $\overline{C}$ is then given by $-X$.

A configuration C is *amphicheiral* if it is isotopic to its mirror $\overline{C}$.

Proposition 3.5. (i) *The crossing matrix X of an amphicheiral configuration of skew lines is switching-equivalent to $-X$. In particular, a crossing matrix of an amphicheiral configuration containing an odd number of skew lines has determinant* 0.

(ii) *Amphicheiral configurations with n lines do not exist if $n \equiv 3$* (mod 4).

Proof. Assertion (i) is obvious.

Assertion (ii) is actually [15, Theorem 1]. We rephrase the proof using some properties of crossing matrices.

Let $\sum_{i=0}^{n} \alpha_i t^i = \det(tI - X)$ be the characteristic polynomial of the crossing matrix X for an amphicheiral configuration with n skew lines. Assertion (i) shows that we have $\alpha_{n-1} = \alpha_{n-3} = \alpha_{n-5} = \cdots = 0$. This implies

$$0 = \alpha_{n-3} = -\sum_{1 \le i,j,k \le n} (x_{i,j}x_{j,k}x_{k,i} + x_{i,k}x_{k,j}x_{j,i}) = -2\left(\sum_{1 \le i<j<k \le n} x_{i,j}x_{j,k}x_{k,i}\right).$$

For $n \equiv 3 \pmod 4$ the number $\binom{n}{3}$ of summands in $\sum_{1\leq i<j<k\leq n} x_{i,j}x_{j,k}x_{k,i}$ is odd. Since all these summands are ± 1, we get a contradiction. $\square$

4. Switching classes and linking numbers

Linking numbers (also called homological equivalence classes or chiral signatures) are a classical and well-known invariant for skew pseudoline diagrams. We sketch below briefly the well-known proof that they correspond to the switching class of a crossing matrix.

In this paper we prefer to work with switching classes mainly because they are easier to handle.

The *linking number* $\mathrm{lk}(L_i, L_j, L_k)$ of three lines ([2],[15]) is defined as the product $x_{i,j}x_{j,k}x_{k,i} \in \{\pm 1\}$ of the signs (after an arbitrary orientation of all lines) of the corresponding three crossings. It is elementary to check that the result is independent of the chosen orientations for L_i, L_j and L_k. It yields hence an invariant

$$\{\text{triplets of lines in skew pseudoline diagrams}\} \longrightarrow \{\pm 1\} .$$

The set of linking numbers is the list of the numbers $\mathrm{lk}(L_i, L_j, L_k)$ for all triplets $\{L_i, L_j, L_k\}$ of lines in a skew pseudoline diagram.

Linking numbers (defining a *two-graph*, see [16]) and switching classes are equivalent. Indeed, linking numbers of a diagram D of skew lines can easily be read of a crossing matrix for D. Conversely, given all linking numbers $\mathrm{lk}(L_i, L_j, L_k)$ of a diagram D, choose an orientation of the first line L_1. Orient lines $L_2, \dots, L_n$ such that they cross L_1 positively. A crossing matrix X for D is then given by $x_{1,i} = x_{i,1} = 1,\ 2 \leq i \leq n$ and $x_{a,b} = \mathrm{lk}(L_1, L_a, L_b)$ for $2 \leq a \neq b \leq n$.

Two skew pseudoline diagrams are called *homologically equivalent* if there exists a bijection between the two sets of lines, which preserves the set of linking numbers. Two diagrams are homologically equivalent if and only if they have switching-equivalent crossing matrices.

A last invariant considered by some authors (see [3, Section 3 and Appendix]) is the *chirality* (γ_+, γ_-) of a skew pseudoline diagram. It is defined as

$$\begin{aligned}\gamma_+ &= \sharp\{1 \leq i < j < k \leq n \mid \mathrm{lk}(L_i, L_j, L_k) = 1\}\\ \gamma_- &= \sharp\{1 \leq i < j < k \leq n \mid \mathrm{lk}(L_i, L_j, L_k) = -1\}\end{aligned}$$

One has of course

$$\gamma_+ = \frac{\binom{n}{3} + c}{2} , \quad \gamma_- = \frac{\binom{n}{3} - c}{2}$$

where

$$c=\sum_{1\le i<j<k\le n} x_{i,j}x_{j,k}x_{k,i}=\frac{1}{6}\mathrm{trace}(X^3)=-\frac{\alpha_{n-3}}{2}$$

is proportional to the coefficient of t^{n-3} in the characteristic polynomial $\det(tI-X)=\sum_{i=0}^n \alpha_i t^i$ of a crossing matrix X.

Let us stress that switching classes are somewhat annoying to work with because of possible signs (-1 entries) in the conjugating matrix between two representatives. For switching classes of odd order there is a very satisfactory answer for this problem which will be addressed in the next section. For even orders, no completely satisfactory way to get rid of signs seems to exist.

One possible normalization consists in choosing a given (generally the first) row of a representing matrix and to make all entries in this row positive by conjugation with a suitable diagonal ± 1 matrix. Such a matrix can then be encoded by a graph on $n-1$ vertices encoding all entries outside the chosen row and the corresponding column. This leads to the notion of graphs which are "cousins" in [3]. This notion is of course equivalent to the notion of switching-equivalence as can be easily checked.

5. Euler partitions

In this section we study some properties of switching classes. They lead to invariants with computational cost of $O(n^2)$ operations. I the number of lines is huge they are easier to compute and to compare than the so-called chirality-invariants with computational cost $O(n^3)$. All these invariants factor of course through switching-classes and they are thus useless for improving known classification results. They seem interesting mainly because of their simplicity and low complexity.

As already mentioned, invariants of switching classes behave quite differently according to the parity of their order n.

Switching classes of odd order $2n-1$ are in bijection with Eulerian graphs. This endows pseudoline diagrams consisting of an odd number of pseudolines with a canonical orientation (up to a global change). We get a partition of the pseudolines into equivalence classes according to the number of positive crossings in which they are involved for an Eulerian orientation.

We consider the case of odd order in Subsection 5.1 below.

For switching classes of even order $2n$ the situation is more complicated. We replace the Euler graphs appearing for odd orders by a suitable kind of planar rooted binary trees which we call *Euler trees*. In this case we loose the injectivity which we have in the odd case.

However, the leaves of the Euler tree induce in this case again a natural partition, also called an *Euler partition*, of the set of pseudolines into equivalence classes of even cardinalities. Subsection 5.2 below addresses the even case.

5.1. **Switching classes of odd order - Euler orientations.** A simple finite graph Γ is *Eulerian* if all its vertices are of even degree. Connected Eulerian graphs are good designs for zoological gardens or expositions since they can be visited by walking exactly once along every edge. The following well-known result goes back to Seidel [12].

Proposition 5.1. *Eulerian graphs on an odd number* $2n-1$ *of vertices are in bijection with switching classes of order* $2n-1$.

We recall here the simple proof since it yields an algorithm for computing Eulerian orientations on configurations with an odd number of skew lines.

Proof. Choose an arbitrary representative X in a given switching class. For $1 \leq i \leq 2n-1$ count the number

$$v_i = \sharp\{j \mid x_{i,j} = 1\} = \sum_{j=1, j\neq i}^{2n-1} \frac{x_{i,j}+1}{2}$$

of entries equal to 1 in the $i-$th row of X. Since X is symmetric, the vector $(v_1, \ldots, v_{2n-1})$ has an even number of odd coefficients and conjugation of the matrix X with the diagonal matrix having diagonal entries $(-1)^{v_i}$ turns X into a matrix X_E with an even number of 1's in each row and column. The matrix X_E is well defined up to conjugation by a permutation matrix and defines an Eulerian graph with vertices $\{1, \ldots, 2n-1\}$ and edges $\{i, j\}$ if $(X_E)_{i,j} = 1$. This construction can easily be reversed. $\square$

Let D be a skew diagram having an odd number of pseudolines. Label and orient its pseudolines arbitrarily thus getting a crossing matrix X. Reverse the orientations of all lines having an odd number of crossings with positive sign. Call the resulting orientation *Eulerian.* It is unique up to globally reversing the orientations of all lines.

An *Eulerian crossing matrix* X_E associated to an Eulerian orientation of D is uniquely defined up to conjugation by a permutation matrix. Its invariants coincide with those of the switching class of X_E but are slightly easier to compute since there are no sign ambiguities. In particular, some of them can be computed using only $O(n^2)$ operations.

An *Eulerian partition* of the set of pseudolines of a diagram consisting of an odd number of pseudolines is by definition the partition of the pseudolines into subsets $\mathcal{L}_k$ consisting of all pseudolines involved in exactly $2k$ positive crossings for an Eulerian orientation.

A few more invariants of Eulerian matrices are:

(1) The total sum $\sum_{i,j} x_{i,j}$ of all entries in an Eulerian crossing matrix X_E (this is of course equivalent to the computation of the number of entries 1 in X_E). The computation of this invariant needs only $O(n^2)$ operations.

(2) The number of rows of X_E whose entries have a given sum (this can also be computed using $O(n^2)$ operations). These numbers yield of course the cardinalities of the sets $\mathcal{L}_0, \mathcal{L}_1, \ldots$.

(3) All invariants of the associated Eulerian graph (having edges associated to entries $x_{i,j} = 1$) defined by X_E, e.g. the number of triangles or of other given finite subgraphs, etc.

For example, for 7 vertices, there are 54 different Eulerian graphs, 36 different sequences (up to a permutation of the vertices) of vertex degrees, and 18 different numbers for the cardinality of 1's in X_E.

Figure 7 shows all seven Eulerian graphs on 5 vertices.

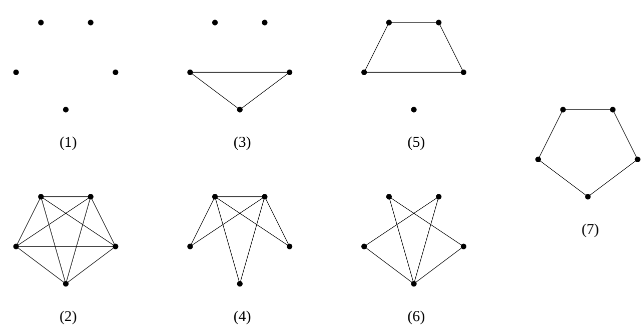


Figure 7. All Eulerian graphs on 5 vertices

5.2. **Switching classes of even order - Euler partitions.** The situation in this case is unfortunately more complicated and less satisfactory.

There exists a natural partition of the rows of X into two subsets A and B according to the parity of $\left(1 + \sum_j x_{i,j}\right)/2$. Conjugation by a diagonal ± 1 matrix D preserves or exchanges these two sets according to the determinant $\prod_i d_{i,i} \in \{\pm 1\}$ of D. This construction can then be iterated: The sets A and B have cardinalities α and β which are both even and they determine thus two switching classes of even order

α and β by erasing all rows and columns not in A, respectively not in B. This stops if one of the subsets A, B is empty. We get in this way a partition of all rows (or columns) of X into subsets $A_1, \dots, A_r$ such that the corresponding symmetric sub-matrices are either *Eulerian*, i.e. represent an Eulerian graph on $|A_i|$ vertices (having an edge between two vertices $L_s, L_t \in A_i$ if and only if $X_{s,t} = 1$) or *anti-Eulerian*, i.e. represent the complement of an Eulerian graph. This situation can be encoded by a planar binary rooted tree with non-zero integral weights as follows: A matrix X of order $2n$ representing a switching class is represented by a root-vertex. If the above partition of the rows into two subsets A and B is non-trivial, then draw a left successor representing all rows containing an even number of coefficients 1 and a right successor representing all rows containing an odd number of coefficients 1. Consider the new vertices as the roots of the corresponding symmetric sub-matrices representing switching classes and iterate. If the partition of all rows of X is trivial then put a weight n on the corresponding leaf if X is an Eulerian matrix of order $2n$ and put a weight of $-n$ if X is anti-Eulerian of order $2n$.

We have yet to understand the effect of switching (conjugation by a diagonal ± 1 matrix) on this decorated planar tree: switching one line in a leave L_i of the above tree changes the weight w of the leaf into $-w$ and has the effect of transposing the sons in all predecessors of the leaf L_i.

There exists thus a unique representative having only Eulerian leaves. We call this representative the *Euler tree* of the switching class. Its weight n is the sum of all (positive) weights of its leaves.

Enumerative digression. The generating function $F(z) = \sum_{n=0} \alpha_n z^n$ enumerating the number α_n of Eulerian trees of weight n satisfies the equation

$$F(z) = \frac{1}{1-z} + (F(z)-1)^2 \ .$$

Indeed, Eulerian trees reduced to a leaf contribute $1/(1-z)$ to $F(z)$. All other Eulerian trees are obtained by gluing two Eulerian trees of strictly positive weights below a root and are enumerated by the factor $(F(z)-1)^2$.

Solving for $F(z)$ we get the closed form

$$F(z) = \sum_{n=0}^{\infty} \alpha_n z^n = \frac{3(1-z) - \sqrt{(1-z)(1-5z)}}{2(1-z)}.$$

showing that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 5 \ .$$

The first terms $\alpha_0, \alpha_1, \ldots$ are given by

$$1, 1, 2, 5, 15, 51, 188, 731, 2950, 12235, \ldots$$

(see also Sequence A7317 in [13]).

The leaves of the Euler tree define a natural partition of the set of rows of X into subsets. We call this partition the *Euler partition*.

Example 5.2. *The symmetric matrix*

$$\begin{pmatrix} 0 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

yields the left tree of Figure 8. It gives hence the Euler partition

$$A = \{1, 2, 4, 7, 8, 9\} \quad \cup \quad B = \{3, 5\} \quad \cup \quad C = \{6, 10\}$$

associated to the Euler tree at the right side of of Figure 8.

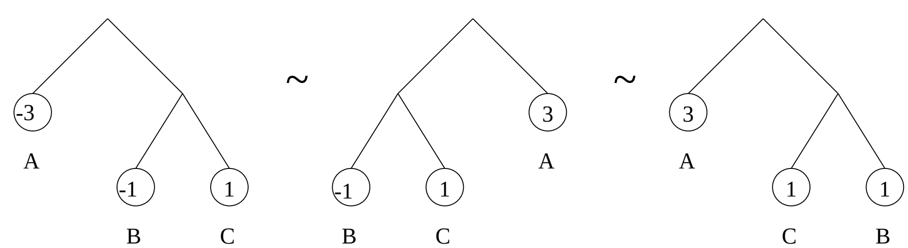


FIGURE 8. Decorated trees and the Euler tree corresponding to Example 5.2

Example 5.3. *The two matrices mentioned in Remark 3.3 are indeed not switching-equivalent: The Euler partition of the first one has four rows in each class (more precisely, the Euler tree has two leaves: the left leaf is associated to rows* $2, 5, 7, 8$ *and the right leaf to rows* $1, 3, 4, 6$*). The second matrix is anti-Eulerian and its Euler partition is hence trivial. It can be turned into an Eulerian matrix by switching an odd number of lines.*

Euler trees can also be used to define fast invariants for skew diagrams of an even number of pseudolines: for instance, compute a crossing matrix X (this needs $O(n^2)$ operations) and compute then the cardinality

$$\alpha = \sharp\{i \mid \sum_{j=1, j\neq i}^{2n} \left(\frac{x_{i,j}+1}{2}\right) \equiv 0 \pmod 2\} \ .$$

The unordered set $\{\alpha, \beta = 2n-\alpha\}$ is an invariant with computational cost $O(n^2)$. It yields the sum of the weights of all leaves at the left, respectively at the right, of the root. Unfortunately, at this stage, one can not decide if α corresponds to the left leaves or not. In order to get rid of this indetermination without computing the complete Euler tree, one can use the following trick: obviously the difference

$$\alpha - \beta$$

changes its sign by switching an odd number of rows. It is straightforward to check that the same holds also for the number

$$\lambda = \prod_{1\leq i<j\leq 2n} x_{i,j} \in \{\pm 1\}.$$

The product

$$\lambda(2\alpha - 2n) \in 2\mathbb{Z}$$

is a well-defined invariant of the switching class of X. If the matrix X represents an Eulerian graph Γ_E, then the sign λ is related to the parity of the number of edges in the Eulerian graph Γ_E.

Let us finally mention a last invariant which is a kind of decoration of the Euler tree for a switching class X of even order $2n$. Suppose X normalized in the obvious way (all leaves of the corresponding Euler tree are Eulerian) and let $A_1, \dots, A_r$ be the Euler partition. For $1 \leq i \leq j \leq r$ define numbers $a_{i,j} \in \{\pm 1\}$ in the following way

$$a_{i,i} = \prod_{s,t\in A_i, s<t} x_{s,t}$$

(this is the definition of the sign λ considered above of the Eulerian graph A_i) and

$$a_{i,j} = \prod_{s\in A_i,\ t\in A_j} x_{s,t}$$

if $i<j$. One checks easily that the numbers $a_{i,j}$ are well-defined.

Let us also remark that this invariant has an even stronger analogue for switching classes of odd order: Given an Eulerian matrix of order $2n+1$ with Euler partition $A_0, \dots, A_r$ (where A_i corresponds to the set of vertices of degree $2i$ in the Euler graph) one can consider the numbers

$$a_{i,j} = \sum_{s \in A_i, t \in A_j} x_{s,t},\ 0 \leq i, j$$

which can easily be shown to be well-defined. They are of course related to the number of edges between vertices of given type.

6. Spindles

This section is devoted to the study of spindles.

6.1. **The definition of a spindle configuration.** Recall that a *spindle* is a configuration of skew lines with all lines intersecting a supplementary line, called the *axis*. A *spindle configuration* (or a *spindle structure*) is a configuration of skew lines isotopic to a spindle.

Orienting the axis A labels the lines of a spindle C according to the order in which they intersect A . Each line $L_i \in C$ defines then a plane Π_i containing L_i and the axis A. The isotopy type of C is now defined by describing the circular order in which the distinct planes $\Pi_1, \dots, \Pi_n$ are arranged around the axis A. A convenient way to summarize this information is to project $\mathbb{R}^3$ along the directed axis A onto an oriented plane, to choose an arbitrary initial plane Π_{i_1} and to read off clockwise the appearance of the remaining planes on the projection thus getting a *spindle permutation* $k \longmapsto \sigma(k) = j$ if $i_j = k$ (Figure 9(a) for instance shows a spindle encoded by the spindle permutation $\sigma(1) = 1,\ \sigma(2) = 3,\ \sigma(3) = 4,\ \sigma(4) = 2$).

This description can also be summarized as follows. Given an extra line B (called a *directrix*) in general position with respect to A and the planes Π_i, one can rotate each line L_i in the plane Π_i around the intersection $L_i \cap A$ until the line L_i hits B. Choose the unique orientation of the directrix B such that it crosses the oriented axis A negatively. This orders the intersection points $L_i \cap B$ linearly. The spindle permutation of C encodes then the relation between the linear orders $L_i \cap A$ and $L_i \cap B$ (see Figure 9(b)).

A crossing matrix X of a spindle C is easily computed as follows. Transform your spindle into a spindle with oriented axis A and directrix B as above. Orient a line L_i from $L_i \cap A$ to $L_i \cap B$. A straightforward computation shows that the crossing matrix X of this labeled oriented configuration has entries

$$x_{i,j} = \mathrm{sign}((i - j)(\sigma(i) - \sigma(j)))$$

where $\mathrm{sign}(0) = 0$ and $\mathrm{sign}(x) = \frac{x}{|x|}$ for $x \neq 0$ and where σ is the associated spindle permutation.

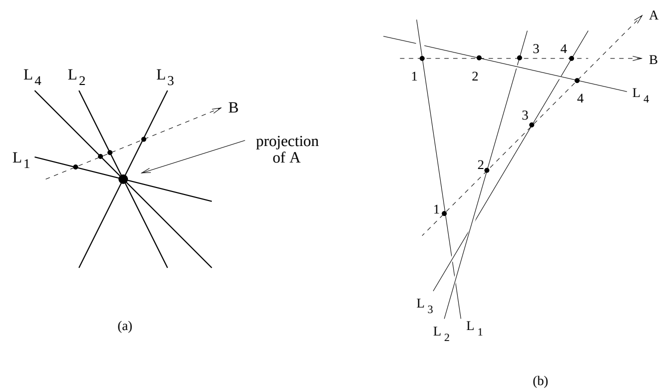


FIGURE 9. A projection of a spindle configuration along the axis A

One can easily see:

Remark 6.1. *A configuration C of n skew lines has a spindle structure if and only if its mirror configuration $\overline{C}$ has a spindle structure. A spindle permutation $\overline{\sigma}$ for $\overline{C}$ is then for instance given by setting $\overline{\sigma}(i) = n+1-\sigma(i),\ 1 \le i \le n$, where σ is a spindle permutation for C.*

For example, the configuration of 5 skew lines depicted in Figure 10 corresponds to the spindle permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$$

and to the crossing matrix

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

6.2. **Spindle-equivalent permutations.** The aim of this subsection is to show that three types of transformations of spindle permutations, defined by Crapo and Penne in [3, Section 15], preserve the isotopy type of the associated spindle configuration.

A *block* of width w ($w \ge 1$) in a permutation σ of $\{1,\dots,n\}$ is a subset $\{i+1,\dots,i+w\}$ of w consecutive integers in $\{1,\dots,n\}$ such that

$$\sigma(\{i+1,\dots,i+w\}) = \{j+1,\dots,j+w\}$$

(i.e. the image under σ of a set $\{i+1,\dots,i+w\}$ of w consecutive integers is again a set of w consecutive integers).

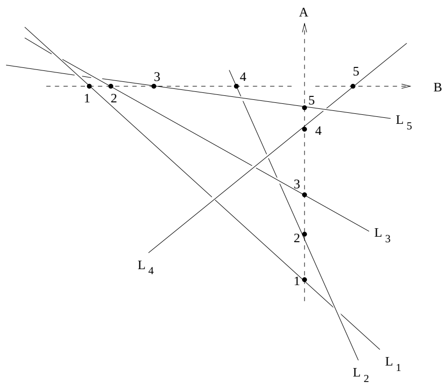


FIGURE 10. A spindle structure

A block containing n lines is *minimal* if it has no sub-block of width $2 \leq w \leq n-1$.

For example, Figure 11 is a graphical representation of the cyclic permutation $\sigma = (1\ 2\ 4\ 3\ 5)$ (i.e., $\sigma(1) = 2, \sigma(2) = 4$, etc.). The three lines contained in the shaded area of Figure 11 form a block of width 3 which is not minimal since it contains $\sigma(\{2,3\}) = \{4,5\}$ as a proper sub-block of width 2.

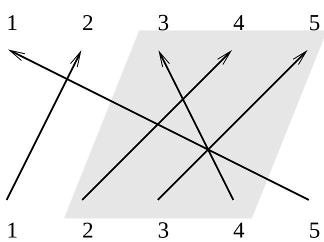


FIGURE 11. An example for a block

Now, recall that two spindle-permutations are *equivalent* (see [3, Section 15]) if they are equivalent for the equivalence relation generated by

(1) (Circular move) $\sigma \sim \mu$ if $\mu(i) = (s + \sigma((i+t) \pmod n)) \pmod n$ for some integers $0 \leq s, t < n$ (all integers are modulo n).
(2) (Vertical reflection of a block or local reversal) $\sigma \sim \mu$ if $\sigma(i) < k$ for $i < k$ and
$$\mu(i) = \begin{cases} k - \sigma(k-i) & i < k \\ \sigma(i) & i \geq k \end{cases}$$
for some integer $k \leq n+1$ (see Figure 12).

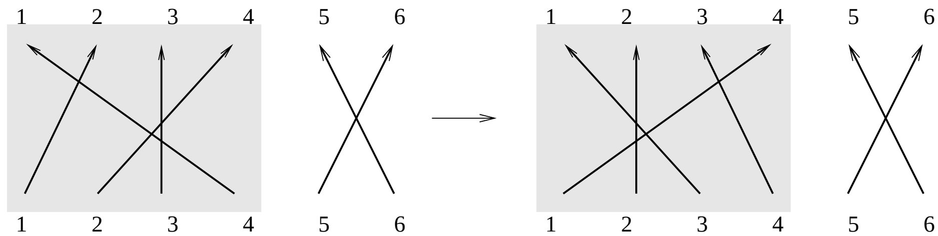


FIGURE 12. Vertical reflection of a block

(3) (Horizontal reflection of a block or local exchange) $\sigma \sim \mu$ if $\sigma(i) < k$ for $i < k$ and

$$\mu(i) = \begin{cases} \sigma^{-1}(i) & i < k \\ \sigma(i) & i \geq k \end{cases}$$

for some integer $k \leq n+1$ (see Figure 13).

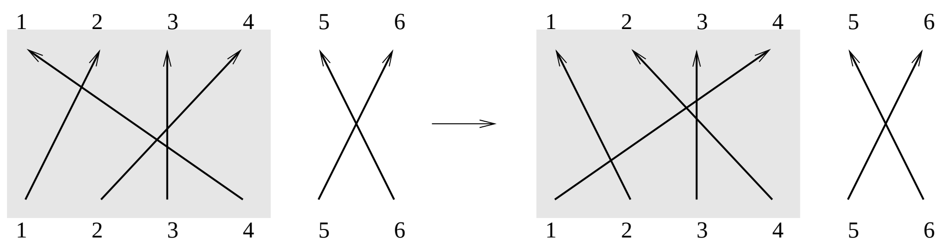


FIGURE 13. Horizontal reflection of a block

One should notice that circular moves behave in a different manner from vertical or horizontal reflections in the following sense: Crossing matrices associated to permutations related by vertical or horizontal reflections are conjugated by a permutation matrix. This is generally no longer true for circular moves: Crossing matrices of spindle permutations related by circular moves are conjugated by signed permutation matrices.

We give now the proof of Theorem 1.1: Equivalent spindle-permutations yield isotopic spindle configurations.

Proof. A transformation of type (1) amounts to pushing the last few lines of the spindle on the axis and/or on the directrix through infinity. This obviously does not change the isotopy type of a spindle.

A transformation of type (2) can be achieved by an isotopy as follows. First, observe that the condition that $\sigma(i) < k$ for $i < k$ implies the existence of two blocks: $\sigma(\{1, \cdots, k-1\}) = \{1, \cdots, k-1\}$ and $\sigma(\{k, \cdots, n\}) = \{k, \cdots, n\}$. All the lines of the first block can then be moved by an isotopy into the interior of a small one-sheeted revolution-hyperboloid whose axis of revolution intersects the axis A and the directrix B at right angles. We may moreover assume that this hyperboloid separates the lines of the first block from all the remaining lines. Rotating the interior (containing all lines of the first block) of this hyperboloid by a half-turn around its axis of revolution accomplishes the requested vertical reflection (see Figure 14).

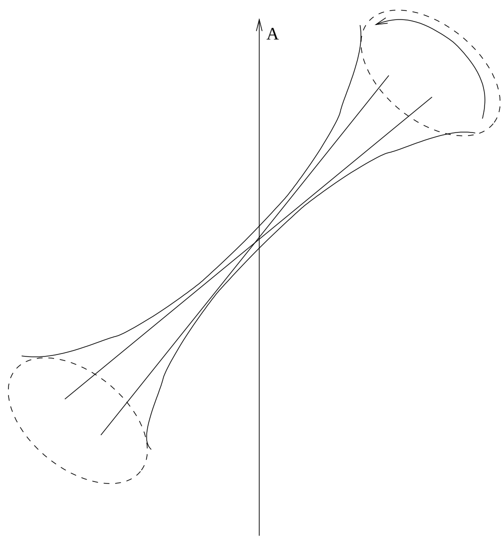


FIGURE 14. Isotopy for a transformation of type (2)

For a transformation of type (3) we start as for type (2) by pushing the k lines of the first block into a small revolution hyperboloid with revolution axis intersecting the axis A and the directrix B orthogonally. Suppose moreover that the axis $A = P_A + \mathbb{R}D_A$ and the directrix $B = P_B + \mathbb{R}D_B$ have orthogonal directions $D_A \perp D_B$. Push the remaining lines into the interior of a second small revolution hyperboloid such that the two hyperboloids are disjoint and the axis H_2 of the second hyperboloid intersects A orthogonally and is parallel to the directrix B.

Rotate now the first hyperboloid together with its interior containing the lines $L_1, \dots, L_{k-1}$ of the first block by 180° around the revolution axis H_2 of the second hyperboloid (see Figure 15). Finally, rotate the first hyperboloid and the lines inside it by 90° around its revolution axis and translate it along its axis until the image of the original directrix B coincides with the axis A. This yields a spindle configuration whose spindle permutation is related by a horizontal reflection and a circular move to the original spindle permutation.

$\square$

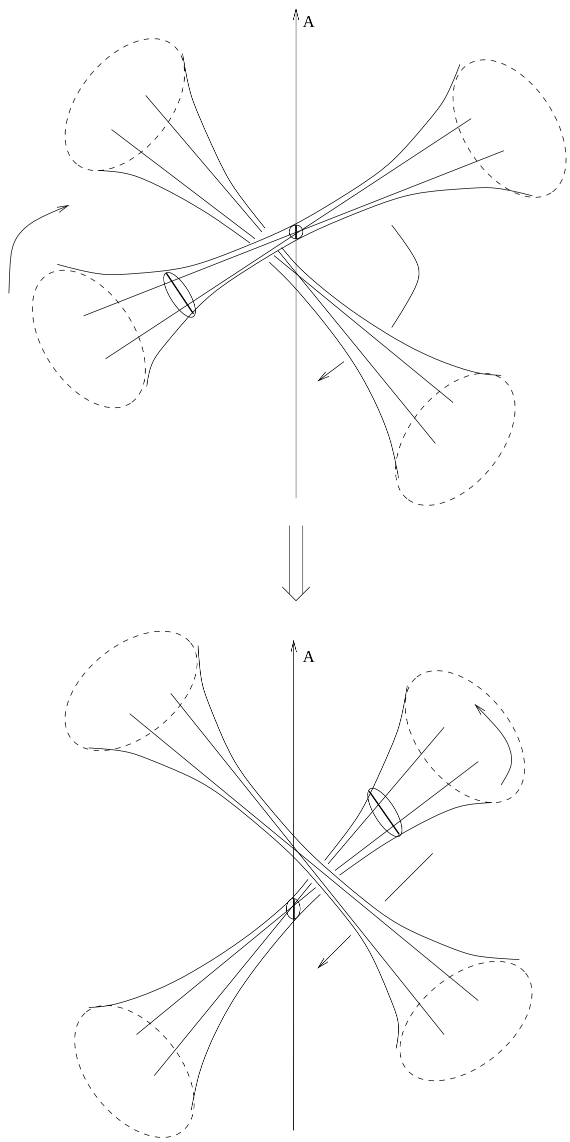


FIGURE 15. Isotopy for realizing a transformation of type (3)

7. Spindlegenus

This short section is devoted to a strange invariant of spindle configurations which we call *spindlegenus*.

Given a permutation $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ one can construct a compact oriented surface by gluing "together along σ" two polygons having n sides. More precisely, consider two oriented polygons P, P' of the oriented plane with clockwise read edges $E_1, \ldots, E_n$, $E'_1, \ldots E'_n$.

Glue the edge E_i of P onto in the edge $E'_{\sigma(i)}$ of P' in the unique way (up to isotopy) which respects all orientations. This produces an oriented compact surface $\Sigma(\sigma)$ of genus $g(\sigma) \in \mathbb{N}$.

Proposition 7.1. *Two permutations σ and σ' which are spindle-equivalent have the same genus (i.e. $g(\sigma) = g(\sigma')$).*

This proposition and Theorem 1.1 show of course that a spindle configuration C has a well-defined spindlegenus $g(C)$: Take an arbitrary spindle-permutation σ with associated spindle isotopic to C and set $g(C) = g(\sigma)$.

Proposition 7.1 yields in fact a second invariant: the *mirror-genus* defined as the genus of the mirror spindle-permutation $\psi = \sigma \circ \mu$ where $\mu(i) = n + 1 - i$ of a permutation σ of $\{1, \ldots, n\}$. The mirror-genus is generally different from the genus (consider for instance the case of the identity permutation). The genus and mirror-genus of amphicheiral permutations (giving rise to amphicheiral spindles) are equal.

The following questions are natural:

(1) Is there a natural extension of the invariant $g(C)$ to non-spindle configurations?
(2) Is there a natural extension of spindlegenus to all switching classes?

Proof of Proposition 7.1. The proposition obviously holds if σ and σ' are related by a circular move.

In order to prove the invariance of $g(\sigma)$ under vertical or horizontal reflections it is useful to recall how $g(\sigma)$ can be computed graphically. By the definition of the Euler characteristic

$$\chi(\Sigma) = 2 - n + S(\sigma) = 2 - 2g(\sigma)$$

where $S(\sigma)$ counts the number of points in $\Sigma(\sigma)$ originating from vertices of P and P', the proposition boils down to the equality $S(\sigma) = S(\sigma')$.

Draw the permutation $\overline{\sigma} : i \longmapsto n+1-\sigma(i)$ graphically with segments joining the points $(i, 0)$ to $(\sigma(i), 1)$ as in Figure 16.

Add the points $(0.5, 0), (1.5, 0), \ldots, (n+0.5, 0)$ and $(0.5, 1), (1.5, 1), \ldots, (n+0.5, 1)$ and join $(0.5, 0)$,$(n + 0.5, 0)$ (respectively $(0.5, 1)$, $(n + 0.5, 1)$) by dotted convex (respectively concave) arcs. Join the points $(i + 0.5, 0)$,$(\overline{\sigma}(i) + 0.5, 1)$ by dotted segments. Do the same with the points $(i - 0.5, 0)$,$(\overline{\sigma}(i) - 0.5, 1)$. One gets in this way a graph with vertices of degree 2 by considering all dotted segments. The number of connected components (3 in Figure 16) of this graph is easily seen to be $S(\sigma)$.

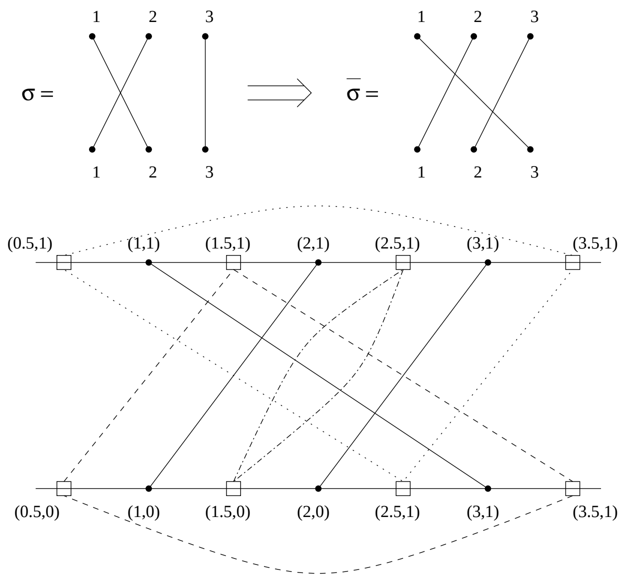


FIGURE 16. Example for the computation of $S(\sigma)$

Let us consider the local situation around a block B of σ. The edges of B connect the four boundary points adjacent B in one of the three ways depicted in Figure 17. The proof is now obvious since each of these three situations is invariant under vertical or horizontal reflections.

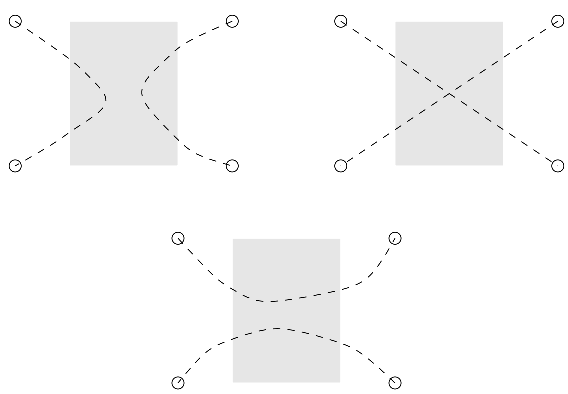


FIGURE 17. Three local situations around a block

$\square$

The following table lists the multiplicities for the spindlegenus of spindle-permutations (normalized by $\sigma(1) = 1$, multiply by n in order to get the corresponding numbers for not necessarily normalized permutations):

n	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
1	1					
2	1					
3	1	1				
4	1	5				
5	1	15	8			
6	1	35	84			
7	1	70	469	180		
8	1	126	1869	3044		
9	1	210	5985	26060	8064	
10	1	330	16401	152900	193248	
11	1	495	39963	696905	2286636	604800
12	1	715	88803	2641925	18128396	19056960

(see also Sequence A60593 of [13]).

8. Spindle structures for switching classes

The existence of a crossing matrix of a spindle in a given switching class is a natural question which we want to address algorithmically in this section.

The following algorithm exhibits a spindle permutation with crossing matrix in a given switching class or proves non-existence of such a permutation.

Algorithm 8.1.

Initial data. *A natural number n and a switching class represented by a symmetric matrix X of order n with rows and columns indexed by $\{1,\dots,n\}$ and coefficients $x_{i,j}$ satisfying*

$$\begin{array}{ll} x_{i,i}=0, & 1\le i\le n\ , \\ x_{i,j}=x_{j,i}\in\{\pm 1\}\ , & 1\le i\ne j\le n\ . \end{array}$$

Initialization. *Conjugate the symmetric matrix X by the diagonal matrix with diagonal coefficients $(1,x_{1,2},x_{1,3},\dots,x_{1,n})$. Set $\gamma(1)=\gamma(2)=1$, $\sigma(1)=1$ and $k=2$.*

Main loop. *Replace $\gamma(k)$ by $\gamma(k)+1$ and set*

$$\sigma(k)=1+\sharp\{j\mid x_{\gamma(k),j}=-1\}+\sum_{s=1}^{k-1}x_{\gamma(s),\gamma(k)}\ .$$

Check the following conditions:

(1) *$\gamma(k)\ne\gamma(s)$ for $s=1,\dots,k-1$.*

(2) $x_{\gamma(k),\gamma(s)} = sign(\sigma(k)-\sigma(s))$ *for* $s=1,\ldots,k-1$ *(where* $sign(0)=0$ *and* $sign(x)=\frac{x}{|x|}$ *for* $x\neq 0$*).*
(3) *for* $j\in\{1,\ldots,n\}\setminus\{\gamma(1),\ldots,\gamma(k)\}$ *and* $1\leq s<k$*:*
if $x_{j,\gamma(s)}\ x_{\gamma(s),\gamma(k)}=-1$ *then* $x_{j,\gamma(k)}=x_{j,\gamma(s)}$.

If all conditions are fulfilled then:
if $k=n$ *print all the data (mainly the spindle permutation* $i\longmapsto\sigma(i)$ *and perhaps also the conjugating permutation* $i\longmapsto\gamma(i)$*) and stop.*
if $k<n$ *then set* $\gamma(k+1)=1$*, replace* k *by* $k+1$ *and redo the main loop.*

If at least one of the above conditions is not fulfilled then:
while $\gamma(k)=n$ *replace* k *by* $k-1$*.*
if $k=1$*: print "no spindle structure exists for this switching class" and stop.*
if $k>1$*: redo the main loop.*

Explanation of the algorithm. The **initialization** is actually a normalization: we assume that the first row of the matrix represents the first line of a spindle permutation σ normalized to $\sigma(1)=1$ (this can always be assumed after a suitable circular move).

The **main loop** assumes that row number $\gamma(k)$ of X contains the crossing data of the $k-$th line (assuming that the rows representing the crossing data of the lines labeled $1,\ldots,k-1$ have already been correctly chosen). The image $\sigma(k)$ of k under a spindle permutation is then uniquely defined and given by the formula used in the main loop.

One has to check three necessary conditions:

- The first condition checks that row number $\gamma(k)$ has not been used before.
- The second condition checks the consistency of the choice for $\gamma(k)$ with all previous choices.
- If the third condition is violated, then the given choice of rows $\gamma(1),\ldots,\gamma(k)$ cannot be extended up to $k=n$.

The algorithm runs correctly even without checking for Condition (3). However, it looses much of its interest: Condition (3) is very strong (especially in the case of non-existence of a spindle structure) and ensures a fast running time.

The algorithm, in the case of success, produces two permutations σ and γ. The crossing matrix of the spindle permutation σ is in the same switching class as X and γ yields a conjugation between these two matrices. More precisely:

$$x_{\gamma(i),\gamma(j)} = \mathrm{sign}((i-j)(\sigma(i)-\sigma(j)))$$

under the assumption $x_{1,i} = x_{i,1} = 1$ for $2 \le i \le n$.

Failure of the algorithm (the algorithm stops after printing "no spindle structure exists for this switching class") proves non-existence of a spindle structure in the switching class of X.

In practice, the average running time of this algorithm should be of order $O(n^3)$ or perhaps $O(n^4)$. Indeed Condition (3) is only very rarely satisfied for a wrong choice of $\gamma(k)$ with $k > 2$. On the other hand, for a switching class containing a crossing matrix of a spindle, checking all cases of Condition (3) needs at least $O(n^3)$ operations (or more precisely $\binom{n-1}{3}$ operations after suppressing the useless comparisons involving $\gamma(1) = 1$).

Remark 8.2. *The algorithm can be improved. Condition (3) can be made considerably stronger.*

9. Computational results and lower bounds for the number of non-isotopic configurations

In this section, we describe some computational results.

The numbers of non-isotopic classes of configurations having at most 7 lines are known (see the survey of Viro and Drobotukhina [15] and the results of Borobia and Mazurovskii [1], [2]):

Lines	Isotopy classes
2	1
3	2
4	3
5	7
6	19
7	74

The following table enumerates the number of switching classes of order $6-9$. The middle row shows the number of distinct polynomials which arise as characteristic polynomials of switching classes (this is of course the same as the number of conjugacy classes under the orthogonal group $O(n)$ of matrices representing switching classes).

Lines	Characteristic polynomials	Switching classes
6	16	16
7	54	54
8	235	243
9	1824	2038

In fact, one can use representation theory of the symmetric groups in order to derive a formula for the number of switching classes of given order (see [6] and Sequence A2854 in [13]).

The map

$$\{\text{isotopy classes of configurations of skew lines}\} \longrightarrow \{\text{switching classes}\}$$

is perhaps not surjective for all n (there seems to be an unpublished counterexample of Peter Shor for $n = 71$, see [3, Section 3]). We rechecked however a claim of Crapo and Penne (Theorem 5 of Section 4) stating that all 243 switching classes of order 8 arise as crossing matrices of suitable skew configurations of 8 lines (the corresponding result holds also for fewer lines). There are thus at least 243 isotopy classes of configurations containing 8 skew lines, 180 of them are spindle classes.

The following table shows the number of spindle permutations, up to equivalence, for $n \leq 13$. We also indicate the number of amphicheiral spindle permutations, up to equivalence.

n	spindle classes	amphicheiral classes
1	1	1
2	1	1
3	2	0
4	3	1
5	7	1
6	15	3
7	48	0
8	180	12
9	985	5
10	6867	83
11	60108	0
12	609112	808
13	6909017	47

Assertion (ii) of Proposition 3.5 explains of course the non-existence of amphicheiral classes for $n \equiv 3 \pmod 4$.

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