HOW IS A GRAPH LIKE A MANIFOLD?

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ABSTRACT. In this article, we discuss some classical problems in combinatorics which can be solved by exploiting analogues between graph theory and the theory of manifolds. One well-known example is the McMullen conjecture, which was settled twenty years ago by Richard Stanley by interpreting certain combinatorial invariants of convex polytopes as the Betti numbers of a complex projective variety. Another example is the classical parallel redrawing problem, which turns out to be closely related to the problem of computing the second Betti number of a complex compact $(\mathbb{C}^*)^n$-manifold.

1. INTRODUCTION

Some recent developments in the theory of group actions on complex manifolds have revealed unexpected connections between the geometry of manifolds and the geometry of graphs. Our purpose in this semi-expository paper is twofold: first, to explore these connections; and second, to discuss some problems in graph theory which "manifold" ideas have helped to clarify. One such problem is that of counting the number of $n-k$ dimensional faces of a simple n-dimensional convex polytope. This number can be expressed as a sum

$$f_{n-k} = \sum_{\ell=0}^{k} \binom{n-\ell}{k-\ell} \beta_\ell, \tag{1.1}$$

where the β_ℓ's are positive integers. A celebrated conjecture of McMullen [Mc1] asserts that these integers satisfy the identities

$$\beta_k = \beta_{n-k} \tag{1.2}$$

and the inequalities

$$\beta_0 \leq \beta_1 \leq \cdots \leq \beta_r, \tag{1.3}$$

where $r = \left\lceil \frac{n}{2} \right\rceil$. In 1980, Richard Stanley [St] solved McMullen's conjecture; his proof involved showing that the β_k's are the Betti numbers of a complex projective variety, and hence that (1.2) is just Poincaré duality and (1.3) is the "hard Lefshetz" theorem.

Another such problem is the classical *parallel redrawing* problem: Given a graph embedded in $\mathbb{R}^n$, how many ways can one reposition the vertices so that the edges of the deformed graph are parallel to the edges of the original graph? We show that the number of such deformations can be counted by a combinatorial invariant involving the zeroth and first "Betti numbers" of the graph.

In the remainder of this section, we describe how graph theoretical structures occur in the study of group actions on complex manifolds. This section provides much of the geometric motivation for the graph theory that follows. Although

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the remainder of the paper does not depend on this section, the reader is strongly encouraged to acquaint herself with the ideas discussed here, particularly with the examples given in Section 1.3. The rest of the paper is graph theoretical. We give purely graph-theoretic definitions of various combinatorial notions associated with the graphs coming from manifolds. However, we make our definitions sufficiently broad so as to apply to many graphs which are *not* associated with group actions on manifolds. In Sections 2 through 4, we define the notions of *connection* and *Morse function* on a regular graph, and using "Morse theory," we define the *Betti numbers* of a graph. Then in Section 5, we take up the problem mentioned above: counting the number of $n-k$ dimensional faces of a simple n-dimensional convex polytope, Δ, and show that the β_ℓ's in Equation (1.1) are just the Betti numbers of Δ. We also provide in this section a brief description of Stanley's proof of the McMullen conjectures.

In Section 6, we discuss another problem in which graph theoretical Betti numbers play an important role. This problem is a polynomial interpolation problem for graphs which generalizes, in some sense, the classical problem of Lagrangian interpolation for polynomials of one variable. For instance, for the complete graph on n vertices, this problem consists of finding a polynomial in one variable,

$$p(z) = \sum_{i=1}^{n-1} g_i z^i$$

whose coefficients g_i are polynomials in $x_1, \dots, x_n$ such that the expressions

$$f_i = p(z)|_{z=x_i} \;\; i = 1, \dots, n \tag{1.4}$$

are pre-assigned polynomials in the x_i's. We will show that (1.4) is solvable if and only if

$$f_i - f_j = 0 \mod x_i - x_j; \tag{1.5}$$

so (1.4) can be regarded as a formula for the general solution of the system of equations (1.5). One encounters systems of equations of the type (1.5) involving more complicated graphs in the area of combinatorics known as spline theory. Furthermore, they come up, somewhat unexpectedly, in the theory of manifolds. The computation of the equivariant cohomology rings of manifolds we describe in the examples below involves solving such equations. We will not attempt to discuss equivariant cohomology in this article, but interested readers can find details in [GKM], [TW], and [GZ2].

In Section 6, we will show how a connection on a graph enables one to construct many explicit solutions to interpolation equations of this type. In Section 7, we will show that the dimension of the space of solutions of degree k is given by a formula similar to the McMullen formula (1.1). Moreover for $k=1$, this formula will solve the parallel redrawing problem for a large class of interesting graphs. In Section 8, we will discuss some connections between this formula and McMullen's formula, and show that this resemblance between the two is not entirely fortuitous. More explicitly, we show that the face counting problem for polytopes can be viewed as an interpolation problem for polynomials in "anti-commuting" variables. Finally, in Section 9, we provide several families of examples of graphs to which this theory applies and some open questions. In the interest of brevity, many details of this section are left to the reader.

We are grateful to many colleagues and friends for enlightening comments about various aspects of this paper. In particular, the proof of Theorem 7.1 is, with minor modifications, identical to that of Theorem 2.4.1 in [GZ2]. We are grateful to Catalin Zara for letting us reproduce this argument here, and also for a number of helpful comments and insights. We would also like to thank Werner Ballmann, Sara Billey, Daniel Biss, Rebecca Goldin, Allen Knutson, Sue Tolman, David Vogan, and Walter Whitely for helpful conversations.

1.1. **GKM Manifolds.** Henceforth, we denote by $\mathbb{C}^*$ the multiplicative group of non-zero complex numbers, and by G the group

$$G = \mathbb{C}^* \times \cdots \times \mathbb{C}^* = (\mathbb{C}^*)^n.$$

Let M be a d-dimensional compact complex manifold, and $\rho : G \times M \to M$ a holomorphic action of G on M. The action ρ is called a *GKM action* if there are a finite number of G fixed points and a finite number of G orbits of (complex) dimension one.

An important example is the Riemann sphere $\mathbb{C}P^1$ with the standard action of $\mathbb{C}^*$. This is the action

$$c \cdot [z_0 : z_1] = [c \cdot z_0 : z_1],$$

where $[z_o : z_1]$ are standard homogeneous coordinates on $\mathbb{C}P^1$. This action has two fixed points, the origin, $[0 : 1]$, and the point at infinity, $[1 : 0]$. The complement of these points is a single $\mathbb{C}^*$ orbit.

When ρ is a GKM action, we let $V = M^G = \{p_1, \dots, p_\ell\}$ be the set of fixed points of ρ and $E = \{e_1, \dots, e_N\}$ the set of one-dimensional orbits. It is not hard to show that thes objects satisfy

(1) The closure of each $e \in E$ is an embedded copy of the complex projective line, $\mathbb{C}P^1$;
(2) If the closure of e_i intersects the closure of e_j, the intersection is a single point p. Moreover, $p \in M^G$;
(3) The closure of each e contains exactly two fixed points;
(4) Every fixed point is contained in the closure of exactly d one-dimensional orbits; and
(5) The action of G on the closure of e is isomorphic to the standard action of $\mathbb{C}^*$ on $\mathbb{C}P^1$.

This last item needs some explaining, which we postpone until the next section.

1.2. **GKM graphs.** Goresky, Kottwitz, and MacPherson point out in [GKM] that the properties (1) through (4) are nicely expressed in the language of graph theory. Let Γ be the graph having vertex set $V = M^G$ and edge set E, where $e \in E$ joins p and q in V exactly when p and q are the two fixed points of G in the closure of e. We call Γ the *GKM graph* of the pair (M, ρ).

Properties (1) through (4) tell us that Γ is a particularly nice graph. For instance, properties (2) and (3) imply that there are no loops and no multiple edges in Γ. Property (4) tells us that Γ is a *d-regular graph*: each vertex has valence d.

Property (5) does not have a purely combinatorial interpretation. It specifies a way to assign a vector in $\mathbb{R}^n$ to each edge of Γ, and so provides a kind of embedding of Γ. Suppose e is the edge joining p and q. To specify an orientation of e, we decide which of p and q is the initial vertex $\iota(e)$ of e and which the terminal vertex $\tau(e)$ of e. Property (5) says that the action of G on the closure of e is isomorphic to the standard action of $\mathbb{C}^*$ on $\mathbb{C}P^1$. This means that there is a group homomorphism

$$\chi_e : G \to \mathbb{C}^*$$

and a bi-holomorphic map

$$f_e : \overline{e} \to \mathbb{C}P^1$$

such that $f_e(g \cdot p) = \chi_e(g) \cdot f_e(p)$ for all $g \in G$ and for all $p \in \overline{e}$. The homomorphism χ_e and the map f_e are more or less unique. The only way that f_e can be altered is by composing it with a bi-holomorphic map of $\mathbb{C}P^1$ which commutes with the action of $\mathbb{C}^*$, and the group of such bi-holomorphisms consists of $\mathbb{C}^*$ itself and the involution

$$\sigma([z_0 : z_1]) = [z_1 : z_0]$$

which flips the fixed points at zero and infinity. Moreover, if f_e is composed with an element of $\mathbb{C}^*$, χ_e is unchanged, and composition with σ interchanges χ_e with χ_e^{-1}. In particular, χ_e is determined when we specify which of the two points of $\overline{e} - e$ maps to zero and which to infinity.

In the coordinate system determined by the expression of G as a product of $\mathbb{C}^*$, this intertwining homomorphism has the form

$$\chi_e(c_1, \dots, c_n) = c_1^{\alpha_1(e)} \cdots c_n^{\alpha_n(e)}.$$

Note that the $\alpha_r(e)$ are integers because they are characters of the group $S^1 \subset \mathbb{C}^*$. So to every oriented edge e of Γ, we can attach the n-tuple of integers

$$\alpha(e) = (\alpha_1(e), \dots, \alpha_n(e)) \in \mathbb{Z}^n.$$

If we reverse the orientation of e, then χ_e becomes χ_e^{-1} and $\alpha(e)$ becomes $-\alpha(e)$. We call the map

$$\alpha : E \to \mathbb{Z}^n \subset \mathbb{R}^n$$

the *axial function* of the graph Γ. The rationale for the name "axial function" is that the map α describes how each of the two-spheres $e = \mathbb{C}P^1 = S^2$ gets rotated about its axis by the torus subgroup

$$G_{\mathbb{R}} = \underbrace{S^1 \times \cdots \times S^1}_{n}$$

of G.

1.3. **Examples.** We describe below a few typical examples of GKM graphs. The reader should make herself familiar with these examples since they will resurface frequently in the remainder of the paper.

EXAMPLE. **The n-simplex.** Let $M = \mathbb{C}P^{n-1}$ be complex projective $n-1$ space, represented using homogeneous coordinates in $\mathbb{C}^n$, and let G act on M by the product action

$$g \cdot [z_1 : \cdots : z_n] = [g_1 \cdot z_1 : \cdots : g_n \cdot z_n].$$

The fixed points of this action are the points $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$, and the one-dimensional orbits are the sets $e_{i,j} = \{[0 : \cdots : z : \cdots : w : \cdots : 0]\}$, points with non-zero entries in just two components. It follows easily that the intersection graph is K_n, the complete graph on n points. Moreover, if we choose zero and infinity in $\overline{e_{i,j}}$ to be the point with ith coordinate zero and jth coordinate zero respectively, then the intertwining homomorphism is

$$\chi_{e_{i,j}}(c_1, \ldots, c_n) = c_i^1 \cdot c_j^{-1},$$

since in homogeneous coordinates,

$$[c_i c_j^{-1} z : w] = [c_i z : c_j w].$$

Thus, $\alpha(e_{i,j}) = (0, \ldots, 1, 0, \ldots, -1, 0, \ldots, 0)$, with a 1 in the ith position and a -1 in the jth position. If we embed the graph K_n in $\mathbb{R}^n$ by sending the vertex $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ to the point $(0, \ldots, 0, 1, 0, \ldots, 0)$, this computation shows that the edge joining two vertices does indeed have the direction $\alpha(e)$. Thus, we have proved the following proposition.

Proposition 1.1. *The graph Γ of M is the one-skeleton (vertices and edges) of the $(n-1)$-simplex*

$$\{(x_1, \ldots, x_n) \in \mathbb{R}_+^n \mid x_1 + \cdots + x_n = 1\},$$

and for every edge e of this embedded graph, $\alpha(e)$ is the embedded image of e.

EXAMPLE. **The hypercube.** Let M be

$$M = \underbrace{\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1}_{n}. \tag{1.6}$$

Then G acts on M by the product action,

$$c \cdot (p_1, \ldots, p_n) = (c_1 \cdot p_1, \ldots, c_n \cdot p_n),$$

and the fixed points for this action are the points

$$(\varepsilon_1, \ldots, \varepsilon_n),$$

where $\varepsilon_i \in \{0, \infty\}$ is zero or the point at infinity of the ith copy of $\mathbb{C}P^1$ in the product (1.6). It is easy to see that the one-dimensional orbits of G are just the sets

$$(\varepsilon_1, \ldots, \varepsilon_{i-1}, p_i, \varepsilon_{i+1}, \ldots, \varepsilon_n),$$

where $p_i \neq \varepsilon_i$.

We leave the proof of the following proposition as an exercise:

Proposition 1.2. *The graph of M is the one-skeleton of the hypercube*

$$\underbrace{[0,1] \times \cdots \times [0,1]}_{n}$$

in $\mathbb{R}^n$, and for every edge e of this embedded graph, $\alpha(e)$ is the embedded image of e.

Note that this embedding of the graph does give us the axial function as well. This construction naturally generalizes to the product of any set of GKM manifolds.

EXAMPLE. **The Johnson graph.** Projective space, $\mathbb{C}P^{n-1}$, is really just the set of lines in complex n-space $\mathbb{C}^n$. This example generalizes that construction. The group G acts on $\mathbb{C}^n$ with the product action

$$c \cdot (z_1, \ldots, z_n) = (c_1 \cdot z_1, \ldots, c_n \cdot z_n). \tag{1.7}$$

Let M be the k-Grassmannian $\mathcal{G}r(k, n)$: the set of all (complex) k-dimensional subspaces of $\mathbb{C}^n$. The action of G on $\mathbb{C}^n$ induces an action of G on M, and it is easy to see that the points of M which are fixed by this action, that is the k-dimensional subspaces of $\mathbb{C}^n$ which are mapped to themselves by (1.7) are the subspaces

$$\mathbb{C}^n_I = \{(z_1, \ldots, z_n) \mid z_i = 0 \text{ for } i \notin I\},$$

where I is a k-element subset of $\{1, \ldots, n\}$. In other words, the fixed points for the action of G on M are indexed by the k-element subsets of $\{1, \ldots, n\}$. The one-dimensional orbits of G are not much harder to describe. Let I and J be k-element subsets of $\{1, \ldots, n\}$ whose intersection has order $k-1$. Then the collection of all k-dimensional subspaces V of $\mathbb{C}^n$ satisfying

$$\mathbb{C}^n_I \cap \mathbb{C}^n_J \subset V \subset \mathbb{C}^n_I + \mathbb{C}^n_J,\ V \neq \mathbb{C}^n_I, \mathbb{C}^n_J \tag{1.8}$$

is a one-dimensional orbit. Moreover, the sets (1.8) are the only one-dimensional orbits. Hence

Proposition 1.3. *The GKM graph Γ of M is the graph whose vertices correspond to k-element subsets of $\{1, \ldots, n\}$, and two such subsets I and J are adjacent if and only if their intersection is a $(k-1)$-element set. If we think of each k-element set as a vector in in $\mathbb{R}^n$ with k 1's and $n-k$ 0's in the obvious way then we have embedded Γ in $\mathbb{R}^n$. The axial function at an edge e is the embedded image of e.*

This graph is known as the *Johnson graph*, $J(k, n)$. $J(2, 4)$ is the 1-skeleton of the octahedron, shown below.

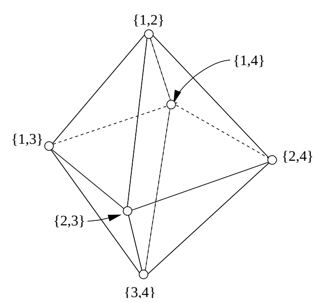


FIGURE 1. The Johnson graph $J(2, 4)$ as an octahedron.

Although $J(2, 4)$ is the cross polytope in dimension three, there is in general no relation between the cross polytopes and the Johnson graphs.

EXAMPLE. **The one skeleton of a simple convex polytope** A convex n-dimensional polytope Δ in $\mathbb{R}^n$ is *simple* if exactly n edges meet at each vertex. Given such a polytope, one can associate with it a complex n-dimensional space M_Δ and an action on M of $(\mathbb{C}^*)^n$ such that the k-dimensional orbits are in one-to-one correspondence with the k-dimensional faces of Δ. In particular, the zero- and one-dimensional orbits correspond to the vertices and edges of Δ. Thus, this action is a GKM action, and its graph is the one-skeleton of Δ. We will describe this example in more detail in Section 5. The first two examples described above, the n-simplex and the hypercube, are special cases of this example, but the polytopes associated with the Johnson graph are not simple.

1.4. **How is a graph like a manifold?** Several recent articles about the topology and geometry of GKM manifolds exploit the fact that topological properties of M have combinatorial implications for Γ and that, conversely, given information about Γ, one can draw conclusions about the topology of M. See [GKM], [TW], [GZ1], [GZ2], [KR], and [LLY]. Throughout these articles, one glimpses a kind of dictionary in which manifold concepts translate to graph concepts and vice versa. Our modest goal in this article is to examine some entries in the "graph" column of this dictionary for their own intrinsic (graph theoretic) interest. For instance, in Section 2 we will give a combinatorial definition of the notion of a *connection* on a graph Γ, *geodesics*, *totally geodesic subgraphs* of Γ, and the *holonomy group* associated to a connection. In Section 3, we define an *axial function* on a graph. In Section 4, we discuss *Morse functions* and define the combinatorial *Betti numbers* of Γ. Then we use those topological notions to prove theorems about graphs. Finally, there are interesting graphs for which these notions make sense which are *not* GKM graphs. For example, the complete bipartite graph $K_{n,n}$ has many of these combinatorial structures, but is not associated with a manifold We discuss this and other examples in more detail in Section 9.

2. Connections and geodesic subgraphs

Let $\Gamma = (V, E)$ be a graph with finite vertex set V and edge set E. We count each edge twice, once with each of its two possible orientations. When x and y are adjacent vertices we write $e = (x, y)$ for the edge from x to y and $e^{-1} = (y, x)$ for the edge from y to x. Given an oriented edge $e = (x, y)$, we write $x = \iota(e)$ for the initial vertex and $y = \tau(e)$ for the terminal vertex.

Definition 2.1. *The* star *of a vertex x, written star(x), is the set of edges leaving x,*

$$\textit{star}(x) = \{e \mid \iota(e) = x\}.$$

The star of a vertex is the combinatorial analogue of the tangent space to a manifold at a point.

Definition 2.2. *A* connection *on a graph Γ is a set of functions $\nabla_{(x,y)}$ or ∇_e, one for each oriented edge $e = (x, y)$ of Γ, such that*

(1) $\nabla_{(x,y)} : \textit{star}(x) \to \textit{star}(y)$,
(2) $\nabla_{(x,y)}(x, y) = (y, x)$, *and*
(3) $\nabla_{(y,x)} = (\nabla_{(x,y)})^{-1}$.

It follows that each $\nabla_{(w,y)}$ is bijective, so each connected component of Γ is regular: all vertices have the same valence. Henceforth we will assume Γ comes equipped with a connection ∇.

Definition 2.3. *A* 3-geodesic *is a sequence of four vertices (x, y, z, w) with edges $\{x, y\}$, $\{y, z\}$, and $\{z, w\}$ for which $\nabla_{(y,z)}(y, x) = (z, w)$. We inductively define a k-geodesic as a sequence of $k + 1$ vertices in the natural way. We may identify a geodesic by specifying either its edges or its vertices, and we will refer to* edge geodesics *or* vertex geodesics *as appropriate. The three consecutive edges (d, e, f) of a 3-geodesic will be called an* edge chain.

Definition 2.4. *A* closed geodesic *is a sequence of edges $e_1, \ldots, e_n$ such that each consecutive triple (e_i, e_{i+1}, e_{i+2}) is an edge chain for each $1 \leq i \leq n$, modulo n.*

A little care is required to understand when a geodesic is closed, since it may in fact use some edges in star(x) multiple times. It is not closed until it returns to the same pair of edges in the same order. That is analogous to the fact that a periodic geodesic in a manifold is an immersed submanifold, not an embedded submanifold. The period completes only when it returns to a point with the same velocity (tangent vector).

Remark 2.5. *Because there is a unique closed geodesic through each pair of edges in the star of a vertex, the set of all closed geodesics completely determines the connection on* Γ. *We will sometimes use this fact to describe a connection.*

We define totally geodesic subgraphs of a graph by analogy to totally geodesic submanifolds of a manifold.

Definition 2.6. *Given a graph* Γ *with a connection* ∇, *we say that a subgraph* $(V_0, E_0) = \Gamma_0 \subseteq \Gamma$ *is* totally geodesic *if all geodesics starting in* E_0 *stay within* E_0.

This definition is equivalent to saying that a totally geodesic subgraph Γ_0 is one in which, for every two adjacent vertices x and y in Γ_0,

$$\nabla_{(x,y)}(\text{star}(x) \cap E_0) \subseteq E_0.$$

Suppose now that $P = \{e_1, \dots, e_n\}$ is any cycle in Γ: $\tau(e_i) = \iota(e_{i+1})$ modulo n. Then following the connection around P leads to a permutation

$$\nabla_P = \nabla_{e_n} \circ \cdots \circ \nabla_{e_1} \circ \nabla_{e_0}$$

of star(x).

Definition 2.7. *The* holonomy group $Hol(\Gamma_x)$ *at vertex* x *of* Γ *is the subgroup of the permutation group of star*(x) *generated by the permutations* ∇_P *for all cycles* P *that pass through* x.

It is easy to see that the holonomy groups $Hol(\Gamma_x)$ for the vertices x in each connected component of Γ are isomorphic. When Γ is connected and d-regular we call that group the holonomy group of Γ and think of it as a subgroup of S_d.

3. Axial functions

We described in Section 1 how a graph arising from a GKM manifold has associated to it an *axial function*, namely an assignment of a vector in $\mathbb{Z}^n$ to each oriented edge e. We need to formalize this definition for abstract graphs.

Definition 3.1. *An* axial function *on a graph with a connection is a map*

$$\alpha : E \to \mathbb{R}^n \setminus \{0\}$$

such that

$$\alpha(e^{-1}) = -\alpha(e)$$

and for each 3*-geodesic* (d, e, f)

$$\alpha(d), \alpha(e), \alpha(f)$$

are coplanar.

It follows immediately that the images under α of all geodesics of Γ are planar. What matters about the axial function is the *direction* of $\alpha(e)$ in $\mathbb{R}^n \setminus \{0\}$, not its actual value. We consider two axial functions α and α' to be equivalent if

$$\frac{\alpha(e)}{||\alpha(e)||} = \frac{\alpha'(e)}{||\alpha'(e)||}$$

for all edges e. Notice that α is *not* equivalent to $-\alpha$.

If $e = (x, y)$ is an edge, we will denote $\alpha(e)$ by $\alpha(x, y)$, rather than using two sets of parentheses. We picture an edge chain as a succession of vectors joined head to tail in their plane, as shown in the figure below. A picture of an equivalent axial

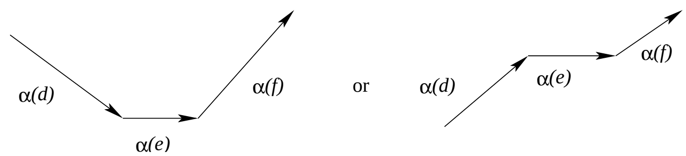


FIGURE 2. This shows how we picture the axial function on an edge chain.

function will show vectors with the same orientations, but different lengths.

Definition 3.2. *An* immersion *of* (Γ, α) *is a map* $F : V \to \mathbb{R}^n$ *such that*

$$\alpha(x, y) = F(y) - F(x).$$

Our picture of an immersed vertex chain (x, y, z, w) is shown below. Here, the

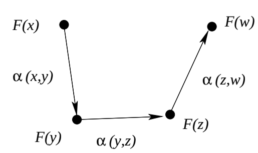


FIGURE 3. This shows how we picture the axial function on an immersed vertex chain.

endpoints of the vectors do make sense, as the vertices are points in $\mathbb{R}^n$.

Definition 3.3. *An axial function is* exact *if for every edge chain* (d, e, f)*,*

$$\alpha(f) + \alpha(d) = c \cdot \alpha(e), \tag{3.1}$$

for $c \in \mathbb{R}$*.*

The figure below shows three exact edge chains. In Figure 4(a), the constant is positive, in (b) it is zero, and in (c) it is negative.

When α is exact, the edges $\alpha(d)$ and $\alpha(f)$ must lie on the same side of $\alpha(e)$ in the plan in which they lie. The examples in Section 1.3 are immersible and exact. So are the axiam functions for the GKM manifolds discussed in Section 1.2. Notice that the product of two immersable (exact) axial functions is again immersable (exact). Under what conditions is an arbitrary axial function equivalent to one that is immersable? exact? We have only partial answers to these interesting questions.

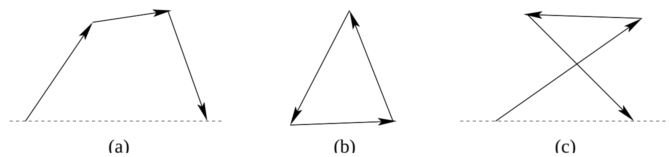


FIGURE 4. This shows three exact edge chains.

Theorem 3.4. *If the axial function α is 3-independent, then it determines the connection.*

Proof. Let d and e be edges with $\tau(d) = \iota(e)$. Then 3-independence implies that there is only *one* edge f with $\tau(e) = \iota(f)$ and and $\alpha(f)$ in the plane determined by $\alpha(d)$ and $\alpha(e)$, □

In Section 7, we will need a special case of the following definition.

Definition 3.5. *The product of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is the graph*

$$\Gamma = \Gamma_1 \times \Gamma_2 = (V, E),$$

with vertex set $V = V_1 \times V_2$. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if

(1) $x_1 = x_2$ *and* $\{y_1, y_2\} \in E_2$*; or*
(2) $y_1 = y_2$ *and* $\{x_1, x_2\} \in E_1$.

Suppose now that each Γ_i is equipped with a connection ∇_i and axial function $\alpha_i : E_i \to \mathbb{R}^{n_i}$. Then we can define a connection ∇ on Γ in a natural way. by specifying the closed geodesics as the closed geodesics in each component and some closed geodesics of length 4 which go between Γ_1 and Γ_2.

The figure below shows one each of the two kinds of geodesics for the example in which Γ_1 is a 3-cycle and Γ_2 is an edge.

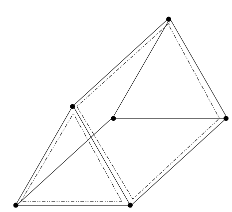


FIGURE 5. This shows the product of two graphs, showing one geodesic of each type.

We define an axial function $\alpha : E \to \mathbb{R}^{n_1+n_2}$ by

$$\alpha((x_1, y_1), (x_2, y_2)) = (\alpha_1(x_1, x_2), \alpha_2(y_1, y_2)),$$

where (by definition) $\alpha_i((x, x)) = 0$. We leave it to the reader to check that ∇ is a well-defined connection, and that α is indeed an axial function compatible with ∇. Note that this generalizes the example of the hypercube, which is an n-fold product of an edge.

4. Betti numbers

Suppose Γ is a graph with a connection ∇ and an axial function α mapping edges to $\mathbb{R}^n$. The images under α of the chains in Γ are planar; we will study how those chains wind in their planes. To that end choose an arbitrary orientation for each such plane P. Then whenever $\alpha(e) \in P$ the direction $\alpha(e)^\perp$ is a well defined direction in P. (If α is immersible then $\alpha(e)^\perp$ is a well defined vector in p.)

Throughout this section we will assume α is 2-independent. That is, no two edges in the star of a vertex of Γ are mapped by α into the same line in $\mathbb{R}^n$. Thus any two edges at a vertex determine a unique plane, which we have assumed is oriented.

Definition 4.1. *The* curvature $\kappa(d,e)$ *of the edges* $d = (x,y)$ *and* $e = (y,z)$ *at* y *is* $\mathrm{sign}(\alpha(d)^\perp \cdot \alpha(e)) \in \{\pm 1\}$.

Definition 4.2. *An edge chain* (d,e,f) *is an* inflection *if* $\kappa(d,e)$ *and* $\kappa(e,f)$ *have opposite signs. We say* α *is* inflection free *if there are no inflections.*

Note that if an axial function is exact, then by (3.1), it is inflection free.

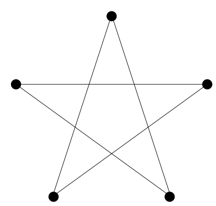


FIGURE 6. The pentagram: a 5-cycle with an unusual immersible axial function that is nonetheless inflection free.

Definition 4.3. *A direction* $\xi \in \mathbb{R}^n \setminus \{0\}$ *is* generic *if for all* $e \in E$, $\xi \not\perp \alpha(e)$.

Definition 4.4. *The* index *of a vertex* $x \in V$ *with respect to a generic direction* ξ *is the number of edges* $e \in star(x)$ *such that*

$$\alpha(e) \cdot \xi < 0.$$

We call those the down *edges. Let* $\beta_i(\xi)$ *be the number of vertices* $x \in V$ *such that the index of* x *is exactly* i.

Theorem 4.5. *If* Γ *is a graph with connection* ∇ *and an inflection free axial function* α, *then the Betti numbers* β_i *do not depend on the choice of direction* ξ.

Proof. Imagine the direction ξ varying continuously in $\mathbb{R}^n$. It is clear from the definitions above that the indices of vertices can change only when ξ crosses one of the hyperplanes $\alpha(x,y)^\perp$. Let us suppose that (x,y) is the only edge of Γ at which the value of the axial function is a multiple of $\alpha(x,y)$. Then at such a crossing only the indices of the vertices x and y can change. Suppose ξ is near $\alpha(x,y)^\perp$. Since α is inflection free, the connection mapping star(x) to star(y) preserves down edges, with the single exception of edge (x,y) itself. That edge is down for one of x and y and up for the other. Thus the vertices x and y have indices i and $i+1$ for ξ on one side of $\alpha(x,y)^\perp$ and indices $i+1$ and i on the other. Thus the number of vertices with index i does not change as ξ crosses $\alpha(x,y)^{\perp}$. If there are several edges of Γ at which the axial function is a multiple of $\alpha(x,y)$, the same argument works, since by the 2-independence of α, none of those edges can share a common vertex. □

Henceforth we will assume α is inflection free. The motivation for the following definitions comes from Morse theory.

Definition 4.6. *When the $\beta_i(\xi)$ are independent of ξ, we call them the* Betti numbers *of Γ (or, more precisely, the Betti numbers of the pair (Γ,α)).*

The following proposition is the combinatorial version of Poincaré duality.

Proposition 4.7. *When the Betti numbers of a graph are independent of the choice of ξ, then $\beta_i(\Gamma) = \beta_{d-i}(\Gamma)$ for $i = 0,\dots,d$.*

Proof. Choose some ξ with which to compute the Betti numbers of Γ. Then simply replace ξ by $-\xi$, and a vertex of index i becomes a vertex of index $d-i$. □

Definition 4.8. *Given a generic ξ, a* Morse function compatible with ξ *on a graph with an axial function α is a map $f : V \to \mathbb{R}$ such that if (x,y) is an edge, $f(x) > f(y)$ whenever $\alpha(x,y)\cdot\xi > 0$.*

There is a simple necessary and sufficient condition for the existence of a Morse function compatible with ξ.

Theorem 4.9. *A Morse function compatible with ξ exists if and only if there exists no closed cycle $(e_1,\dots,e_n)$ with $e_1 = e_n$, in Γ for which all the edges e_i are "up" edges.*

Proof. The necessity of this condition is obvious since f has to be strictly increasing along such a path. To prove sufficiency, for every vertex p, define $f(p)$ to be the length N of the longest path $(e_1,\dots,e_N)$ in Γ of up edges $\tau(e_N) = p$ The hypothesis that there is no cycle of up edges guarantees that this function is well-defined, and it is easy to check that it is a Morse function. □

Remark 4.10. *One can easily arrange for f in the above proof to be an* injective *map of V into $\mathbb{R}$ by perturbing it slightly.*

When α is immersible, so that $\alpha(x,y) = f(y) - f(x)$, then we can define a Morse function on Γ by setting $m(x) = f(x)\cdot\xi$ for any generic direction ξ. Then $m(x)$ increases along each up edge. The vertices with index i resemble critical points of Morse index i in the Morse theory of a manifold. We call the β_i Betti numbers because when a graph we are studying is the GKM graph of a manifold, the β_i indeed correspond to the Betti numbers of the manifold, and they are the dimensions of the cohomology groups of the manifold.

Remark 4.11. *When an inflection-free 2-independent axial function is projected generically into a plane, it retains those properties, so the Betti numbers of Γ can be computed using a generic direction in a generic plane projection. In most of our examples α is immersible. In these cases we are of course drawing a planar embedding of Γ. Thus the figures in this paper are more than mere suggestions of some high dimensional truth. They actually capture all the interesting information about Γ.*

Definition 4.12. *The generating function β for the Betti numbers of Γ is the polynomial*

$$\beta(z) = \sum_{i=0}^{n} \beta_i z^i.$$

Remark 4.13. *When Γ is d-regular, β is of degree d. The sum of the Betti numbers, $\beta(1)$, is just the number of vertices of Γ*

Remark 4.14. *It is clear that $\beta_0 > 0$ if a Morse function exists, because the vertex at which the Morse function assumes its minumum value has no down edges.*

We can relate the Betti numbers of the product of two graphs to the Betti numbers of the two multiplicands as follows. The proof is left to the reader.

Proposition 4.15. *Let Γ and Δ be graphs with Betti numbers generated by $\beta_\Gamma(z)$ and $\beta_\Delta(z)$ respectively. Then the generating function for the Betti numbers of the product graph $\Gamma \times \Delta$ is the polynomial product*

$$\beta_\Gamma(z) \cdot \beta_\Delta(z).$$

5. Simple convex polytopes

Recall that a polytope Δ in $\mathbb{R}^n$ is simple if each vertex has degree n. Every plane polygon is simple. Three of the five Platonic polyhedra are simple: the tetrahedron, the cube and the dodecahedron. So are the higher dimensional analogues of the simplex and the cube we have already encountered.

Let Γ be the embedded graph whose vertices and edges are the one-skeleton of the simple polytope Δ. Every two dimensional face of Δ is a plane polygon. Since Δ is simple, every pair of adjacent edges belongs to exactly one such polygon. These polygons thus serve to define a natural connection on Γ whose geodesics are those polygons. That connection is compatible with the exact axial function α defined by the embedding. The totally geodesic subgraphs naturally correspond the the faces of Δ of every dimension. If we assume Δ is convex (and we shall) then α is inflection free and the Betti numbers of Γ are well defined.

Let us now turn to the identities (1.1). We will prove that the β_k's in these identities are just the Betti numbers of Γ. Since the equations (1.1) can be solved for the β_k's as expressions in the terms on the left hand side, this will give us another proof of Proposition 4.7.

Proof of (1.1). Fix a generic vector $\xi \in \mathbb{R}^n \setminus \{0\}$ ($\xi \not\perp \alpha(e)$ for all $e \in E$) and let $f : \mathbb{R}^n \to \mathbb{R}$ be the function $f(x) = x \cdot \xi$. By choosing ξ appropriately, we can assume that $f : V \to \mathbb{R}$ is injective. Then f on V is an injective Morse function compatible with ξ sense of Section 4. Now to count the faces of Δ we exploit the fact that every $n-k$-dimensional face F has a unique vertex p_F at which f takes its minimum value. At this vertex, the index of p_F with respect to ξ is at most k, since the $n-k$ edges of F at p_F are upward-pointing with respect to ξ.

Let ℓ be a number between 0 and k, and let p be a vertex of Γ of index ℓ. We can count the number of $n-k$-dimensional faces F for which $p_F = p$. At p, there are exactly $n-\ell$ edges which are upward pointing with respect to ξ, and every $n-k$-element subset of this set of edges spans an $n-k$-dimensional face F with

$p_F = p$. Thus, there are $\binom{n-\ell}{n-k}$ such faces in all. Summing over p, we find

$$\binom{n-\ell}{n-k}\beta_\ell$$

$n-k$-dimensional faces F for which the index of p_F equals ℓ. Summing over ℓ we get

$$f_{n-k} = \sum_{\ell=0}^{k} \binom{n-\ell}{n-k}\beta_\ell$$

for the total number of $n-k$-dimensional faces of Δ. □

We noted in the introduction that Stanley proved the McMullen conjectures by showing that the graph theoretical Betti numbers β_k in the identites (1.1) are actually the Betti numbers of a complex projective variety, and hence that the identities (1.2) follow from Poincaré duality and the inequalities (1.3) from the hard Lefshetz theorem. We are part way there: the identites (1.2) are Proposition 4.7. A combinatorial proof of (1.3) exists ([Mc2], [K]), but it is much harder than the proof we gave above of (1.2). So for completeness (and because it is so elegant) we will sketch Stanley's argument.

The McMullen inequalities follow from the following two assertions:

(1) If M is a GKM manifold, its manifold Betti numbers coincide with its graph theoretical Betti numbers.
(2) Given a simple convex polytope, there exists a GKM manifold whose graph and axial function are the graph and axial function of the polytope.

The proof of the first of these assertions is by "manifold" Morse theory and resembles our "graph" Morse theory computations of Section 4. We will briefly describe how to prove the second assertion. If we perturb the vertices of Δ slightly it will still be simple and convex, so without loss of generality, we may assume that the vertices of Δ are rational points in $\mathbb{R}^n$. For each facet (*i.e.* $n-1$-dimensional face) F of Δ, Then we can find an outward pointing normal vector m_F to $\mathbb{F}$ with relatively prime integer entries. Let $F_1, \dots, F_d$ be the facets of Δ, and let $m_{F_i} = m_{1,i}, \dots, m_{n,i})$. The map

$$(\mathbb{C}^*)^d \to (\mathbb{C}^*)^n,$$

mapping $(z_1, \dots, z_d)$ to $(w_1, \dots, w_n)$, where

$$w_i = z_1^{m_{i,1}} \cdots z_d^{m_{i,d}},$$

is a group homomorphism. Let N be its kernel Clearly,

$$G = (C^*)^n = (\mathbb{C}^*)^d/N. \tag{5.1}$$

Now let $(\mathbb{C}^*)^d$ act on $\mathbb{C}^d$ by coordinate-wise multiplication,

$$a \cdot z = (a_1 \cdot z_1, \dots, a_d \cdot z_d).$$

Since N is a subgroup of $(\mathbb{C}^*)^d$ it acts on $\mathbb{C}^d$. Then (5.1) induces an action of G on the quotient

$$\mathbb{C}^d/N.$$

This quotient space, with its G action, is, morally speaking, the manifold M that we are looking for. Unfortunately, it is not a manifold, and as a topological space, it is not even Hausdorff. Fortunately, it is easily desingularized, by deleting from it a finite number of "non-stable" orbits of G. More explicitly, for every subset I of $\{1, \ldots, d\}$, let

$$\mathbb{C}^d_I = \left\{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_i = 0 \Leftrightarrow \ i \in I\right\}.$$

It is easy to see that these sets are exactly the $(\mathbb{C}^*)^d$-orbits in $\mathbb{C}^d$. In particular, if F is a face of Δ, there is a unique subset I of $\{1, \ldots, d\}$ for which

$$F = \cap_{i \in I} F_i$$

and to that face, we attach the orbit

$$\mathbb{C}^d_F := \mathbb{C}^d_I.$$

Finally, we define

$$\mathbb{C}^d_\Delta = \cup_F \mathbb{C}^d_F. \tag{5.2}$$

Theorem 5.1. *The space $\mathbb{C}^d_\Delta$ is an open subset of $\mathbb{C}^d$ on which N acts properly and locally freely, and the quotient space*

$$M = \mathbb{C}^d_\Delta / N$$

is a compact orbifold.

Remark 5.2. *The term "orbifold" means that M is not quite a manifold. It does have singularities, but they are fairly benign, and its topological properties, including the behavior of its Betti numbers, are the same as those of a manifold.*

Remark 5.3. *There is a simple condition equivalent to the assertion that M is a manifold: for each vertex p of Δ the n vectors m_{F_i} normal to the n facets containing p form a* lattice *basis of $\mathbb{Z}^n$. That is, every vector in $\mathbb{Z}^n$ can be written as a linear combination of the m_{F_i}'s with integer coefficients.*

Since each of the summands in (5.2) is a $(\mathbb{C}^*)^d$ orbit, the G-orbits in M are the sets

$$M_F = \mathbb{C}^d_F / N,$$

and a simple computation shows that $\dim M_F = \dim F$. Thus, the zero and one dimensional orbits of M correspond to the vertices and edges of Δ. In other words, the action ρ of G on M is a GKM action and the graph of (M, ρ) is the one-skeleton, Γ, of Δ.

EXAMPLE.Let Δ be the n-simplex

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ \mid x_1 + \cdots + x_n = 1\}.$$

Then

$$m_{F_i} = (1, \ldots, -n, \ldots, 1)$$

(a "$-n$" in the i^th slot), and N is the diagonal subgroup of $(\mathbb{C}^*)^{n+1}$: the n-tuples for which $z_1 = \cdots = z_{n+1}$. Moreover, $\mathbb{C}^{n+1}_\Delta$ is $\mathbb{C}^{n+1} \setminus \{0\}$. Note that the action of N on $\mathbb{C}^{n+1}$ is non-Hausdorff, as every N orbit contains 0 in its closure. However the action of N on $\mathbb{C}^{n+1} \setminus \{0\}$ is free and proper, and the quotient $\mathbb{C}^{n+1}_\Delta / N$ is just $\mathbb{C}P^n$. Thus, in this case, the construction we have just outlined reproduces the first of the examples discussed in Section 1.3. For more information about this construction, see, for example, [G, p. 109–130].

6. Polynomial interpolation schemes

6.1. **Polynomial interpolation schemes.** Let S be the polynomial ring in the variables $x = (x_1, \dots, x_n)$ and S^k the k^{th} graded component S: the space of homogeneous polynomials of degree k. S^k has dimension $\binom{n+k-1}{n-1}$.

Let $\Gamma = (V, E)$ be a regular d-valent graph with connection ∇ and axial function $\alpha : E \to \mathbb{R}^n$. Then for every edge e, of Γ we will identify the vector, $\alpha(e) \in \mathbb{R}^n$, with the linear function $\alpha_e(x) = \alpha(e) \cdot x$ so we can think of α_e as an element of S^1. Finally, for g and $h \in S$ we will say that

$$g \equiv h \mod \alpha$$

when $g - h$ vanishes on the hyperplane, $\alpha(x) = 0$.

Definition 6.1. *Let m be the numbers of vertices of Γ. An m-tuple of polynomials*

$$g_p \in S, \quad p \in V$$

is a polynomial interpolation scheme *if for every $e = (p, q) \in E$*

$$g_p \equiv g_q \mod \alpha_e. \tag{6.1}$$

Henceforth we will write $< g >$ for such an m-tuple. We say that this scheme has degree k *if for all p, $g_p \in S^k$, and we will denote the set of all polynomial interpolation schemes of degree k by $H^k(\Gamma, \alpha)$.*

It's clear that every interpolation scheme is in the space

$$H^*(\Gamma, \alpha) = \bigoplus_{k=0}^{\infty} H^k(\Gamma, \alpha)\,.$$

Moreover this space is clearly a graded module over the ring S. That is, if $< g_p >$ satisfies (6.1) then for every $h \in S$, so does $< hg_p >$. More generally, if $< g_p >$ and $< h_p >$ satisfy (6.1) then so does $< g_p \cdot h_p >$. So $H^*(\Gamma, \alpha)$ is not just a module, but in fact a *graded ring*, and S sits in this ring as the subring of constant interpolation schemes

$$g_p = g \text{ for all } p.$$

In this section we describe some methods for constructing solutions of the interpolation equations (6.1). These methods will rely heavily on the ideas that we introduced in Sections 2 and 3.

Let $F : V \to \mathbb{R}^n$ be an immersion of Γ. If we identify vector $F(p)$ with the monomial

$$f_p(x) = F(p) \cdot x$$

then (6.1) is just a rephrasing of the identity (3.1), so $< f_p >$ is an interpolation scheme of degree 1. More generally if

$$\mathfrak{p}(z) = \sum_{i=0}^{k} \mathfrak{p}_i(x) z^i$$

is a polynomial in z whose coefficients are polynomials in $x = (x_1, \dots, x_n)$ then the m-tuple of polynomials

$$< \sum_{i=0}^{k} \mathfrak{p}_i(x) f_p^i > \tag{6.2}$$

is a polynomial interpolation scheme. $< f_p >$ itself corresponds to the case $k = 1$, $\mathfrak{p}_0 = 0, p_1 = 1$.

6.2. **The complete graph.** In one important case this construction gives *all* solutions of (6.1). Namely let $K_{n+1} = (V, E)$ be the complete graph on $n + 1$ vertices with the natural connection. Then every immersion $F : V \to \mathbb{R}^n$ defines an axial function compatible with that connection by setting

$$\alpha(p, q) = F(q) - F(p) \tag{6.3}$$

for every oriented edge $e = (p, q)$. We will prove

Theorem 6.2. *If the axial function (6.3) is two-independent, every interpolation scheme can be written* uniquely *in the form*

$$< g_p >=< \sum_{i=0}^{k} \mathfrak{p}_i(x) f_p^i > \tag{6.4}$$

for some polynomials $\mathfrak{p}_i$.

Proof. By induction on n. Let $\{p_1, \ldots, p_{n+1}\}$ be the vertices of V, and let $< g_p >$ be an interpolation scheme. By induction there exists a polynomial

$$\mathfrak{p}(z) = \sum_{i=0}^{k} \mathfrak{p}_i z^i$$

with coefficients in S such that $\mathfrak{p}(g_{p_i}) = f_{p_i}$ for $i = 1, \ldots, n$. Hence the $n + 1$-tuple of polynomials

$$< f_p - p(g_p) > n$$

is an interpolation scheme vanishing on $p_1, \ldots, p_n$. Therefore, by (6.1) and the two-independence of the axial function (6.2)

$$f_{p_n} - \mathfrak{p}(g_{p_n}) = h \prod_{i<n} (g_{p_n} - g_{p_i})$$

for some polynomial $h \in S$. Let

$$\mathfrak{q}(z) = h \prod_{i \leq n} (z - g_{p_i}) .$$

Then

$$\mathfrak{q}(g_{p_{n+1}}) \quad = \quad f_{p_{n+1}} - \mathfrak{p}(g_{p_{n+1}})$$

and

$$\mathfrak{q}(g_{p_{n+1}}) \quad = \quad 0$$

for $i < n$. Thus the theorem is true for $\#V = n + 1$ with $\mathfrak{p}$ replaced by $\mathfrak{p} + \mathfrak{q}$.

The uniqueness of $\mathfrak{p}$ follows from the Vandermonde identity

$$\det \begin{pmatrix} 1 & g_{p_1} & \cdots & g_{p_1}^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & g_{p_n} & \cdots & g_{p_n}^{n-1} \end{pmatrix} = \prod_{i>j} g_{p_i} - g_{p_j} ,$$

the right-hand side of which is non-zero by the two-independence of the axial function (6.2).

□

6.3. **Holonomy and polynomial interpolation schemes.** The complete graph is the only example we know of for which the methods of the previous section give *all* the polynomial interpolation schemes. In this section we describe an alternative method which is effective in examples in which one has information about the holonomy group of the graph Γ. To simplify the exposition below we will confine ourselves to the case in which the axial function α is exact.

Let p_0 be a vertex of Γ. The holonomy group, $\mathrm{Hol}(\Gamma_{p_0})$ is by definition a subgroup of the group of permutations of the elements of star(p_0), so if we enumerate its elements in some order

$$e_i^0 \in \text{star}(p_0) \quad i = 1, \dots d$$

we can regard $\mathrm{Hol}(\Gamma_0)$ as a subgroup of the permutation group S_d on $\{1, \dots, d\}$. Let $\mathfrak{q}(z_1, \dots, z_d)$ be a polynomial in d variables with scalar coefficients. We will say that $\mathfrak{q}$ is $\mathrm{Hol}(\Gamma_{p_0})$ *invariant* if for every $\sigma \in \mathrm{Hol}(\Gamma_{p_0})$

$$\mathfrak{q}(z_{\sigma(1)}, \dots, z_{\sigma(n)}) = \mathfrak{q}(z_1, \dots, z_n)\,.$$

Now fix such a $\mathfrak{q}$ and construct a polynomial assignment $< g_p >$ as follows. Given a path, γ in Γ joining p_0 to p the connection gives us a holonomy map

$$\nabla_\gamma : \text{star}(p_0) \to \text{star}(p)$$

mapping $e_1^0, \dots, e_d^0$ to $e_1, \dots, e_d$. Set

$$g_p = \mathfrak{q}(\alpha_{e_1}(x), \dots, \alpha_{e_d}(x))\,. \tag{6.5}$$

The invariance of $\mathfrak{q}$ guarantees that this definition is independent of the choice of γ. Let us show that $< g_p >$ satisfies the interpolation conditions (6.1). Let $e = (p, q)$. The map Γ

$$\nabla_p : \text{star}(p) \to \text{star}(q)$$

maps $e_1, \dots, e_d$ to $e_1', \dots, e_d'$ and by the exactness of α

$$\alpha_{e_i'} \equiv \alpha_{e_i} \mod \alpha_e\,.$$

Hence

$$\mathfrak{q}(\alpha_{e_1'}, \dots, \alpha_{e_d'}) \equiv \mathfrak{q}(\alpha_{e_1}, \dots, \alpha_{e_d}) \mod \alpha_e\,.$$

If the holonomy group is small this construction provides many solutions of (6.1). Even if $\mathrm{Hol}(\Gamma_{p_0})$ is large this method yields some interesting solutions. For instance if $\mathfrak{q}$ is a symmetric polynomial in $z_1, \dots, z_d$, (6.5) is a solution of (6.1).

6.4. **Totally geodesic subgraphs and polynomial interpolation schemes.** A third method for constructing solutions of (6.1) makes use of totally geodesic subgraphs. Whenever $\Gamma_0 = (V_0, E_0)$ is a totally geodesic subgraph of degree j then for every $p \in V_0$, star(p) is a disjoint union of star(p, Γ_0) and its complement, which we can regard as the *tangent* and *normal* spaces to Γ_0 at p. Let

$$g_p = \prod_{e \perp \Gamma_0} \alpha(e)\,, \tag{6.6}$$

a homogeneous polynomial of degree $d - j$. By the results of Section 6.3, the assignment $< g_p >$ sending $p \to g_p$, is an interpolation scheme on V_0, and we can extend this scheme to V by setting

$$g_p = 0 \tag{6.7}$$

for $p \in V - V_0$. Then (6.6) and (6.7) *do* define an interpolation scheme on V. Clarly the interpolation conditions (6.1) are satisfied if $e = (p, q)$ is either an edge of V_0 or if p and q are both in $V - V_0$. If $p \in V_0$ and $q \in V - V_0$ then $\alpha(p, q)$ is one of the factors in the product (6.6); so in this case the interpolation condition (6.1) is also satisfied.

7. Polynomial interpolation schemes and Betti numbers

In many examples, all solutions of the interpolation equations (6.1) can be constructed by the methods discussed in Section 6. However, to check this, one needs an effective way of counting the total number of solutions of these equations. Modulo some hypotheses which we will make explicit below, we will prove the following theorem.

Theorem 7.1. *The dimension of the space of solutions, $H^r(\Gamma, \alpha)$, of the interpolation equations* (6.1) *is equal to*

$$\sum_{\ell=0}^{r} \binom{r-\ell+n-1}{n-1} \beta_\ell. \tag{7.1}$$

We will show in the next section that the resemblance between (7.1) and the McMullen formula (1.1) is not entirely an accident.

Let $\xi \in \mathbb{R}^n$ be a generic vector. Without loss of generality, we may assume that ξ is the vector $(1, 0, \dots, 0)$. Hence,

$$\alpha(e) = m_e \cdot (x_1 - \delta(e)), \tag{7.2}$$

where $m_e \neq 0$ and

$$\delta(e) = d_{e,2}x_2 + \cdots + d_{e,n}x_n. \tag{7.3}$$

Our proof of Theorem 7.1 will make three unnecessarily stringent assumptions about Γ, ∇, and α. That these assumptions *are* unnecessarily stringent is shown in [GZ2]. The proof given in [GZ2] of Theorem 7.1 follows the lines of the proof below, but is much more complicated.

The three assumptions that we will make are the following. The first is that the axial function α is 3-independent. That is, for each $p \in V$, the set of vectors

$$\alpha(\text{star}(v)) \tag{7.4}$$

is 3-independent: any set of three of these vectors is linearly independent. This assumption guarantees that every plane in $\mathbb{R}^n$ will contain the image under α of a disjoint set of closed geodesics. Namely, let W be a two-dimensional subspace of $\mathbb{R}^n$ and let $\Gamma_W = (V, E_W)$ be the subgraph whose oriented edges $e \in E_W$ satisfy $\alpha(e) \in W$. Then 3-independence guarantees that if Γ_W is non-empty, its connected components are either edges or totally geodesic subgraphs of valence two. In particular, if e_1 and e_2 are edges of Γ having a common vertex p, there is a *unique* connected totally geodesic subraph of valence two containing e_1 and e_2. To see this, let $W = \text{span}\{\alpha(e_1), \alpha(e_2)\}$ and let F be the connected component of Γ_W containing p.

To avoid repeating the phrase "connected totally geodesic subgraphs of valence two," we will henceforth refer to such objects as *two-faces*. Our second assumption about Γ, ∇ and α is that for every two-face F, the zeroth Betti number $\beta_0(F)$ is one. In other words, if a totally geodesic subgraph of valence two is connected in the graph theoretical sense, it is connected in the homological sense as well.

Finally we will asume that there exists a ξ-compatible Morse function,

$$f : V \to \mathbb{R}, \tag{7.5}$$

as defined in Section 4. By perturbing f slightly, we may assume that the map (7.5) is injective.

Definition 7.2. *The* critical values *of f are the numbers $f(p) \in \mathbb{R}$ for $p \in V$. The* regular values *of f are the numbers which are* not *critical values.*

Our goal for the remainder of this section is to prove the following theorem.

Theorem 7.3. *Suppose Γ is a graph equipped with a connection ∇, a 3-independent axial function $\alpha : E \to \mathbb{R}^n$, and a ξ-compatible Morse function f. Suppose further that each two-face of Γ has zeroth Betti number equal to 1. Then the dimension of $H^r(\Gamma, \alpha)$ is given by the formula* (7.1)*.*

7.1. **The cross sections of a graph.** Let $c \in \mathbb{R}$ be a regular value of f, and let V_c be the set of oriented edges e of Γ having $f(\iota(e)) < c < f(\tau(e))$. Intuitively, these are the edges of Γ which intersect the "hyperplane" $f = c$. The main goal of this section is the construction of the *cross section of Γ at c* which will be a graph Γ_c having V_c as its vertex set. This construction will make strong use of the second of our three hypotheses. This hypothesis implies the following.

Lemma 7.4. *For every two-face, F, the restriction of f to V_F has a unique maximum, p, and a unique minimum, q. Moreover, if $f(p) > c$ and $f(q) < c$, there are exactly two edges of F contained in V_c.*

We will now decree that a pair $(e_1, e_2) \in V_c \times V_c$ is in E_c if and only if they are contained in a common two-face. The following is an immediate corollary of the lemma.

Theorem 7.5. *The pair $\Gamma_c = (V_c, E_c)$ is a $(d-1)$-regular graph.*

Remark 7.6. *One can intuitively think of the edges of Γ_c as being the intersections of two-faces with the "hyperplane" $f = c$. In this way, each vertex of Γ_c corresponds to an edge of Γ, and each edge of Γ_c corresponds to a two-face of Γ.*

We will now show how to equip Γ_c with a connection and an axial function. Let F be a two-face, p a vertex of F and e_1 and e_2 the two edges of F meeting in p. We define two natural connections on Γ_c.

The first connection is the *up connection* ∇_{up} on Γ_c. Let $\tau(e_1) = q_1$ and $\tau(e_2) = q_2$. Suppose $e \neq e_1$ is an edge of Γ with $\iota(e) = \tau(e_1)$. If q is the maximum point of f on F, then there is a unique geodesic path on F joining q_1 to q_2 passing through q. By applying ∇ to e along this path, we can associate to e an edge e' of Γ with $\tau(e') = q_2$. Let E be the unique two-face containing e_1 and e, and E' the unique two-face containing e_2 and e'. Then the correspondence ∇_{up} mapping $E \mapsto E'$ defines a connection on Γ_c.

We next define the *down connection* ∇_{down} on Γ_c. Let $\iota(e_1) = p_1$ and $\iota(e_2) = p_2$. Suppose $e \neq e_1$ is an edge of Γ with $\iota(e) = \iota(e_1)$. If p is the minimum point of f on F, then there is a unique geodesic path on F joining p_1 to p_2 passing through p. By applying ∇ to e along this path, we can associate to e an edge e' of Γ with

$\iota(e') = p_2$. Let E be the unique two-face containing e_1 and e, and E' the unique two-face containing e_2 and e'. Then the correspondence ∇_{down} mapping $E \mapsto E'$ defines a connection on Γ_c.

Now we will give a natural axial function on Γ_c. Let W be the two-dimensional space $\operatorname{span}\{\alpha(e_1), \alpha(e_2)\}$.

Lemma 7.7. *For every edge e of F, $\alpha(e) \in W$.*

Proof. Since F is totally geodesic, it is a connected component of Γ_W. Furthermore, since F has valence 2, it must, in fact, be a geodesic, and so its image with respect to α must lie in a plane. This completes the proof. □

We will identify $\mathbb{R}^{n-1}$ with the orthogonal complement of $\xi = (1, 0, \dots, 0)$ in $\mathbb{R}^n$ and let δ_F be a fixed basis vector of the one-dimensional space $W \cap \mathbb{R}^{n-1}$. For every edge e of Γ, let $\delta(e)$ be the vector in $\mathbb{R}^{n-1}$ defined by (7.2) and (7.3).

Lemma 7.8. *For every pair of edges e and e' of F, $\delta(e) - \delta(e')$ is a multiple of δ_F.*

Proof. By Lemma 7.7, $\alpha(e)$ and $\alpha(e')$ are in W, so

$$\delta(e') - \delta(e) = m_{e'}^{-1} \cdot \alpha(e') - m_e^{-1} \cdot \alpha(e) \tag{7.6}$$

is in $W \cap \mathbb{R}^{n-1}$. □

Theorem 7.9. *Let e_1 and e_2 be adjacent elements of V_c and let F be the unique two-face with e_1 and e_2 as edges. The assignment $(e_1, e_2) \mapsto \delta((e_1, e_2)) = \delta_F$ is an axial function on Γ_c taking its values in $\mathbb{R}^{n-1}$. This is an axial function compatible with ∇_{up}* and *with ∇_{down}.*

Proof. We will prove that this is an axial function compatible with ∇_{up}. The proof that it is compatible with ∇_{down} is nearly identical.

Let $\tau(e_1) = q_1$ and $\tau(e_2) = q_2$ and let q be the unique maximum point of f on F. Suppose $e \neq e_1$ is an edge of Γ with $\iota(e) = \tau(e_1)$. In this case, we apply ∇ to e along the unique path from q_1 to q_2 through q, and associate to e an edge e' of Γ with $\tau(e') = q_2$. If E is the unique two-face containing e_1 and e, and E' is the unique two-face containing e_2 and e', then ∇_{up} maps E to E'. Thus, (E, F, E') is an edge chain under ∇_{up} in Γ_c. The three dimensional subspace of $\mathbb{R}^n$ spanned by $\alpha(e_1)$, $\alpha(e_2)$ and $\alpha(e)$ is the same as the three dimensional subspace of $\mathbb{R}^n$ spanned by $\alpha(e_1)$, $\alpha(e_2)$ and $\alpha(e')$, since e was obtained from e' by the original connection ∇. Hence the intersections of these three-dimensional subspaces with $\mathbb{R}^{n-1}$ are the same two-dimensional subspaces. Therefore, the image of (E, F, E') under δ lies in a plane, and thus δ is an axial function compatible with ∇_{up}. □

In Section 7.6, we will need the following lemma.

Lemma 7.10. *The axial function δ is 2-independent.*

Proof. Let e be in V_c and let e_1 and e_2 be edges of Γ with $p = \tau(e) = \iota(e_i)$. Let F_i be the two-face with edges e and e_i, for $i = 1, 2$. Then up to scalar multiple,

$$\beta_{F_i} = \beta(e_i) - \beta(e)$$

by Lemma 7.8, and by 3-independence, $\alpha(e)$, $\alpha(e_1)$, and $\alpha(e_2)$ are linearly independent. Hence, by (7.2) and by (7.3), β_{F_1} and β_{F_2} are linearly independent. □

7.2. **A Morse lemma for cross sections.** A classical theorem in Morse theory describes how the level sets of a Morse function change as one passes through a critical point. The goal of this section is to prove a combinatorial analogue of this theorem.

Theorem 7.11. *Let $p \in V$ be a vertex of index k and let $c = f(p)$ and $c^{\pm} = c \pm \varepsilon$. Then for small ε, Γ_{c^+} can be obtained from Γ_{c^-} by deleting a complete totally geodesic subgraph isomorphic to K_k and inserting in its place a complete totally geodesic subgraph isomorphic to K_{d-k}.*

Remark 7.12. *In the previous section, we defined* two *canonical connections on a cross section, the up connection and the down connection. The subgraph that we delete from Γ_{c^-} will be totally geodesic with respect to the up connection, and the subgraph of Γ_{c^+} that we insert in its place will be totally geodesic with respect to the down connection.*

Proof. We will orient the edges of Γ by assigning to each edge the orientation for which $\alpha(e) \cdot \xi > 0$. Thus, because f is compatible with our choice of ξ,

$$f(\tau(e)) > f(\iota(e)).$$

Let $e_1, \dots, e_k$ be the oriented edges with $\tau(e_r) = p$ and $e'_1, \dots, e'_\ell$ be the oriented edges with $\iota(e'_s) = p$. Here, $\ell = d - k$, and k is by definition the idex of p. Then, if ε is sufficiently small, $f(\iota(e_r)) < c^-$ and $f(\tau(e'_s)) > c^+$. See the figure below. Thus, the edges $e_1, \dots, e_k$ are vertices of Γ_{c^-} and the edges $e'_1, \dots, e'_\ell$ are vertices

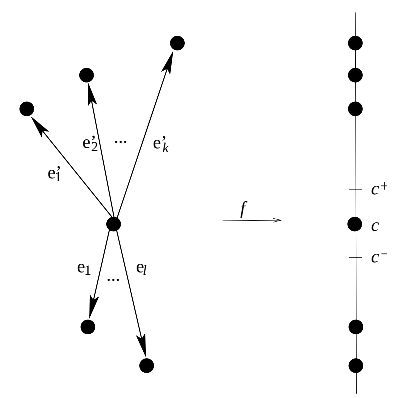


FIGURE 7. This shows the edges appearing as vertices in Γ_{c^-} and in Γ_{c^+}.

of Γ_{c^+}. By 3-independence, there exists a unique two-face $F_{i,j}$ having e_i and e_j as edges, so the e_i regarded as vertices of Γ_{c^-} form a subgraph Δ_- of Γ_{c^-} isomorphic to K_k. Moreover, the "up" geodesic path in $F_{i,j}$ joining $\tau(e_i) = p$ to $\tau(e_j) = p$ consists of the point p itself. So, ∇ on Γ simply maps e_m, $m \neq i, j$, along the path p to itself, and thus ∇_{up} maps the two-face $F_{i,m}$ to the two-face $F_{j,m}$. Thus, Δ_- is a totally geodesic subgraph of Γ_{c^-} with respect to the up connection. Note that this connection on K_k agrees with the usual connection on K_k, as described in Section 9.

Similarly, the edges, $e'_1, \dots, e\ell'$ are the vertices of a subgraph $\Delta_+ \cong K_\ell$ of Γ_{c^+} which is totally geodesic with respect to the down connection.

Now let e be an edge of Γ not having p either as an initial or terminal vertex and satisfying

$$f(\iota(e)) < c < f(\tau(e)).$$

If we choose ε small enough, all of these edges e belong to both V_{c^-} and V_{c^+}. Thus, there is a bijection

$$V_{c^-} - \{e_i \mid i = 1, \dots, k\} \to V_{c^+} - \{e'_i \mid i = 1, \dots, \ell\}.$$

In other words, Γ_{c^+} is obtained from Γ_{c^-} by deleting Δ_- and inserting Δ_+. □

7.3. **A Morse lemma for $H^r(\Gamma_c)$.** We will compute in this section the change in dimension of $H^r(\Gamma_c)$ as c passes through a critical value of indes k. As in Subsection 7.2, let $c = f(p)$ and let $\Gamma_{c^\pm} = \Gamma_\pm$ be the cross-sections of Γ just above and just below $f = c$. Let $V_\pm$ be the vertices of $\Gamma_\pm$ and $V^c_\pm$ the vertices of the subgraphs $\Delta_\pm$ of $\Gamma_\pm$. We will prove the following change of dimension formula.

Lemma 7.13.

$$\dim H^r(\Gamma_+) - \dim H^r(\Delta_+) = \dim H^r(\Gamma_-) - \dim H^r(\Delta_-). \tag{7.7}$$

Proof. An element of $H^r(\Gamma_+)$ is a function which assigns to each vertex e of Γ_+ a homogeneous polynomial in $x_2, \dots, x_n$ of degree r and for each edge of Γ_+ satisfies the Γ_+ analogue of the interpolation conditions (6.1). Therefore, since Δ_+ is a totally geodesic subgraph of Γ_+, the restriction of this function to V^c_+ is an element of $H^r(\Delta_+)$, so there is a natural mapping

$$H^r(\Gamma_+) \to H^r(\Delta_+). \tag{7.8}$$

We claim that this mapping is surjective. Indeed by Theorem 6.2 and the formula (7.6), every element of $H(\Delta_+)$ is the restriction to V^c_+ of an element of $H(\Gamma_+)$ of the form $\mathfrak{p}(\delta)$ where δ is given by (7.3) and

$$\mathfrak{p} = \mathfrak{p}(z) = \sum_{i=1}^{\ell-1} \mathfrak{p}_i z^i$$

is a polynomial in z of degree $\ell - 1$, where $\ell = d - k$, having as coefficients polynomials $\mathfrak{p}_i$ in $x_2, \dots, x_n$.

The analogous mapping

$$H^r(\Gamma_-) \to H^r(\Delta_-). \tag{7.9}$$

is also surjective, so if we denote the kernels of (7.8) and (7.9) by $H^r(\Gamma_+, \Delta_+)$ and $H^r(\Gamma_-, \Delta_-)$ respectively, the proof of (7.7) reduces to showing that

$$\dim H^r(\Gamma_+, \Delta_+) = \dim H^r(\Gamma_-, \Delta_-). \tag{7.10}$$

By Theorem 7.11,

$$V_+ - V^c_+ = V_- - V^c_-,$$

so if

$$f^-_p,\ p \in V_- \tag{7.11}$$

is an interpolation scheme that vanishes on V^c_- we can associate with it a function

$$f^+_p,\ p \in V_+ \tag{7.12}$$

by setting

$$f^+_p = f^-_p \tag{7.13}$$

for $p \in V_= - V_+^c$ and

$$f_p^+ = 0 \tag{7.14}$$

for $p \in V_+^c$. Let us show that (7.12) is an interpolation scheme for the graph Γ_+. To prove this, we must show that

$$f_{e_1}^+ = f_{e_2}^+ \mod \beta_F$$

for every edge $F = (e_1, e_2)$ of Γ_+. If e_1 and e_2 are in $V_+ - V_+^c$, this follows from (7.13). If e_1 and $e + 2$ are in V_+^c, then if follows from (7.14). So the only case we have to consider is the case where $e_1 \in V_+^c$ and $e_2 \in V_+ - V_+^c$. For the moment, let us regard e_1 and e_2 as edges of Γ and F as a 2-face of Γ containing these edges. The critical point p on the level set $f = c$ is a vertex of F since p is the initial vertex of e_1 and the two edges of F meeting in p must be e_1 and one of the e_j's in V_-^c, since if it were in V_+^c, e_2 would have to be that e_j. Now note that

$$f_{e_2}^+ = f_{e_2}^-$$

by (7.13) and

$$f_{e_1}^+ = f_{e_j}^- = 0$$

by (7.14) and by the fact that f^- is zero on V_-^c. Hence,

$$f_{e_2}^+ - f_{e_1}^+ = f_{e_2}^- - f_{e_j}^- = 0 \mod \delta_F,$$

since e_2 and e_j lie on the common edge F of Γ_-. Thus, the natural map (7.8) is indeed a surjection.

The dimensions of $H^r(\Delta_+)$ and $H^r(\Delta_-)$ can be computed directly from Theorem 7.11. Namely, by Theorem 7.11, the dimension of $H^r(\Delta_+)$ is the dimension of the space of homogeneous polynomials of degree r in $z, x_2, \ldots, x_n$ of the form

$$\sum_{i=0}^{\ell-1} \mathfrak{p}_i(x_2, \ldots, x_n) z^i, \ \ell = d - k,$$

so if we let $d_r(n)$ be the dimension of the space of homogeneous polynomials in $x_1, \ldots, x_n$ of degree r, we get the formula

$$\dim H^r(\Delta_+) = \sum_{i=1}^{\ell-1} d_{r-i}(n-1). \tag{7.15}$$

Similarly, we get the formula

$$\dim H^r(\Delta_-) = \sum_{i=1}^{k-1} d_{r-i}(n-1). \tag{7.16}$$

Noting that

$$d_s(n) = \sum_{i=0}^{s} d_i(n-1),$$

we can rewrite (7.15) and (7.16) as

$$\dim H^r(\Delta_+) = d_r(n) - d_{r-\ell}(n) \tag{7.17}$$

and

$$\dim H^r(\Delta_-) = d_r(n) - d_{r-k}(n). \tag{7.18}$$

Hence, from (7.7), we get the identity

$$\dim H^r(\Gamma_+) - \dim H^r(\Gamma_-) = d_{r-k}(n) - d_{r-\ell}(n). \tag{7.19}$$

□

7.4. **The proof of Theorem 7.3.** We are now ready to prove the main theorem of this section.

Proof of Theorem 7.3. Regard the unit interval $I = [0, 1]$ as the graph consisting of a single edge. This graph has a unique connection. Furthermore, equip it with the axial function $\alpha(0) = x$ and $\alpha(1) = -x$, where x is the unit vector 1 in $\mathbb{R}$. Now let Γ be a graph with a connection and axial function, and let $\tilde{\Gamma}$ be the product graph $\Gamma \times I$ with its product axial function $\tilde{\alpha} : \tilde{\Gamma} \to \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^n$, as defined in Section 3. Let $f : V_\Gamma \to \mathbb{R}$ be our given Morse function, and extend f to a Morse function $\tilde{f} : V_{\tilde{\Gamma}} \to \mathbb{R}$ by setting

$$\tilde{f}(p, o) = f(p) \tag{7.20}$$

and

$$\tilde{f}(p, o) = f(p) + C, \tag{7.21}$$

where C is larger than the maximum value of f. Notice that when

$$\max(f) < c_0 < C - min(f),$$

the cross section $\tilde{\Gamma}_{c_0}$ is just the graph Γ itself. Moreover, it is easy to see that the "up" connection on $\tilde{\Gamma}_{c_0}$ coincides with the original connection on Γ and the axial function on $\tilde{\Gamma}_{c_0}$ with the original axial function. Now let's count $\dim \Pi(\tilde{\Gamma}_{c_0})$ using Theorem 7.11. The critical points of $\tilde{f}$ with critical value less than c_0 are just the points $(p, 0)$, with $p \in V_\Gamma$ and the index of each of these points is simply the index of p. Therefore, since $\tilde{\Gamma}$ is a $(d+1)$-valent graph, the dimension of $H^r(\tilde{\Gamma}_c)$ changes by

$$d_{r-k}(n+1) - d_{r-(d+1-k)}(n+1) \tag{7.22}$$

every time one passes through a critical point of index k. Thus the total change in dimension as one goes from $c < \min \tilde{f}$ to c_0 is

$$\sum_{k=0}^{d} d_{r-k}(n+1)\beta_k - \sum_{k=1}^{d+1} d_{r-k}(n+1)\beta_{d+1-k} \tag{7.23}$$

However, by Poincaré duality, $\beta_{d+1-k} = \beta_{k-1}$, so we can rewrite the second sum in (7.23) as

$$\sum_{k=1}^{d+1} d_{r-k}(n+1)\beta_{k-1}$$

or as

$$\sum_{k=0}^{d} d_{r-(k-1)}(n+1)\beta_k.$$

But by (7.16),

$$d_{r-k}(n+1) - d_{r-(k-1)}(n+1) = d_{r-k}(n)$$

so the combined sum (7.23) is just

$$\sum_{k=0}^{d} d_{r-k}(n)\beta_k.$$

This completes the proof of the main theorem. □

7.5. **Parallel redrawings.** For $r = 1$, the formula (7.1) tells us that β_0 is equal to the number of connected components of Γ, a fact which is not completely without interest. Much more interesting, however, is the case $r = 1$.

Definition 7.14. *A* parallel redrawing of Γ *is a map* $\pi : V \to \mathbb{R}^n$ *such that*

$$\pi(p) - \pi(q) = \lambda\alpha(p,q), \text{ for some } \lambda \in \mathbb{R}, \tag{7.24}$$

for every edge $(p,q) \in E$.

If Γ is embedded graph in $\mathbb{R}^n$, the identity (7.24) asserts that the deformation $\Gamma \mapsto \Gamma_\varepsilon$ defined by replacing each vertex p by $p + \varepsilon\pi(p)$ leaves the edges of the deformed graph parallel to the edges of the original.

Every exact immersion is a parallel redrawing, but there are others. In particular, the Euclidean translations are parallel redrawings. So is dilation.

The set $\Pi(\Gamma)$ of all parallel redrawings is clearly a vector space, and the "number of parallel redrawings" is its dimension.

Since the condition that $\pi(p) - \pi(q)$ be parallel to $\alpha(p,q)$ is identical to the interpolation condition

$$\pi(p) \equiv \pi(q) \mod \alpha(p,q).$$

the dimension of $\Pi(\Gamma)$ is the dimension of $H^1(\Gamma,\alpha)$. For the graphs we considered in Theorem 7.3, this dimension is just

$$n\beta_0 + \beta_1. \tag{7.25}$$

We write the count in this form even though we assumed in our proof that $\beta_0 = 1$ since the count in that more general form is still sometimes correct.

We can think of the first term $n\beta_0 = n$ as counting the n translations. The dilation must be a linear combination of the other β_1 parallel redrawings.

We observed earlier that the betti numbers are preserved under projection. So is the number of nontrivial parallel redrawings. Only the number of translations changes, and the first term accounts for that exactly.

Suppose Γ is the one-skeleton of a simple polytope P. Then every facet (face of codimension 1) determines a parallel redrawing: just move the hyperplane containing that facet parallel to itself, as in Figure 8.

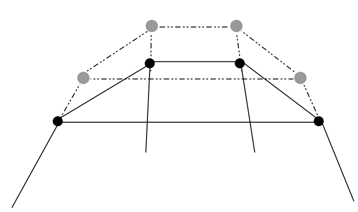


FIGURE 8. This shows a parallel redrawing by moving a facet parallel to itself.

Formula 1.1, proved in Section 5, counts the faces of P. When $k = 1$ it tells us that P has $n\beta_0+\beta_1$ facets, so our construction has found all the parallel redrawings.

7.6. **Morse inequalities.** The identites (7.1) imply trivially that

$$\dim H^r(\Gamma, \alpha) \leq \sum_{\ell=0}^{r} \dim S^{r-\ell}\beta_\ell. \tag{7.26}$$

In the next section, we will study an analogue of $H^r(\Gamma, \alpha)$ for which these inequalities hold, but in general the equalities do not. We will give a new proof of these inequalities here. This proof is much simpler than the proof of (7.1), and in fact (7.1) is true under somewhat weaker hypotheses.

Theorem 7.15. *Suppose that* (Γ, α) *is a graph with a two-independent axial function, and that* Γ *admits a* ξ*-compatibile Morse function,* $f : V \to \mathbb{R}$*. Then the inequalities* (7.26) *hold.*

Proof. Let $H^r(\Gamma, c)$ be the set of all interpolation schemes $< g_p >$ of degree r for which $g_p = 0$ whenever $f(p) < c$. As in Section 7.2, let $c = f(p)$ be a critical value of index k and let $c_\pm = c \pm \varepsilon$ for ε small. We claim that there is a short exact sequence

$$0 \to H^r(\Gamma, c_+) \to H^r(\Gamma, c_-) \to S^{r-k}\alpha_{e_1}\cdots\alpha_{e_k} \tag{7.27}$$

where e_i, $i = 1, \ldots, k$, are the k edges in star(p) with $\alpha(e_i) \cdot \xi < 0$. It is clear that $H^r(\Gamma, c_+)$ is the kernel of the map

$$< g >\in H^r(\Gamma, c_-) \mapsto g_p \in S^r.$$

But if g is in $H^r(\Gamma, c_-)$ then for every down-pointing edges e_i with terminal vertex q_i, $f(q_i) < c_-$, so $g_{q_i} = 0$. Hence, the interpolation conditions

$$g_p \equiv g_{q_i} \mod \alpha(e_i)$$

imply that g_p is divisible by $\alpha(e_i)$. Moreover, since the $\alpha(e_i)$'s are two-independent, g_p is divisible by $\alpha(e_1) \cdot \cdots \cdot \alpha(e_k)$, completing the proof that the sequences (7.27) is exact.

This short exact sequence implies

$$\dim H^r(\Gamma, c_-) - \dim H^r(\Gamma, c_+) \leq \dim S^{r-k}\ .$$

If $c_0 < \min(f) < \max(f) < c_1$, summing these inequalities yields the desired

$$\dim H^r(\Gamma, \alpha) = \dim H^r(\Gamma, c_0) - \dim H^r(\Gamma, c_1) \leq \sum \dim(S^r - k)\beta_k\ .$$

□

8. Interpolation schemes involving polynomials in "anti-commuting" variables

The interpolation schemes that we considered in Sections 6 and 7 involved polynomials, $f \in S^r$ of the form

$$f = \sum_{i_1+\cdots+i_n=r} a_{i_1\ldots i_n} x_1^{i_1} \ldots x_n^{i_n}$$

in "commuting" variables: i.e., $x_i x_j = x_j x_i$. One can also, however, consider interpolation schemes involving polynomials in "anti-commuting" variables

$$f = \sum_{i_1+\cdots+i_n=r} a_{i_1}\dots a_{i_n} x_{i_1}\dots x_{i_n}$$

where $x_i x_j = -x_j x_i$; in other words interpolation schemes, $< f_p >$, for $p \in V$ in which f_p sits in the r^{th} exterior power, $\Lambda^r(\mathbb{R}^n)$, of the vector space, $\mathbb{R}^n$. The interpolation conditions

$$f_p \equiv f_q \mod \alpha_e \tag{8.1}$$

still make sense in this anti-commuting context if we interpret this equation as saying that $f_p - f_q \in \alpha_e \wedge \Lambda^{r-1}$.

Let us denote by $\tilde{H}^r(\Gamma, \alpha)$ the space of all r^{th} degree solutions of the equations (8.1). The sum

$$\tilde{H}(\Gamma, \alpha) = \oplus_{r=0}^n \tilde{H}^r(\Gamma, \alpha)$$

is, like its "bosonic" counterpart, $H(\Gamma, \alpha)$, a graded ring; and, in particular, a graded module over the exterior algebra $\Lambda(\mathbb{R}^n)$. We claim that the following analogue of Theorem 7.15 is true.

Theorem 8.1. *Suppose that α is r-independent and that there exists an injective ξ-compatible Morse function, $f : V \to \mathbb{R}$. Then*

$$\dim \tilde{H}^r(\Gamma, \alpha) \leq \sum_{k=0}^{r} \binom{n-k}{n-r} b_k \,. \tag{8.2}$$

Proof. The "fermionic" analogue of (7.27) asserts that there is an exact sequence

$$0 \to \tilde{H}^r(\Gamma, C_+) \to \tilde{H}^r(\Gamma, C_-) \to \Lambda^{r-k}\alpha_{e_1} \wedge \dots \wedge \alpha_{e_k} \,.$$

Hence

$$\dim \tilde{H}^r(\Gamma, C_-) - \dim \tilde{H}^r(\Gamma, C_+) \leq \binom{n-k}{n-r}$$

and by adding up these identities as in Section 7.6 one gets (8.2). □

Are these Morse inequalities ever equalities? We will show that they are if Γ is the one-skeleton of a simple convex n-dimensional polytope, Δ; and, in fact, we will show in this case that these identities are identical with the McMullen identities (1.1). The idea of our proof will be to show that for every $n-r$ dimensional face, F, one can associate an r^{th} order solution of the interpolation identities (6.1) by mimicking the constructing in Section 6.4, and by showing that the solutions constructed this way form a basis of $\tilde{H}^r(\Gamma, \alpha)$. Fix a set of vectors, $v_1, \dots, v_r \in \mathbb{R}^n$ such that the v_r 's are normal to the face F and are linearly independent; and for every vertex, p, of Γ define

$$f_p^F = 0 \tag{8.3}$$

if p is not a vertex of F and

$$f_p^F = c_p \alpha_{e_1} \wedge \dots \wedge \alpha_{e_r} \tag{8.4}$$

if p is a vertex of F, the e_i's being, as in Section 6.4, the edges of Γ normal to F at p and c_p being defined by the normalization condition

$$c_p \det(\alpha_{e_i}(v_j)) = 1 \,. \tag{8.5}$$

It is easy to check that (8.3)–(8.5) is a solution of the interpolation equations (8.2). We will prove the following theorem.

Theorem 8.2. *Let $F_i, i = 1, \dots, N$ be the $(n-r)$-dimensional faces of Λ. Then the interpolation schemes, $f^{F_i}, i = 1, \dots, N$ are a basis of $\tilde{H}^r(\Gamma, \alpha)$.*

Proof. By McMullen's identity the right-hand side of (8.2) is equal to N so it suffices to prove that the f^{F_i}'s are linearly independent; and hence it suffices to prove that at each vertex, p, the vectors

$$f^{F_i}(p)\,, \quad p \text{ a vertex of } F_i$$

are linearly independent. However it is clear that these vectors are in fact a basis of $\Lambda^r(\mathbb{R}^n)$. □

9. Examples

In this final section we review the examples we have been following through the text and introduce some new ones that suggest new directions to explore. When proofs are short we include them. Some will be found in [H]. Others we leave as exercises.

9.1. **The complete graph K_n.** Our standard view K_n embeds with vertices the standard basis vectors in $\mathbb{R}^n$. That embedding is a regular simplex in the $n-1$-dimensional subspace $\Sigma x_i = 1$. The exact axial function is determined by assigning to each vertex the difference between its end points. The following figure shows a part of the connection determined by that axial function for K_4: it moves edges across the triangular faces.

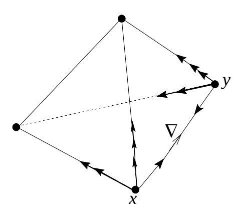


FIGURE 9. This shows the connection we defined above on the graph K_4.

When we think of K_4 just as an abstract 4-regular graph we find that it has 10 different connections (up to graph automorphism). But in each of these connections other than the standard one there is at least one geodesic of length at least 4, so none of those connections has a 3-independent immersion. So we will study only the standard view.

Proposition 9.1. *The geodesics of K_n are the triangles. The connected totally geodesic subgraphs are the complete subgraphs.*

Proof. It's clear that the geodesics are the triangles. Let Γ_0 be a connected totally geodesic subgraph and p and q two vertices of Γ_0. Then transporting edge $e = (p, q)$ along a path in Γ_0 from p to q we eventually reach a triangle containing q. At that point the image of e transports to an edge of Γ_0 so e must have been part of Γ_0 to begin with. □

It's easy to compute the holonomy of K_n.

Proposition 9.2. $Hol(K_n) \cong S_{n-1}$.

Proof. If you follow the connection along triangle (p, q, r) from p back to itself you interchange (p, q) and (p, r). Thus the holonomy group acting on star(p) contains all the transpositions. □

Proposition 9.3. *The Betti numbers of K_n are invariant of choice of direction ξ and are* $(1, 1, \ldots, 1)$.

Proof. The geodesics are triangles, hence convex. hence inflection free, so the Betti numbers are well defined. Let $\xi = (1, 2, \ldots, n)$. Then the number of down edges at the vertex corresponding to the i^{th} coordinate vector is the number of j's less than i. □

9.2. **The Johnson graph** $J(n, k)$**.** Recall that the Johnson graph $J(n, k)$ is the graph with vertices corresponding to k-element subsets of $\{1, 2, \ldots, n\}$; two vertices $S, T \in V$ are adjacent if $\#(S \cap T) = k - 1$. Then we can think of an oriented edge as an ordered pair (i, j): to get from S to T we remove i and add j. We naturally embed $J(n, k)$ in $\mathbb{R}^n$ by representing each vertex as a vector with k 1's and $n - k$ 0's. That embedding is a $k \times (n - k)$-regular polytope in the $n - 1$-dimensional subspace $\Sigma x_i = k$. The exact axial function is determined by assigning to each vertex the difference between its end points.

The easiest way to describe the natural connection is to describe its geodesics. They are the triangles $Q \cup \{a\}$, $Q \cup \{b\}$, $Q \cup \{c\}$ for $k - 1$ element sets Q and distinct a, b, c and the planar squares $Q \cup \{a, b\}$, $Q \cup \{b, c\}$, $Q \cup \{c, d\}$, $Q \cup \{d, a\}$, for $k - 2$ element sets Q and distinct a, b, c, d.

The triangles are actual faces of the polytope. The squares are more like equators, as in the picture of the octahedron $J(4, 2)$ below.

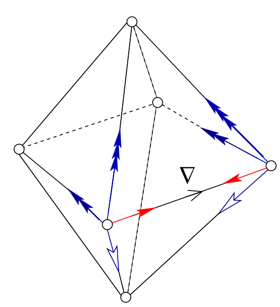


FIGURE 10. This shows the connection we defined on the Johnson graph $J(4, 2)$.

Knowing this connection it is not hard to find all the totally geodesic subgraphs of $J(n, k)$.

Proposition 9.4. *If Γ_0 is a totally geodesic subgraph of $\Gamma = J(n, k)$, then*

$$\Gamma_0 \cong J(A_1, \ell_1) \times \cdots \times J(A_r, \ell_r),$$

where the A_i are subsets of $\{1, \ldots, n\}$ of size $a_i \geq \ell_i$, and $\{1, \ldots, n\}$ is the disjoint union of the A_i.

We can also compute the holonomy of $J(n, k)$:

Proposition 9.5. $Hol(J(n, k)) \cong S_k \times S_{n-k}$.

The proof is similar to that for the complete graph.

Since the geodesics are convex the Betti numbers are well defined. For the octahedron they are $(1, 1, 2, 1, 1)$. Since $n\beta_0 + \beta_1 = 3 \times 1 + 1 = 4$ the octahedron has only trivial parallel redrawings.

We leave the computation of the Betti numbers of the general Johnson graph as an exercise for the reader.

9.3. **The permutahedron S_n.** The permutahedron is the Cayley graph of the symmetric group S_n, generated by reflections. Its vertices correspond to permutations of $\{1, \ldots, n\}$. Two vertices (permutations) are adjacent if they differ by left multiplication by a transposition (i, j): the vertices joined to $\sigma \in S_n$ are the permutations $\tau \cdot \sigma$ for all transpositions τ. This construction defines a natural labeling of the edges in star(σ) by the transpositions. In turn that defines a natural connection: map an edge in the star of a vertex to the edge labelled by the same transposition in the star of an adjacent vertex.

To embed S_n we think of its vertices as permutations of the entries of the vector $(1, \ldots, n)$. The convex hull of that embedding is a simple polytope in the $n-1$-dimensional subspace $\Sigma x_i = n(n+1)/2$, but the convex hull is only a small part of the story. The graph is $\binom{n}{2}$-regular. Most of the embedded edges are internal to the polytope. When so embedded the natural axial function is exact, with inflection free geodesics.

S_3 is a regular hexagon together with its diagonals. Figure 11 shows S_3 and one of its three geodesics. Since each of the geodesics is inflection free (although not convex!) the Betti numbers are well defined. They are $(1, 2, 2, 1)$. In the next section we will discuss the parallel redrawings of S_3.

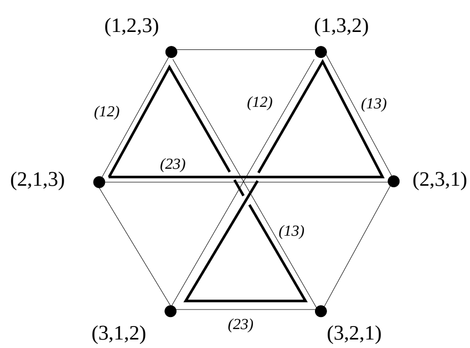


FIGURE 11. An immersion of S_3, with one of the three geodesics in bold. The edges are labelled by transpositions.

The permutahedron S_4 is the truncated octahedron in R^3 together with the necessary internal edges. It has hexagonal and square faces (internal and external) corresponding to natural subgroups $S_3 \times S_1$ and $S_2 \times S_2$ of S_4. These are totally geodesic subgraphs. The reader can decide whether the analagous construction produces all totally geodesic subgraphs. S_4 has Betti numbers $(1, 3, 5, 6, 5, 3, 1)$. In general

Proposition 9.6. *The generating polynomial $B_n(z)$ for the Betti numbers of S_n is*

$$B_n(z) = (1 + z + \cdots + z^{n-1})B_{n-1}(z)$$

$$= \prod_{k=0}^{n-1}(1 + z + \cdots + z^k)$$

Proof. If we compute the Betti numbers for S_n using the generic direction $\xi = (1, 2, \ldots, n)$ then the number of down edges at σ is the number of inversions in σ. The generating function in the proposition is the one that counts permutations according to the number of inversions. [Sl] □

The holonomy of S_n is trivial:

Proposition 9.7. $Hol(S_n) \cong 0$.

Proof. The connection just matches edges labelled by the same transposition, so following a chain from a vertex back to itself permutes nothing in the star of that vertex. □

The permutahedra are examples from the class of Cayley graphs which have a GKM graph structure which is compatible with their structure as a Cayley graph. S_n corresponds to the full flag manifold of all subspaces of CP^n. Cayley graphs are discussed more thoroghly in [H].

9.4. **The complete bipartite graph $K_{n,n}$.** Let $\mathcal{D} = \mathcal{D}_n$ be the group of symmetries of the regular n-gon: the dihedral group with $2n$ elements. Then $\mathcal{D}_n$ is a reflection group of type $I_2(n)$, following the notational conventions of Humphreys [Hu]. It is generated by two reflections, and contains n reflections and n rotations. If we let Δ be the set of reflections in $\mathcal{D}_n$, then the Cayley graph $\Gamma = (\mathcal{D}_n, \Delta)$ has vertices corresponding to elements of $\mathcal{D}_n$. $\sigma \in \mathcal{D}_n$ is connected to $\tau\sigma$ for every $\tau \in \Delta$. Just half the vertices of Γ correspond to symmetries that preserve the orientation of the n-gon, and σ preserves orientation if and only if $\tau\sigma$ reverses it. Thus the graph is bipartite. The only n-regular bipartite graph on $2n$ vertices is $K_{n,n}$.

D_n has a natural holonomy free connection defined just as for the permutahedron, using the reflection generating one vertex from another as the label for the corresponding edge. The natural embedding of D_n as the vertices of a regular $2n$-gon produces an exact axial function with inflection free geodesics for that connection.

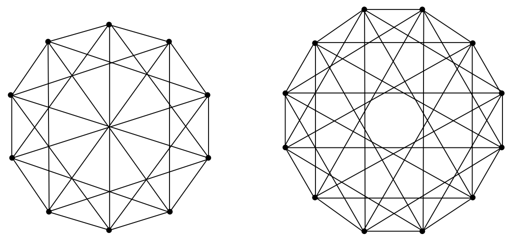
$\mathcal{D}_3$ is $K_{3,3}$ and also the permutahedron S_3 discussed above. The figure below shows two more examples.

This class of graphs is particularly interesting because $\mathcal{D}_n = K_{n,n}$ is the graph associated with a manifold in the sense described in Section 1.1 only when $n = 1, 2, 3, 4, 6$, so they provide examples where combinatorics may go further than differential geometry.

We will leave as an exercise the following Betti number count.

Proposition 9.8. *The Betti numbers of $K_{n,n}$ are invariant of choice of direction ξ and are* $(1, 2, \ldots, 2, 1)$.

Note that $\mathcal{D}_n$ is far from 3-independent. Nevertheless $n\beta_0 + \beta_1 = 2 \times 2 + 2 = 4$ counts the number of parallel redrawings. There are the three trivial ones (two



(a) (b)

FIGURE 12. This shows the Cayley graphs for (a) $\mathcal{D}_5$ and (b) $\mathcal{D}_6$.

translations and the dilation) and one significant one: rotate the sense preserving symmetries clockwise and others counterclockwise around the circle on which they lie. Figure 13 shows the resulting deformation of the hexagon.

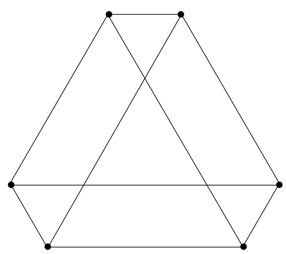


FIGURE 13. A parallel redrawing of $K_{3,3}$.

Explaining the existence of this deformation is worth a digression. The first remark is that the hypothesis of 3-independence in Theorem 7.3 can be dropped if we assume instead that the axial function is exact, as it is in these cases. For a proof in the context of GKM manifolds, see [GZ2]. Thus we do expect to see a nontrivial parallel deformation here.

The theory of rigidity predicts the same deformation. An *infinitesimal motion* of an embedded graph is an assignment of a velocity vector to each vertex in such a way that the length of each edge is (infinitesimally) unchanged. The space of infinitesimal motions includes the Euclidean motions and perhaps others. For precise definitions see [AR]. In the plane there is a one to one correspondence between infinitesimal motions and parallel redrawings: rotating all the vectors of an infinitesimal motion through a quarter of a turn converts that motion into a parallel redrawing. Translations remain translations. Rotation becomes dilation. Nontrivial motions become nontrivial parallel redrawings.

A nineteenth century theorem (reproved and generalized in [BR]) says that a plane embedding of $K(m,n)$ (for $m,n \geq 3$) is rigid (no infinitesimal motions) except when it lies on a conic. In this case that's just what happens. The regular $2n$-gon lies on a circle. The single nontrivial infinitesimal motion moves the odd permutations radially outward while the even ones move inward at the same velocity. Rotating that motion a quarter of a turn produces the parallel redrawing: half the vertices move clockwise, half counterclockwise. Figure 13 shows the result for S_3.

Finally, even without the picture we could have deduced the existence of the conic on which S_3 lies from the the exactness of the axial function together with the dual of Pascal's theorem in projective geometry.

9.5. **Several more examples.** We conclude our bestiary with several final suggestive examples.

Whenever the plane containing two adjacent edges of a polytope intersects that polytope in a cycle of edges the one skeleton of the polytope is an embedded graph with a natural axial function. The polytope need not be simple. The cuboctahedron, shown below, provides one example. It is 4-regular with 6 square and 8 triagular faces. Three hexagonal plane sections define three more geodesics. Its Betti numbers are $(1, 2, 6, 2, 1)$ so it has one nontrivial parallel redrawing, which dilates four of the triangular faces, converting the square faces to rectangles.

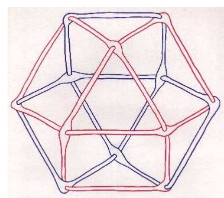
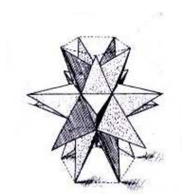
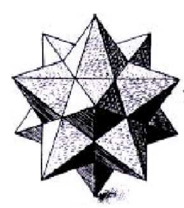


FIGURE 14. The cuboctahedron.

Figure 15(a) shows the *great stellated dodecahedron*, from Kepler's 1619 *Harmonice Mundi*. It is in fact a stellated icosahedron. It's a simple polytope with pentagrams for faces. These are the geodesics. Since these are inflection free the Betti numbers are well defined. They are $(5, 5, 5, 5)$. In this case $n\beta_0 + \beta_1 = 3 \cdot 5 + 5 = 20$ does properly count the number of parallel redrawings, since there is one for each of the 20 faces. However, Theorem 7.3 does not apply because the zeroth Betti number for each geodesic is 2, not 1.



(a)



(b)

FIGURE 15. This shows (a) the great stellated dodecahedron (a stellated icosahedron) and (b) small stellated dodecahedron (a stellated dodecahedron).

Figure 15(b) shows Kepler's *small stellated dodecahedron*, which *is* a stellated dodecahedron, Its geodesics are 12 pentagrams and 20 equilateral triangles. The invariant Betti numbers are $(3, 1, 2, 2, 1, 3)$. In this case $n\beta_0+\beta_1 = 3\cdot3+1 = 10$ counts neither the number of faces in Kepler's sense (as it would for a simple polytope) nor the number of parallel redrawings.

In the plane, however, we can often get a correct count even when hypotheses fail, Any n-gon has n parallel redrawings, one for each edge, and $n = 2\beta_0 + \beta_1$

as long as Poincare duality holds, even when the Betti numbers are not invariant. Figure 16 below shows the dart, whose Betti numbers are $(1, 2, 1)$ or $(2, 0, 2)$ depending on the choice of ξ.

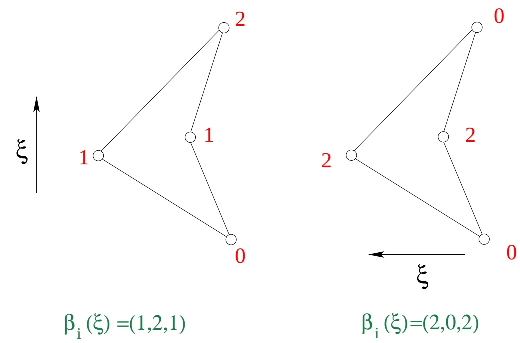


FIGURE 16. This shows the Betti numbers for the dart, with its Betti numbers $\beta_i(\xi)$ for two choices of ξ. The number at each vertex is the index, and the Betti numbers are given below each figure.

Finally Figure 17 shows the Petersen graph in the plane with two inflection free geodesics that define a connection. The well defined Betti numbers are $(1, 4, 4, 1)$. Since the axial function is exact we can count parallel redrawings even though it is not 3-independent. There are $2{\cdot}1{+}4 = 6$. Five correspond to edges of the enclosing pentagon which, when moved independently, force parallel redrawing of the inner pentagram. The sixth is a dilation of the inner pentagram while the outer pentagon remains fixed. It corresponds to the infinitesimal motion that rotates the inner pentagram relative to the outer pentagon - an infinitesimal motion possible only because exactness means the radial edges will meet in a point when extended.

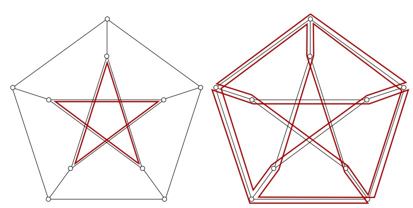


FIGURE 17. The two geodesics of the Petersen graph.

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