a priori Probability that Two Qubits are Unentangled

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In a previous study (Slater, P. B. (2000) *Eur. Phys. J. B.* 17, 471-480), several remarkably simple *exact* results were found, in certain specialized m-dimensional scenarios ($m \leq 4$), for the *a priori* probability that a pair of qubits is unentangled/separable. The measure used was the volume element of the Bures metric (identically one-fourth the statistical distinguishability [SD] metric). Here, making use of a newly-developed (Euler angle) parameterization of the 4×4 density matrices of Tilma, Byrd and Sudarshan, we extend the analysis to the complete 15-dimensional convex set (C) of *arbitrarily* paired qubits — the *total* SD volume of which is known to be $\frac{\pi^8}{1680} = \frac{\pi^8}{2^4 \cdot 3 \cdot 5 \cdot 7} \approx 5.64794$. Using advanced quasi-Monte Carlo procedures (scrambled Halton sequences) for numerical integration in this high-dimensional space, we approximately (5.64851) reproduce that value, while obtaining an estimate of .416302 for the SD volume of separable states. We *conjecture* that this is but an approximation to $\frac{\pi^6}{2310} = \frac{\pi^6}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} \approx .416186$. The ratio of the two volumes, $\frac{8}{11\pi^2} \approx .0736881$, would then constitute the exact Bures/SD probability of separability. The SD area of the *14*-dimensional *boundary* of C is $\frac{142\pi^7}{12285} = \frac{2 \cdot 71 \pi^7}{3^3 \cdot 5 \cdot 7 \cdot 13} \approx 34.911$, while we obtain a numerical estimate of 1.75414 for the SD area of the boundary of separable states.

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Życzkowski, Horodecki, Sanpera, and Lewenstein [1], giving various "philosophical", "practical" and "physical" motivations, were the first apparently to pose the question of "how many entangled ... states there are in the set of all quantum states". In a sequel to [1], Życzkowski examined to what extent the choice of a measure in the space of density matrices describing m-dimensional quantum sytems affects conclusions regarding the relative frequency of entangled and unentangled (separable/classically correlated) states [2]. Also, Życzkowski and Sommers analyzed several product measures in the space of mixed quantum states, in particular measures induced by the operation of partial tracing [3].

In this general context, it was argued by Slater [4,5] — in analogy to the use classically in Bayesian theory of the volume element of the Fisher information metric as Jeffreys' prior [6] — that a natural measure on the quantum states would be the volume element of the Bures metric [7–12]. "... the Bures metric is locally equivalent to a Riemannian metric defined by the quantum analogue of the Fisher information matrix" [13, p.60]. "This metric provides a unitarily invariant measure for distinguishing between two quantum states, and has been strongly motivated as physically relevant both on measurement and statistical grounds" [14]. Hall found compelling evidence, at least in the case of the *two*-dimensional quantum systems, that the Bures metric induces the "minimal-knowledge ensemble" over the space of density matrices [14]. An additional distinguishing feature of the Bures metric is that the associated connection form (gauge field) satisfies the source-free Yang-Mills equation [15,16]. Chen, Fu, Ungar, and Zhao have interpreted the Bures fidelity between possible states of a qubit in terms of the hyperbolic geometry applicable to special relativity [17]. While the Bures metric fulfills the role of the *minimal* monotone metric, there are a nondenumerable number of other monotone metrics as well, all satisfying certain desirable statistical properties [18]. It appears that all these other (non-minimal) monotone metrics would lead to *lower* estimates of the proportion of states that are separable/nonquantum in nature [4]. So, in this sense, the Bures metric provides upper bounds on reasonable/acceptable estimates of separability.

In [5], specific use was made of the volume element of the Bures metric as a measure to address the question initially posed by Życzkowski *et al* [1]. This led to a number of quite surprisingly simple probabilities of separability, such as $\frac{1}{4}$ (Werner states), $\frac{1}{2}$, $\sqrt{2} - 1$ and $\frac{2}{\pi} - \frac{1}{2}$, when applied to pairs of quantum bits (qubits) in certain restricted low-dimensional scenarios (cf. [19]) for which exact integrations could be performed. (In several of these instances, the two *individual* qubits were in the fully mixed or classical state and constraints were placed on possible correlations between the two qubits.) In the present study, we seek to remove any such limitations and determine the Bures probability of separability of two qubits in the full 15-dimensional framework. However, due to the increased dimensionality/complexity, it appears necessary (at least with the current state of development of appropriate mathematical software) to have recourse to numerical methods for the requisite integrations. For this purpose, we rely upon recent developments in quasi-Monte Carlo procedures [20] — namely, the use of scrambled Halton sequences [21,22]. Upon the basis of our numerical results, we formulate a conjecture (still awaiting *formal* proof or disproof), having interesting number-theoretic properties, that the Bures probability of separability of two arbitrarily paired qubits is $\frac{8}{11\pi^2} \approx .0736881$.

The joint state of two qubits is describable by a 4×4 density matrix (ρ)— that is, a Hermitian matrix, having trace 1 and nonnegative eigenvalues. Such a state is separable (that is nonentangled or, equivalently, classically correlated) if it can be expressed as the convex sum of tensor products of pairs of 2×2 density matrices (which, in turn, represent the states of individual qubits) [23]. Ensembles of separable states, as well as of *bound entangled* states can *not* be "distilled" to obtain pairs in singlet (total spin 0) states for *quantum* information processing [24,25]. Let us note that the Bures metric (and the related concept of *fidelity*) has been an important instrument in the currently widespread study of bipartite and multipartite quantum systems [26,28,29].

As a practical matter, the question of what parameterization of the 4×4 density matrices to employ is quite important for computational purposes. In [5], we used the "polarization matrix density technique" parameterization (based on tensor products of Pauli matrices) [30, eq. (1)] [31], focusing on certain m-dimensional subsets ($m \leq 4$) of the fifteen-dimensional space of 4×4 density matrices. We were then able to obtain, as already indicated, several simple *exact* values for the Bures probabilities of separability of two qubits, the possible joint states of which were restricted to these low-dimensional spaces.

In our initial study on the question of relative separability/entanglement [4], we had relied upon the *naive* parameterization — $\rho_{ij} = a_{ij} + ib_{ij}$, where the a's and b's are real — of the 15-dimensional convex set of 4×4 density matrices, in order to obtain a *number* of estimates of the full, general Bures probability (as well as the corresponding probability of separability for the 35-dimensional convex set of 6×6 density matrices, representing the joint state of a qubit and *qutrit*). Because no analytical expressions are known for the highly complicated boundary of the domain using this parameterization (cf. [32]) (as well as for the polarization density matrix technique), it was necessary in [4] to reject many points of the imposed lattices used for numerical sampling since they turned out to be incompatible with the *positivity* requirement for density matrices. Additionally, diagonal entries (a_{ii}) — because they all sum to unity — had to be sampled differently (that is, from a probability simplex rather than a regular lattice) than the off-diagonal entries ($a_{ij}, b_{ij}, i \neq j$). This led us to report *several* estimates, each depending upon the particular resolutions used for selecting candidate diagonal and off-diagonal entries of the 4×4 density matrices [4, Tables 1-3]. Most of the resultant estimates of the Bures probability of separability were in the neighborhood of .1. (For an analogous study of the Gaussian two-party quantum states, see [33].)

In contrast to this somewhat nonideal situation, the recently-reported Euler angle parameterization of Tilma, Byrd and Sudarshan [34] (based on the diagonalization $\rho = U\Lambda U^{\dagger}$, where U is unitary) appeared to yield a domain — relatively easy to numerically integrate over — that is simply a 15-dimensional hyperrectangle. However, there was an erroneous claim made in [34] regarding this, requiring rectification before we can proceed correctly. It was stated that for the choice of ranges of the three spherical angles [34, eq. (36)] (the other twelve variables being the Euler angles parameterizing the unitary matrix U drawn from the Lie algebra $SU(4)/Z(4)$),

$$\frac{\pi}{4} \leq \theta_1 \leq \frac{\pi}{2}; \quad \cos^{-1}\frac{1}{\sqrt{3}} \leq \theta_2 \leq \frac{\pi}{2}; \quad \frac{\pi}{3} \leq \theta_3 \leq \frac{\pi}{2}, \tag{1}$$

the vector of (nonnegative) eigenvalues,

$$(\lambda_1, \lambda_2, \lambda_3, 1 - \lambda_1 - \lambda_2 - \lambda_3) = (\sin^2\theta_1 \sin^2\theta_2 \sin^2\theta_3, \cos^2\theta_1 \sin^2\theta_2 \sin^2\theta_3, \cos^2\theta_2 \sin^2\theta_3, \cos^2\theta_3), \tag{2}$$

would be strictly ordered, that is, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1 - \lambda_1 - \lambda_2 - \lambda_3$. However, simple testing of ours revealed that while λ_1 is, in fact, always at least as great as the other three eigenvalues, these last three do not necessarily conform to any particular order within the ranges designated. (We were not yet aware of this difficulty in our earlier study [35], and were led to erroneously assert there that the desired Bures/SD probability of separability of arbitrarily paired qubits was $\sqrt{2}/24 \approx .0589256$.)

In [38] we had specifically addressed the question of generating an *ordered* vector (lying in the unit simplex). In terms of the parameterization (2), these conditions can be expressed as

$$\frac{\pi}{4} \leq \theta_1 \leq \frac{\pi}{2}; \quad f(\theta_1) \leq \theta_2 \leq \frac{\pi}{2}; \quad f(\theta_2) \leq \theta_3 \leq \frac{\pi}{2}; \quad f(x) = \cot^{-1}\Big(\cos x\Big). \tag{3}$$

The Lebesgue measure of the Euclidean domain defined by the three-dimensional volume determined by the ranges (3) is .0564221, while that defined by the ranges (1) is 4.48593 times larger, that is .253106.

We could proceed further with these ranges (3), but in order to avoid loss of the convenient hyperrectangular structure posited in [34], we have chosen to simply employ

$$0 \leq \theta_1, \theta_2, \theta_3 \leq \frac{\pi}{2}. \tag{4}$$

(The Lebesgue measure of which is, of course, $(\frac{\pi}{2})^3 \approx 3.87578$.) These ranges generate all possible four-vectors (2) in a unique manner. (Integrals using the ranges (4) would then, in the context here, simply be $4! = 24$ times those obtained using the set of angular ranges (3).)

Given the (restored/modified) hyperrectangular structure, it is simple to rescale each of the fifteen coordinates, so as to obtain a hyper*cubic* domain, with all edges equal to unity in length. Most available quasi-Monte Carlo computer routines are written with such a regular framework in mind [20].

Having so converted to the unit hypercubic structure, we placed 65 million ("low discrepancy") points over the hypercube, devised so as to be near to *uniformly* distributed. The specific method employed was that of scrambled Halton sequences [21,22]. One of the classical low-discrepancy sequences is the van der Corput sequence in base b, where b is any integer greater than one. The uniformity of the van der Corput numbers can be further improved by permuting/scrambling the coefficients in the digit expansion of m in base b. The scrambled Halton sequence in m-dimensions — which we employ — is constructed using the so-scrambled van der Corput numbers for b's ranging over the first m prime numbers [22, p. 53].

Let us now discuss an issue largely of terminology, but important to keep in mind in implementing various formulas. Braunstein and Caves [9] showed that the Bures metric (as stipulated in [7,8]) was equal to identically *one-quarter* of their statistical distinguishability (SD) metric (cf. [14, eq. (2.29)]). However, Hall [14], citing [9], spoke in terms of the Bures metric, but actually employed the formulas for the SD metric [14, eq. (24)]. This, of course, is a matter of no consequence if one computes weighted averages or probabilities with respect to the volume element of one metric or another. It is pertinent, however, when absolute rather than relative volumes are to be determined, with a factor of $4^{\frac{m(m-1)}{2}}$ difference occurring for m-level systems. Thus, the Bures volumes themselves will be 4^{-6} times smaller than the SD ones (of a somewhat more appealing form) given below for our case of $m = 4$.

We computed the corresponding statistical distinguishability (SD) volume element ("quantum Jeffrey's prior" [12]) at each point of the scrambled Halton sequence in 15 dimensions. This volume element is the product of the *Haar* volume element over the *twelve* Euler angles parameterizing $SU(4)/Z(4)$ [34, eqs. (24), (25)] and the *conditional* SD volume element over the *three*-dimensional simplex of eigenvalues [38, eqs. (16), (17)].

The *conditional* SD volume element ($\mathrm{d}D_n$) over an $(m-1)$-dimensional simplex of (nonnegative) eigenvalues, constrained to sum to 1, can be expressed as (cf. [14, eq. (24)]),

$$\mathrm{d}D_m = \frac{\mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_{m-1}}{\sqrt{\Pi_{i=1}^m \lambda_i}} \Pi_{1\leq i<j}^m \frac{4(\lambda_i - \lambda_j)^2}{(\lambda_i + \lambda_j)}. \tag{5}$$

(If the factor of 4 is omitted, this becomes the conditional Bures volume element.) Integrating (5) over the simplices for various m, we obtain, as far as we have been able to compute exactly,

$$D_2 = 2\pi \approx 6.28319; \quad D_3 = \frac{64\pi}{35} = \frac{2^6\pi}{5\cdot 7} \approx 5.74463; \quad D_4 = \frac{2\pi^2}{35} = \frac{2\pi^2}{5\cdot 7} \approx .563977; \tag{6}$$

$$D_5 = \frac{8388608\pi^2}{156165009} = \frac{2^{23}\pi^2}{3\cdot 7^2 \cdot 11\cdot 13\cdot 17\cdot 19\cdot 23} \approx .530159.$$

(Also, $D_6 \approx \frac{4^{15}\pi^3}{1.53636\cdot 10^{16}} \approx 2.1643610^{-6}$.)

The "truncated" Haar volume of the Lie algebra $SU(4)/Z(4)$, using the Euler angle parameterization [34], is $\frac{\pi^6}{96}$ [34], so the total (separable plus nonseparable) SD volume (V^{s+n}) of the 15-dimensional convex set of four-level quantum systems is the *product* of this term and the term D_4 [14, eq. (24)],

$$V^{s+n} = \frac{\pi^6}{96} \cdot \frac{2\pi^2}{35} = \frac{\pi^8}{1680} = \frac{\pi^8}{2^4\cdot 3\cdot 5\cdot 7} \approx 5.64794. \tag{7}$$

"Truncation" occurs because three of the fifteen Euler angles, corresponding to diagonal Lie generators, become irrelevant (that is, "drop out") in the formation of ρ. Without truncation, the appropriate Haar volume would equal $\frac{\pi^9}{288\sqrt{2}}$ [34, eq. (B24)].

We also determined whether or not the density matrix corresponding to each of the 65 million points of the scrambled Halton sequences was separable. ("Essentially, the mathematical context is one of two nested compact convex sets and the determination of whether a point in the larger set is in the smaller set" [43].) For this purpose, we employed the *partial transposition* criterion of Peres [39] and the Horodecki trio [40]. (The partial transpose of a 4×4 density matrix can be obtained by transposing in place each of its four 2×2 blocks.) In fact, since no more than one eigenvalue of the partial transpose of a 4×4 density matrix can be negative [41, Thm. 5] [34,42], one could simply employ the sign of the determinant of the partial transpose as the test for separability, rather than the positivity of all four eigenvalues (cf. [44]). That is, a positive determinant of the partial transpose informs us that the density matrix which has been partially transposed is separable in nature, while a negative determinant tells us it is nonseparable or entangled.

For the scrambled Halton sequence of 65 million points, distributed in a near-to-uniform manner over the 15-dimensional unit hypercube, we obtained 5.64851 as an estimate of the (known) total Bures volume (5.64794) and .416302 for an estimate of the (unknown) Bures volume of the separable states. (For the initial 10 million points, the analogous figures were 5.64615 and .415716, and for the initial 20 million, 5.64829 and .416775.)

Multiplying V^{s+n} by the *probability* $\frac{8}{11\pi^2} \approx .0736881$, we obtain what — on the basis of our numerical evidence plus considerations of mathematical simplicity/elegance, buttressed by our previous findings of simple exact solutions in low-dimensional settings [5] — we *conjecture* to be the SD volume of the *separable* 4×4 density matrices, that is,

$$V^s = \frac{8}{11\pi^2} V^{s+n} = \frac{\pi^6}{2310} = \frac{\pi^6}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} = \frac{\pi^6}{11\#} \approx .416186. \tag{8}$$

The notation $p\#$ denotes the products of the primes less than or equal to p [48], so V^{s+n} can be expressed as $\frac{\pi^8}{2^3 \cdot 7\#}$. Let us point out that the pair of integers (1680, 2310), occurring in the denominators of V^{s+n} and V^s, are the *last* two members of a certain sequence (denoted A064377) of number-theoretic interest [45], in that it has been conjectured by E. Lábos [45] that 1680 and 2310 are the two largest integers for which the sum of the *fourth* power of their divisors exceeds the *fifth* power of the number (Euler's totient function) of positive integers relatively prime to them.

Since our numerical (quasi-Monte Carlo) estimate of V^{s+n}, that is $5.64851 > 5.64794$, errs on the positive side, it is rather natural also to expect that our numerical estimate of V^s would err in the same direction. Subject to the validity of our conjecture (8), this is, in fact, the case, since $.416302 > .416186$.

In the computational process described above, we also determined the average Bures/SD entanglement of each of the 65 million density matrices generated, using two possible measures — the *negativity* and the *concurrence*, the former always being no greater than the latter [46]. For the mean negativity we obtained .177162 and for the mean concurrence, .197284 (cf. [3, Figs. 4(b), 5(a)]). (Another interesting measure of entanglement to similarly study would be the Bures distance of a quantum state to the separable states [26] (cf. [27,43,47]).)

Since the Bures/SD probability of separability conjectured here ($\frac{8}{11\pi^2} \approx .0736881$) of a pair of qubits is somewhat less in value than those estimates ($\approx .1$) arrived at previously in [4], using the naive parameterization ($\rho_{ij} = a_{ij} + ib_{ij}$) of the 4×4 density matrices, we are led to assert that the "world is even more quantum" [1] than previously indicated. In [4] it appeared that the Bures probability of separability provides a natural *upper* bound on an entire class (corresponding to the monotone metrics [18]) of possible prior probability measures. (The Bures metric serves as the *minimal* monotone metric [18].) On a qualitative level, then, knowledge that the joint state of two qubits is separable would be considerably more "informative" — in allowing us to estimate the underlying parameters of the state — than knowledge that it is entangled.

In a study [10] of the *eight*-dimensional convex set of the 3×3 density matrices, we developed explicit formulas for the entries (g_{ij}) of the Bures metric tensor, based on the Euler angle parameterization of $SU(3)$ [49]. Several pairs (i, j) of the eight variables there were found to be mutually orthogonal (that is, $g_{ij} = 0, i \neq j$). We have examined the question of mutual orthogonality also in the present $SU(4)$ case [34] (but have not yet attempted to obtain simplified explicit formulas for the g_{ij}'s, in general). Our conclusions based on strong *numerical* evidence — obtained by implementing the "explicit formulae for the Bures metric" of Dittmann [11] — are that: the twelve Euler angles α's (in the notation of [34]) are each mutually orthogonal to the three (mutually orthogonal) θ's, parameterizing the eigenvalues (2); α_6 is orthogonal to α_5 and also to all $\alpha_i, i > 6$; α_9 is orthogonal to both α_{10} and α_{12}; and $\alpha_{10}, \alpha_{11}, \alpha_{12}$ are all mutually orthogonal. In the important inverse matrix ($||g_{ij}||^{-1}$) the twelve α's are again, obviously, all orthogonal with the three θ's, but no other such pairs were found (unlike the SU(3) case [10,49]).

Additionally, of course, we would like to study in similar ways *higher*-dimensional bipartite and multipartite quantum systems than that examined here. Tilma and Sudarshan have given, along with other higher-dimensional systems, Euler angle-based parameterizations of the 8×8 density matrices of *three* qubits [51], and indicated to the author the parameterization for the 6×6 density matrices, corresponding to a paired qubit and *qutrit*. (It would be desirable for any such analyses to know beforehand the precise conditional SD volumes, D_m, $m > 5$, seeing that knowledge of D_4

was crucial in our being able to formulate the conjecture as to the Bures volume of separable 4×4 density matrices.) We recall that for $m > 6$, the Peres-Horodecki partial transposition criterion provides a necessary, but not sufficient condition for separability [24,39]. As the Hilbert space dimensions of coupled l and m-dimensional quantum systems *increase*, we expect the corresponding Bures/SD probability of separability to *decrease* [1,4].

Let us apply the formula (5) for the conditional SD volume $\mathrm{d}D_m$, with $m = 4$, but now setting, say, $\lambda_1 \to 0$ (by taking $\theta_1 \to 0$). Then, integrating the resulting expression, to high accuracy, over the 2-dimensional simplex, we obtain .871513859457. Multiplying this by a factor of four (to account for the possibility that $\lambda_2, \lambda_3, \lambda_4$ or $1 - \lambda_1 - \lambda_2 - \lambda_3$ is the zero eigenvalue) and then by the truncated Haar volume, $\frac{\pi^6}{96}$, we get 34.9110002722. This is the *14*-dimensional SD surface *area* (A^{s+n}) of the boundary ($|\rho| = 0$) of the 15-dimensional convex set of 4×4 density matrices. It appears overwhelmingly convincing that

$$A^{s+n} = \frac{142\pi^7}{12285} = \frac{2 \cdot 71\pi^7}{3^3 \cdot 5 \cdot 7 \cdot 13} \approx 34.91100027222665. \tag{9}$$

(The denominator 12,285 is the number of permutations of 15 items in which exactly 4 of them change places [45, seq. A060008].) For the 3×3 density matrices, the SD area of the boundary $|\rho| = 0$ is $3(512/63)(\pi^3/2) = 256\pi^3/21 \approx 377.981$, and for the 5×5 density matrices, $5(.00736276442200)(\pi^{10}/18432) \approx .187041154554$. Note that $2439209213\pi/5716630 \approx .00736276442200$ with $5716630 = 2 \cdot 5 \cdot 41 \cdot 73 \cdot 191$ and $2439209213 = 7 \cdot 348458459$. For the 6×6 density matrices — forming a 35-dimensional space — we have $6(3.85759 \cdot 10^{-6})(\pi^{15}/35389440) \approx .00001874312$. Observe that $168\pi^2/v \approx .38548 \cdot 10^{-6}$, where $v = 430,137,400$ is the number of permutations of 35 items in which exactly 6 of them change places. (For the simplest case of the 2×2 density matrices, we get $2\pi^2$ for the SD volume and 16π for the SD area of the pure state boundary.)

We have also estimated A^s itself — based on the first 11,800,000 points given by scrambled Halton sequences, now in 14-dimensional space. Of these 11,800,000, we found 8,083,953 of them for which there existed at least one acceptable value (that is, lying between 0 and 1) of the 15-th coordinate (taken to be the rescaled form of θ_3) for which the corresponding density matrix lay on the separable-nonseparable boundary (as indicated by a zero determinant of its partial transpose). (DiVincenzo, Terhal, and Thapliyal considered situations is which a mixed state is "marginally separable, in the first case because the partial transpose of the state has zero eigenvalues, and in the second because the state is defined as the complement (in a larger Hilbert space) of a barely completable product basis" [52].) In total, we obtained 15,330,369 such (boundary) density matrices. We then determined the associated SD area elements (identically π^{-1} times in value the corresponding SD volume elements). The computations gave an estimate of $A^s \approx 1.75414$. (Based on just the first 3,200,000 points of this sequence, the estimate of A^s was 1.74893.) If the SD area does, in fact, have a simple exact expression, it might possibly be

$$A^s = \frac{\pi^5}{175} = \frac{\pi^5}{5^2 \cdot 7} \approx 1.74868. \tag{10}$$

(But also $\pi^6/548 \approx 1.75436$.)

It would be of interest to study the relations between A^{s+n} and V^{s+n}, as well as between A^s and V^s, in terms of isoperimetric inequalities [53–56] — taking into account known curvature properties of the Bures metric [57–59] (cf. [61]). The scalar curvature of the Bures metric on a 4×4 density matrix (ρ) has been expressed as [58],

$$S^1 = 6\frac{63e_4 + 35e_3^2 - 43e_2e_3 - 7e_3 - 3e_2^2}{e_4 + e_3^2 - e_2e_3}, \tag{11}$$

where e_i is the elementary invariant of degree i of ρ (that is, $\Pi_{i=1}^m(\lambda_i - t) = \sum_{i=0}^m e_{m-i}(-t)^i$), so that the scalar curvature depends only on the eigenvalues (λ's) of ρ. It is unbounded for $m > 2$ and achieves its minimum, $(5m^2 - 4)(m^2 - 1)/2$, at the fully mixed or classical state, having the $m \times m$ density matrix $\rho = \frac{1}{m}\mathbf{1}$, so this minimum is 570 for $m = 4$ [58]. (The sectional curvature is also always greater than 1 [60, eq. (6.2)].) Also, the $(m^2 - 1)$-dimensional space of $m \times m$ density matrices, representing the m-level quantum systems, is *not* locally symmetric for $m > 2$ [57]. (We might also observe that a Euclidean 15-sphere (having a radius of 1.19682) with volume equal to V^{s+n} has a surface area of 70.7865 (cf. 34.911), while such a 15-sphere (having a radius of 1.01128) with volume equal to V^s has an area 6.20661 (cf. 1.74893).)

If, in addition, to the scalar curvature (11), the Ricci curvature were also bounded below, in particular, by $(m^2 - 1) - 1 = 14$, then we could directly apply the "Levy-Gromov" isoperimetric inequality [62, p. 520]. In this case, the ratio (.318581) of the area $A^s \approx 1.75414$ of the boundary of separable states to the volume $V^{s+n} = \pi^8/1680$ would be *greater* than the ratio (w) of $s(\alpha)$ to the volume $\tilde{V} = 256\pi^7/2027025$ of a unit ball in 15-dimensional space. Now

α itself is the ratio $V^s/V^{s+n} = 8/11\pi^2$ and $s(\alpha)$ is the area of the boundary of a ball in 15-dimensional space having a volume equal to $\alpha\tilde{V}$. This gives us $w = 1.31521$, so the Levy-Gromov inequality fails, since $.318581 \not\geq 1.31521$ and we are left to conclude that the Ricci curvature for the qubit-qubit states endowed with the Bures metric must somewhere assume a value less than 14. (However, for the qubit-qutrit case [63], no contradiction with the corresponding inequality appears to hold.) To continue along these lines, there is an extension [64, Thm. 6.6] of the Levy-Gromov result from the case where the Ricci curvature is bounded below not simply by m^2-1 but by $(m^2-1)\kappa$ (where κ is interpreted as the constant sectional curvature of a 15-dimensional sphere of radius $1/\sqrt{\kappa}$). Then, the corresponding isoperimetric inequality would not be inconsistent with our particular values of V^s, V^{s+n} and A^s if the Ricci curvature were bounded below by $\approx .780703$ (but nothing higher).

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