Identities for Tribonacci-related sequences

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Abstract

We establish some identities relating two sequences that are, as explained, related to the Tribonacci sequence. One of these sequences bears the same resemblance to the Tribonacci sequence as the Lucas sequence does to the Fibonacci sequence. Defining a matrix that we call Tribomatrix, which extends the Fibonacci matrix, we see that the other sequence is related to the sum of the determinants of the 2nd order principal minors of this matrix.

1 Antefacts

Let S_n be the generalized Lucas sequence, also called generalized Tribonacci sequence, that is

$$S_{n+1} = S_n + S_{n-1} + S_{n-2}, \qquad S_0 = 3,\ S_1 = 1,\ S_2 = 3.$$

S_n is sequence A001644 in [2]. Let $\{\alpha,\ \beta,\ \gamma\}$ be the roots of the characteristics polynomial $x^3 - x^2 - x - 1 = 0$ (for an explicit expression see [1]). Let us assume that α is the real root, β and γ are the complex conjugate roots. We have $\alpha = 1.8392286...$, $|\beta| = |\gamma| = 0.737353...$ (see [3]).
The Binet's formula (see [1]) is

$$S_n = \alpha^n + \beta^n + \gamma^n,$$

and the ordinary generating function $A(x)$ is

$$A(x) = \frac{3 - 2x - x^2}{1 - x - x^2 - x^3}.$$

Let us consider the following matrix, that we might call Tribomatrix,

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of this matrix are $\{\alpha, \beta, \gamma\}$. Using the relationships between eigenvalues and coefficients of the characteristic equation we have

1. $$\alpha + \beta + \gamma = 1,$$

2. $$\alpha\beta + \alpha\gamma + \beta\gamma = -1,$$

3. $$\alpha\beta\gamma = 1.$$

By induction we get

$$\mathbf{A}^n = \begin{bmatrix} T_{n+1} & T_n & T_{n-1} \\ T_n + T_{n-1} & T_{n-1} + T_{n-2} & T_{n-2} + T_{n-3} \\ T_n & T_{n-1} & T_{n-2} \end{bmatrix},$$

where T_n are the Tribonacci numbers (sequence A000073 in [2])

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad T_0 = 0,\, T_1 = 1,\, T_2 = 1.$$

Then

$$\begin{aligned} \mathrm{tr}(\mathbf{A}^n) &= S_n \\ &= T_n + 2T_{n-1} + 3T_{n-2}, \end{aligned}$$

where $\mathrm{tr}(\cdot)$ is the trace operator.
From the generating function we get also immediately

$$S_n = 3T_{n+1} - 2T_n - T_{n-1}.$$

Define

$$C_n = \alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n.$$

Then C_n is the sum of the determinants of the principal minors of order 2 of $\mathbf{A}^n$ and we obtain

$$\begin{aligned} C_n &= 2T_{n+1}T_{n-2} + T_{n+1}T_{n-1} - T_n^2 - 2T_nT_{n-1} - T_{n-1}T_{n-3} + T_{n-2}^2 \\ &= -T_n^2 + 2T_{n-1}^2 + 3T_{n-2}^2 - 2T_nT_{n-1} + 2T_nT_{n-2} + 4T_{n-1}T_{n-2}. \end{aligned}$$

The sequence C_n is sequence A073145 in [2].

2 A Recurrence for C_n

We have

$$\begin{aligned} & -C_{n-1} - C_{n-2} + C_{n-3} = \\ = \; & -\alpha^{n-1}\beta^{n-1} - \alpha^{n-1}\gamma^{n-1} - \beta^{n-1}\gamma^{n-1} - \alpha^{n-2}\beta^{n-2} - \alpha^{n-2}\gamma^{n-2} \\ & \quad -\beta^{n-2}\gamma^{n-2} + \alpha^{n-3}\beta^{n-3} + \alpha^{n-3}\gamma^{n-3} + \beta^{n-3}\gamma^{n-3} \\ = \; & \alpha^{n-3}\beta^{n-3}(1 - \alpha\beta - \alpha^2\beta^2) + \alpha^{n-3}\gamma^{n-3}(1 - \alpha\gamma - \alpha^2\gamma^2) \\ & \quad +\beta^{n-3}\gamma^{n-3}(1 - \beta\gamma - \beta^2\gamma^2). \end{aligned}$$

Using relationships among roots we get

$$\begin{aligned} 1 - \alpha\beta - \alpha^2\beta^2 &= \alpha\beta\gamma - \alpha\beta - \alpha^2\beta^2 \\ &= \alpha\beta(\gamma - 1 - \alpha\beta) \\ &= \alpha\beta(\gamma + \alpha\beta + \alpha\gamma + \beta\gamma - \alpha\beta) \\ &= \alpha\beta(\gamma + \alpha\gamma + \beta\gamma) \\ &= \alpha\beta\gamma(1 + \alpha + \beta) \\ &= 1 + 1 - \gamma \\ &= 2 - \gamma. \end{aligned}$$

Upon repeating the same calculations for the other quantities we get

$$\begin{aligned} -C_{n-1} - C_{n-2} + C_{n-3} &= 2\alpha^{n-3}\beta^{n-3} - \alpha^{n-3}\beta^{n-3}\gamma + 2\alpha^{n-3}\gamma^{n-3} \\ & \quad -\alpha^{n-3}\gamma^{n-3}\beta + 2\beta^{n-3}\gamma^{n-3} - \beta^{n-3}\gamma^{n-3}\alpha \\ &= 2C_{n-3} - \alpha^{n-4}\gamma^{n-4}\alpha\beta\gamma - \alpha^{n-4}\beta^{n-4}\alpha\beta\gamma \\ & \quad -\beta^{n-4}\gamma^{n-4}\alpha\beta\gamma \\ &= 2C_{n-3} - C_{n-4}, \end{aligned}$$

that is

$$C_{n-1} = -C_{n-2} - C_{n-3} + C_{n-4}.$$

So we got the recurrence

$$C_n = -C_{n-1} - C_{n-2} + C_{n-3}, \tag{1}$$

with $C_0 = 3$, $C_1 = -1$, $C_2 = -1$.

In this way we obtain easily the ordinary generating function for C_n

$$A(x) = \frac{3 + 2x + x^2}{1 + x + x^2 - x^3}. \tag{2}$$

3 A Recurrence for C_{2n}

$$\begin{aligned}
C_{2n} &= -C_{2n-1} - C_{2n-2} + C_{2n-3} \\
&= C_{2n-2} + C_{2n-3} - C_{2n-4} + C_{2n-3} + C_{2n-4} \\
&\quad -C_{2n-5} - C_{2n-4} - C_{2n-5} + C_{2n-6} \\
&= C_{2n-2} + 2C_{2n-3} - C_{2n-4} - 2C_{2n-5} + C_{2n-6} \\
&= -C_{2n-2} - 3C_{2n-4} + C_{2n-6} + 2C_{2n-2} + 2C_{2n-3} \\
&\quad +2C_{2n-4} - 2C_{2n-5} \\
&= -C_{2n-2} - 3C_{2n-4} + C_{2n-6} + 2C_{2n-2} \\
&\quad -2(-C_{2n-3} - C_{2n-4} + C_{2n-5}) \\
&= -C_{2n-2} - 3C_{2n-4} + C_{2n-6} + 2C_{2n-2} - 2C_{2n-2} \\
&= -C_{2n-2} - 3C_{2n-4} + C_{2n-6}.
\end{aligned}$$

So we got the recurrence

$$C_{2n} = -C_{2n-2} - 3C_{2n-4} + C_{2n-6}, \tag{3}$$

with $C_0 = 3$, $C_2 = -1$, $C_4 = -5$. The ordinary generating function is

$$A(x) = \frac{3 + 2x + 3x^2}{1 + x + 3x^2 - x^3}.$$

4 Identities

Let $n \geq m$. Then

$$\begin{aligned}
S_n S_{n+m} &= (\alpha^n + \beta^n + \gamma^n)(\alpha^{n+m} + \beta^{n+m} + \gamma^{n+m}) \\
&= \alpha^{2n+m} + \alpha^n\beta^{n+m} + \alpha^n\gamma^{n+m} + \alpha^{n+m}\beta^n + \beta^{2n+m} \\
&\quad +\beta^n\gamma^{n+m} + \alpha^{n+m}\gamma^n + \beta^{n+m}\gamma^n + \gamma^{2n+m} \\
&= S_{2n+m} + \alpha^n\beta^n(\alpha^m + \beta^m) + \alpha^n\gamma^n(\alpha^m + \gamma^m) \\
&\quad +\beta^n\gamma^n(\beta^m + \gamma^m) \\
&= S_{2n+m} + \alpha^n\beta^n(S_m - \gamma^m) + \alpha^n\gamma^n(S_m - \beta^m) \\
&\quad +\beta^n\gamma^n(S_m - \alpha^m) \\
&= S_{2n+m} + S_m(\alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n) + \\
&\quad -\alpha^m\beta^m\gamma^m(\alpha^{n-m}\beta^{n-m} + \alpha^{n-m}\gamma^{n-m} + \beta^{n-m}\gamma^{n-m}) \\
&= S_{2n+m} + S_m C_n - C_{n-m}. \qquad (4)
\end{aligned}$$

On the other hand let $n < m$. Then everything goes the same with the exception of the next-to-last line, where we collect in the third sum $\alpha^n\beta^n\gamma^n$. Then the result is

$$S_nS_{n+m} = S_{2n+m} + S_mC_n - S_{m-n}. \tag{5}$$

5 Consequences

1. If we put $n = n - 1$ and $m = 1$ then we get

$$S_nS_{n-1} = S_{2n-1} + C_{n-1} - C_{n-2}.$$

2. If we put $m = n$ we get

$$S_nS_{2n} = S_{3n} + S_nC_n - 3.$$

3. Generally we have

$$S_nS_{nm} = S_{n(m+1)} + S_{n(m-1)}C_n - S_{n(m-2)}.$$

4. If we put $m = 0$ we get

$$\begin{aligned} S_n^2 &= S_{2n} + 3C_n - C_n \\ &= S_{2n} + 2C_n. \end{aligned} \tag{6}$$

5. For the cube we have

$$\begin{aligned} S_n^3 &= S_n^2S_n \\ &= (S_{2n} + 2C_n)S_n \\ &= S_nS_{n+n} + 2S_nC_n \\ &= S_{2n+n} + S_nC_n - C_{n-n} + 2S_nC_n \\ &= S_{3n} + 3S_nC_n - 3. \end{aligned}$$

6. For the 4-th power

$$\begin{aligned} S_n^4 &= (S_n^2)^2 \\ &= (S_{2n} + 2C_n)^2 \\ &= S_{2n}^2 + 4C_n^2 + 4S_{2n}C_n \\ &= S_{4n} + 2C_{2n} + 4C_n^2 + 4S_{2n}C_n. \end{aligned} \tag{7}$$

But also

$$\begin{aligned}
S_n^4 &= S_n^3 S_n \\
&= (S_{3n} + 3S_nC_n - 3)S_n \\
&= S_nS_{n+2n} + 3S_n^2C_n - 3S_n \\
&= S_{4n} + S_{2n}C_n - S_n + 3S_{2n}C_n + 6C_n^2 - 3S_n \\
&= S_{4n} - 4S_n + 4S_{2n}C_n + 6C_n^2. \qquad (8)
\end{aligned}$$

Confronting Equation 7 and Equation 8 we get this other identity

$$2S_n = C_n^2 - C_{2n}. \qquad (9)$$

References

[1] M. Elia (2001), "Derived Sequences, The Tribonacci Recurrence and Cubic Forms." *The Fibonacci Quarterly* **39.2** (2001): 107-109.

[2] N.J.A. Sloane, Editor (2002), The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/˜njas/sequences/.

[3] Eric Weisstein's World of Mathematics, published electronically at http://mathworld.wolfram.com/Fibonaccin-StepNumber.html