

GENERATING FUNCTIONS, FIBONACCI NUMBERS AND RATIONAL KNOTS

This is a preprint. I would be grateful for any comments and corrections!

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Abstract. We describe rational knots with any of the possible combinations of the properties (a)chirality, (non-)positivity, (non-)fiberedness, and unknotting number one (or higher), and determine exactly their number for a given number of crossings in terms of their generating functions. We show in particular how Fibonacci numbers occur in the enumeration of fibered achiral and unknotting number one rational knots. Then we show how to enumerate rational knots by crossing number and genus and/or signature. This allows to determine the distribution of these invariants among rational knots. We give also an application to the enumeration of lens spaces.

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1 Introduction

A natural question one can ask in knot theory is how many different knots or links, possibly of some special class, there are of given crossing number (that is, minimal number of crossings in any of their diagrams). Clearly, to have a satisfactory approach to such a problem, a good understanding of the class in question is necessary. For an arbitrary knot or link, the problems to identify it from a given diagram, and (hence also) to determine its crossing number, although solved in theory by Haken [Ha], are impracticably complicated. Thus an even approximate enumeration of general knots and links by crossing number seems so far impossible. However, some bounds are known. In [W], Welsh proved that this number is exponentially bounded in the crossing number n , with an upper bound to the base of the exponential of 13.5.

Even if a class of links is well-understood, still its enumeration may be difficult. An example of such a class are the prime alternating links. Such links have been classified (contrarily to Haken, in a very practicable manner) in [MT], and the determination of their crossing number was settled in [Ka, Mu, Th] (both results having been conjectured decades before by Tait). Thus one can algorithmically generate the table of links of certain (not too high) crossing number n , and hence in particular determine (by “brute force”) how many of them there are [HTW]. However, a reasonable expression for the numbers thus obtained is not known, and possibly does not exist. Only recently, Sundberg and Thistlethwaite [ST] obtained asymptotical estimates, accurate up to a linear factor in n . (This slight inexactness was later removed in a note of Schaeffer and Kunz-Jacques [KS].) In particular, they determined the base of the exponential growth of these numbers to about 6.14. There has been other recent work [ZZ], which exhibits a deep connection to statistical mechanics. This approach, however, even if more effective than brute force enumeration, is still very involved, and not yet made mathematically rigorous. In [St6], I used the Sundberg-Thistlethwaite method to improve Welsh’s upper bound on the rate of growth of the number of arbitrary links to about 10.3.

Using quite different methods, basing on the theory of Wicks forms, in joint work with A. Vdovina [SV], we determined the asymptotical behaviour of the number of alternating knots of given genus up to a scalar (depending on the genus).

All of these results are asymptotical and do not give exact formulas. The only so far known such formulas concern the special class of rational knots. In [ES], Ernst and Sumners gave formulas for the exact number of arbitrary and achiral rational knots and links of given crossing number. The method they applied is again different from the previously mentioned, and bases on Schubert’s classification [Sh] in terms of iterated (or continued) fractions.

In this paper, we will refine the results of Ernst and Sumners for knots by considering three further properties: positivity, fiberedness, and unknotting number one. Together with achirality, these four properties subdivide the class of rational knots into 16 subclasses, given by demanding or excluding any of the properties. Only some of these subclasses are easy to understand, since the properties defining them are causally dependent (for example, positivity and achirality are mutually exclusive). Still many of the classes are non-trivial, and apparently nothing about them was so far known. We will obtain a description of all of these classes, which allows to find an exact formula for their size by crossing number. (In case there are only few knots in the class, we will give them directly.) Usually, it will be most convenient to give the numbers by means of their generating functions, which turn out to be all rational functions (a new, rather unexpected, justification for the designation of these knots as rational).

Most interestingly, two of our enumeration problems turn out to be directly related to Fibonacci numbers. This way we have the possibly first explicit appearance of this common integer sequence in a knot theoretically related enumeration problem. A previous good candidate for such a problem was

the dimension of the space of primitive Vassiliev knot invariants by degree. The apparent occurrence of Fibonacci numbers therein originally led to some excitement, until computer calculation [BN] gave a disappointing result in degree 8, where the dimension in question was 12, and not 13. (Now this problem is known to be extremely hard and, if at all, will unlikely offer such an elegant solution, see [CD, Za].)

We start with some preliminaries in §1.1, which occupies the rest of this section, containing standard definitions, facts, and conventions. The enumeration results concerning knots with the aforementioned four properties will be discussed then in §2–4. Our method will be to study the effect of (combinations of) these properties on the form of the iterated fractions associated to the rational knots. It will be in particular decisive to understand, how the two normal forms, of all integers positive, and of all integers even, transform into each other. While the description of fibered and achiral rational knots in terms of their iterated fraction is classical, the property of unknotting number one has been made very approachable only by the more recent work of Kanenobu and Murakami [KM]. For positive rational knots a convenient description will have to be worked out below.

One of the two enumeration results involving Fibonacci numbers will be presented here first only as an inequality. We will remark that the other (reverse) inequality depends on the truth of a conjecture of Bleiler for fibered rational knots (henceforth considered and meant only for unknotting number one). Namely, in [B1] he conjectured that any rational knot realizes its unknotting number in a rational diagram corresponding to the expression of its iterated fraction with all integers even. This conjecture was disproved by Kanenobu and Murakami [KM], quoting the counterexample 8_{14} , which is not fibered. Since Bleiler’s conjecture now again turns out to be relevant in the fibered case, it will be the matter of new consideration.

In a note [St5], written after this paper was begun, but already published, we announce and complete the results of this paper, by giving a first proof of the “fibered” Bleiler conjecture. There we also formulate a statement about unimodular matrices which is related to this conjecture. For the historical reason to explain a part of the result of [St5], we include here a section §8. In this section we establish a relation between the conjecture a products of certain unimodular matrices, and use this relation to obtain some results related to it. Before this, in §7.2, we discuss how to enumerate counterexamples to Bleiler’s conjecture (of which the Kanenobu–Murakami knot 8_{14} is the simplest one), by classifying their even-integer notations. This also leads to a new proof of (a generalization of) the “fibered” Bleiler conjecture.

In §5 and §6, we give a few other formulas, including one determining the number of rational knots of given genus or signature. The formulas arising here contain several variables and are much more involved. Some of them are not rational, but all can be given in closed form. They yield by substitutions the Ernst–Sumners result, and also several formulas obtained previously in this paper. We apply an integration method allowing to build the generating function of the product of two sequences and to “select” certain parts of a multivariable generating function.

The final enumeration results will concern lens spaces by fundamental group, by using their correspondence to rational knots of given determinant, of which they are the 2-fold branched coverings. In the enumeration some exceptional (duplication) series of determinants occur, and the question whether they intersect non-trivially is related to the integer solutions of a certain hyperelliptic equation.

We conclude the introduction with a remark addressing rational links. We decided to leave them completely out of the discussion in this paper. One reason is that there will be already enough to say on knots. Secondly, at least most of the arguments can be adapted to links. (In fact, links occur naturally jointly with knots at some places, and we will have then to artificially get disposed of them.) However, for links also unpleasant questions connected with orientation come in, and would make the approach more technical than methodical.

1.1 Preliminaries and notations

The Fibonacci numbers F_n are a very popular integer sequence. These numbers can be defined recursively by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, explicitly by

$$F_{n-1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right],$$

and also by the generating function

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{1-x-x^2}.$$

See your favorite calculus textbook, or [Sl, sequence A000045] for an extensive compilation of references. Due to this simple property Fibonacci numbers appear very often in many unrelated situations and it is always amazing to see them come up in some mathematical problem.

$\#S = |S|$ are alternative designations for the cardinality of S .

A *knot* is a C^1 embedding $K : S^1 \hookrightarrow S^3$ (for convenience henceforth identified with its image) up to isotopy. Usually knots are represented by diagrams, plane curves (images of K under the projection of $\mathbf{R}^3 = S^3 \setminus \{*\}$ onto a generic hyperplane) with transverse self-intersections (crossings) and distinguished (over)crossing strand (a connected component of the preimage under K of a neighborhood of the crossing).

A knot K is called *fibred*, if $S^3 \setminus K$ is a bundle over S^1 with fiber being a Seifert surface for K , an embedded in S^3 punctured compact orientable surface S with $\partial S = K$ (see [Ga]). In this case, by the theorem of Neuwirth-Stallings, S has minimal genus among all Seifert surfaces for K (called the *genus* $g(K)$ of K) and is unique up to isotopy.

A knot has *unknotting number* one if it has some diagram, such that a crossing change $\times \rightarrow \times$, creates (a diagram of) the unknot (the knot with diagram \bigcirc). More generally, one defines the *unknotting number* $u(K)$ of a knot K as the minimal number of crossing changes in any diagram of K needed to turn K into the unknot (see e.g. [KM, Li]).

The *crossing number* of a knot is the minimal crossing number of all its diagrams.

A knot K is called *achiral* (or *amphicheiral*) if there exists an isotopy turning it into its mirror image in S^3 , otherwise K is called *chiral*.

The *writhe* is a number (± 1), assigned to any crossing in a link diagram. A crossing as in figure 1(a) has writhe 1 and is called *positive*. A crossing as in figure 1(b) has writhe -1 and is called *negative*.

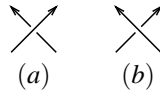


Figure 1

A knot is called *positive* if it has a positive diagram, i. e. a diagram with all crossings positive. See for example [Cr, CM, St].

The *braid index* of a knot is the minimal number of strands of a braid which closes up to the knot; see [Mu2]. In [Mu3], one finds a definition and properties of the *signature*.

A knot K is *rational* (or 2-bridge), if it has bridge number 2, where the bridge number is half of the smallest number of critical points of a Morse function on K . In [Sh], rational knots have been

classified by the iterated fractions corresponding to their *Conway notation* [Co]. See Goldman and Kauffman [GK] for a more modern account.

Let the *iterated fraction* $[[s_1, \dots, s_m]]$ for integers s_i be defined inductively by $[[s]] = s$ and

$$[[s_1, s_2, \dots]] = s_1 + \frac{1}{[[s_2, \dots]]}.$$

Note: there is another convention of building iterated fractions, in which the ‘+’ above is replaced by a ‘−’. See e.g. [Mu2]. Latter is more natural in some sense (see the proof of theorem 7.2), but the permanent sign switch makes it (at least for me) more unpleasant to work with in practice. Thus we stick to the version with ‘+’.

The rational knot or link $S(p, q)$ in Schubert’s [Sh] notation has the Conway [Co] notation $c_n c_{n-1} \dots c_1$, when the c_i are chosen so that

$$[[c_1, c_2, c_3, \dots, c_n]] = \frac{p}{q}. \quad (1)$$

Since when replacing integers with variables the Conway notation $a_1 a_2 \dots a_n$ in its original form becomes somewhat illegible, we will sometimes put the sequence into parentheses, or also use the alternative designation $C(a_1, \dots, a_n)$ for this sequence. Thus $S(p, q) = C(c_n, c_{n-1}, \dots, c_1)$. We also abbreviate repeating subsequences as powers, for example $(4(12)^2 1^3 3) = (412121113)$. We call the numbers c_i also *Conway coefficients* of the notation.

Note that $S(-p, -q)$ is the same knot or link as $S(p, q)$, while $S(-p, q) = S(p, -q)$ is its mirror image. $S(p, q)$ is a knot for p odd and a 2-component link for p even. The number p is the *determinant* of K , given by $|\Delta_K(-1)|$, where Δ is the Alexander polynomial [Al].

Without loss of generality one can assume that $(p, q) = 1$, $|q| < |p|$, and that (exactly) one of p and q is even. (If both p and q are odd, we replace q by $q \pm |p|$, the sign being determined by the condition $|q| < |p|$.)

Then we can choose all c_i in (1) to be even (and non-zero). It is known that, with this choice of c_i , their number $n = 2g(S(p, q))$ is equal to twice the genus of $S(p, q)$ for p odd (i.e. a rational knot). To fix a possible ambiguity between a diagram and its mirror image, we consider the crossings corresponding to the entry c_i in the Conway notation to have writhe $(-1)^{i-1} \text{sgn}(c_i)$.

For the purpose of calculating with iterated fractions, it will be helpful to extend the operations ‘+’ and ‘1/.’ to $\mathbf{Q} \cup \{\infty\}$ by $1/0 = \infty$, $1/\infty = 0$, $k + \infty = \infty$ for any $k \in \mathbf{Q}$. The reader may think of ∞ as the fraction $1/0$, to which one applies the usual rules of fraction arithmetics and reducing. In particular reducing tells that $-1/0 = 1/0$ so that for us $-\infty = \infty$. This may appear at first glance strange, but has a natural interpretation in the rational tangle context.

It will be helpful to introduce some notation for subsequences of the sequence of integers giving the Conway notation for some rational knot. We most commonly denote such subsequences by letters towards the end of the alphabet like x or y , while single integers will be called a, b, \dots . Define for a finite sequence of integers $x = (a_1, \dots, a_n)$ its *reversion* (or *transposition*) $\bar{x} := (a_n, \dots, a_1)$ and its *negation* by $-x := (-a_1, \dots, -a_n)$. If $x = \pm \bar{x}$, we call x (*anti*)*palindromic*. For $y = (b_1, \dots, b_m)$ the term x, y denotes the concatenation of both sequences $(a_1, \dots, a_n, b_1, \dots, b_m)$. Similarly one defines concatenation with a single integer, for example $x, b = (a_1, \dots, a_n, b)$ etc.

Figure 2 shows how to obtain a diagram of the rational knot or link from its Conway notation. It is the closure of the rational tangle with the same notation. The convention in composing the tangles is that a Conway notation with no negative integers gives an alternating diagram. For more details see [Ad, §2.3].

Since by [Ka, Mu, Th] (reduced) alternating diagrams have minimal crossing number, the crossing number of a rational knot is the sum of the integers in its Conway notation with all integers positive.

A good reference on generating function theory is [Wi].

Before we start with our results, we make two remarks.

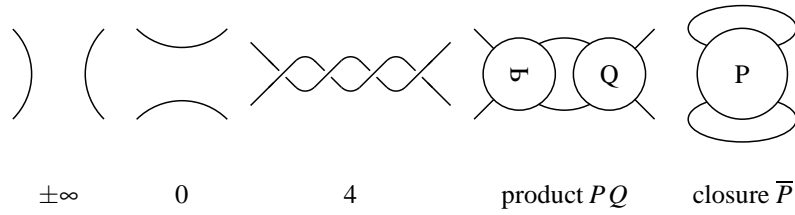


Figure 2: Conway's primitive tangles and tangle operations.

First, we will adopt the convention of considering rational knots *up to mirroring*. This is, a rational knot and its mirror image will be considered equivalent, and hence counted once. Since we will have formulas for the number of rational knots with specific properties (counted up to mirroring) and for the number of achiral rational knots with the same specific properties, one can easily obtain from both numbers the number of rational knots with the same properties with chiral knots and their mirror images counted separately. An exception to this convention will be §6 and §7.1, where the sensitivity of the signature under mirroring forces care to be taken. There we will specify in each statement whether we count chiral pairs once or twice.

When considering rational knots up to mirroring, the Conway notation with all integers positive is determined up to reversal and the up to the ambiguity $\dots, n-1, 1 \leftrightarrow \dots, n$ at the end. The Conway notation with all integers even and non-zero is determined up to reversal and the simultaneous negation of all entries. In both cases the reversal of notation corresponds to the identity $S(p, q) = S(p, \pm q^{-1})$. Here q^{-1} is the multiplicative inverse of q in \mathbf{Z}_p^* , the group of units of $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$, and the sign is positive or negative depending on whether the Conway notation has odd or even length.

Also, we will content ourselves only with interesting combinations of the four properties. For some of the remaining combinations, the results are known, sometimes even in greater generality than just for rational knots. We refer to [St2, St] for the treatment of these cases, and do not consider them here. (One could certainly prove some of these results also from our setting, but such an attempt does not seem any longer relevant.) For other combinations of properties, the description easily follows from what we will prove below. For some of them, a subset of the properties already restricts sufficiently the class, and the check of the remaining properties on the few knots is easy. The remaining enumerations follow by simple inclusion-exclusion arguments. In such cases we mostly waive on presenting the results explicitly here and leave them to the reader.

2 Rational knots of unknotting number one

We start with the description and enumeration of rational knots of unknotting number one for given crossing number. Here, unlike in subsequent sections, we first use the notation with positive Conway coefficients. (We will return to unknotting number one later, when armed with a more effective method.)

Theorem 2.1 If K is a rational knot of unknotting number one, then it has a Conway notation with all coefficients positive, which is in at least one of the types listed below. (In the first five cases the entry ‘ -1 ’ indicates the crossing to be switched to unknot the knot.)

- i) $x, n, -1, 1, n-1, \bar{x}, c$
- ii) $x, n-1, 1, -1, n, \bar{x}, c$
- iii) $a, x, n, -1, 1, n-1, \bar{x}, a \pm 1$

- iv) $n + 1, -1, 1, n - 1, 1, c$ and $n - 1, 1, -1, n - 1, 1, c$ (degenerate cases of i) and ii))
- v) $n - 1, 1, -1, n \pm 1$ (degenerate case of iii))
- vi) $2, n$

Here x denotes a (possibly empty) sequence of positive integers, and a, c and n are positive integers, so that all entries in the above sequences (except the ‘ -1 ’) are positive. Also, unlike elsewhere, x is considered up to the ambiguity $n, \dots = 1, n - 1, \dots$ for $n > 1$. (Thus for example the sequence $(5, 2, 3, 1, 1, 4, 2, 4, 1, 7)$ is considered of type ii) with $c = 7, n = 4$ and $x = (5, 2) = (1, 4, 2)$.)

Proof. It was proved in [St4, §3.1], that a rational knot of unknotting number one has an alternating diagram of unknotting number one, and hence all alternating diagrams have this property. Consider the alternating diagram of the Conway notation with all integers a_1, \dots, a_k positive. If the (unknotting) crossing change occurs in a group of ≥ 2 half-twists, then the only such case is vi). Else we need to switch ‘ 1 ’ \rightarrow ‘ -1 ’. In this case after this change we obtain modulo mirroring a closed rational tangle with iterated fraction $1/n$ for some $n \in \mathbf{N}$. Modulo transposition of the notation, we may assume $n = \pm 1$ (case iii)) or that the (sub)tangle with Conway notation a_1, \dots, a_{k-1} turns into the 0-tangle under the crossing change (giving cases i) and ii) with $c = a_k$). The almost-symmetry in the first three cases arises when analyzing the iterated fraction from left and right until the crossing changed. Up to a correction $n, \dots \rightarrow 1, n - 1, \dots$ in their inner ends, and the ambiguity $\dots, p = \dots, p - 1, 1$ at the outer ends (because only their iterated fraction is relevant) they must be transposed. This explains the occurrence of x and \bar{x} . (The ambiguity at the outer end of \bar{x} changes the knot if not at outermost position in the notation.) The degenerate cases iv) and v) occur when the fraction expression has length one. \square

From the theorem (and the lack of essential restrictions to x) the enumeration of unknotting number one rational knots of given crossing number is straightforward (but rather tedious by virtue of having to take care of duplicatedly counted cases and the ambiguity for x). Thus it is clear how to obtain the following corollary, which was suggested empirically. However, instead of going now into unpleasant details, we will later give a much more elegant proof.

Corollary 2.1 If c_n denotes the number of rational unknotting number one knots of c crossings (chiral pairs counted only once), then these numbers are given basically by powers of 2, namely via the generating (rational) function

$$\sum_{n=1}^{\infty} c_n x^n = x^3 + x^4(x+1) \left[\frac{2}{1-2x^2} + \frac{1}{x^2-1} \right] + \frac{x^8}{x^4-1}.$$

In particular, $\lim_{n \rightarrow \infty} \sqrt[n]{c_n} = \sqrt{2}$.

It is worth mentioning that for every fourth crossing number the number of rational unknotting number one knots does not increase compared to the next crossing number – this is possibly not what one may expect!

Corollary 2.2 The number of achiral unknotting number one rational knots of c crossings is 2 for $c = 10 + 6k, k \geq 0$, and 1 for other even $c \geq 4$. More exactly, these knots are those with Conway notation $(n11n)$ and $(3(12)^k 1^4 (21)^k 3)$.

Proof. It is known that $C(a_1, \dots, a_n)$ with all $a_i > 0$ is achiral iff the sequence a_1, \dots, a_n is (up to the ambiguity $\dots, n - 1, 1 \leftrightarrow \dots, n$) palindromic of even length. The result then is a direct verification of the palindromicity of the patterns of the above 6 cases. The series $(n11n)$ clearly comes from case v), while $(3(12)^k 1^4 (21)^k 3)$ for $k > 0$ is in case i) and for $k = 0$ in case iv). The other cases only give at best alternative representations for $4_1 = (22)$ and $6_3 = (2112)$. \square

Corollary 2.3 Except for the trefoil and figure eight knot, all unknotting number one rational knots have in their alternating diagrams at most two crossings, such that switching any single one of them unknots the knot.

Proof. The theorem shows that if a ‘1’ is changed to ‘−1’ to unknot, then the number of integers left and right from it differs by at most three. This leaves at most 4 (neighbored) positions. The degenerate cases are easily excluded, and considering i), ii) and iii), one finds that only the edge ‘1’ in a subsequence of ‘1’s can unknot, and at most one of these edge ‘1’s does, if the sequence is of length two (except for case v)). \square

Clearly the knots where (exactly) two such crossings exist include the achiral ones given in corollary 2.2. We leave it to the reader to modify the proof of corollary 2.2 and to show that the remaining knots are of the forms $(32^k 1^3 2^{k+1})$ and $(2^k 1^3 2^k)$. (This result was again suggested by computer calculation, and I have not carried out a rigorous proof.)

3 Fibered rational knots

For the following results it is more convenient to work with the (unique up to reversal and negation) expression of the iterated fraction by even (non-zero) integers rather than natural numbers. (The number of all these even integers is always even and equal to the double genus of the knot.) The key point is how to extract the crossing number out of this representation. The result is given in the following lemma, which will be of central importance throughout the rest of the paper.

Lemma 3.1 If a_1, \dots, a_{2g} are even (non-zero) integers, then the crossing number of $C(a_1, \dots, a_{2g})$ is

$$\sum_{i=1}^{2g} |a_i| - \#\{1 \leq i < 2g : a_i a_{i+1} < 0\}.$$

(In fact, the formula still holds if all $|a_i| \geq 2$ not necessarily even.)

Proof. We remarked that the crossing number result for alternating diagrams [Ka, Mu, Th] implies that the crossing number of a rational knot is the sum of the integers in its Conway notation with all integers positive. Thus we need to account for the change of the sum of the $|a_i|$, when transforming the Conway notation with all integers even into the one with all integers positive. This is a repeated application of the iterated fraction identity $[[x, a, -b, y]] = [[x, a-1, 1, b-1, -y]]$ (with x and y subsequences and a and b integers). The claim then follows by induction on the number of such applications needed. \square

Theorem 3.1 If c_n denotes the number of fibered rational knots of n crossings (chiral pairs counted only once), then these numbers are given by the generating function

$$\sum_{n=1}^{\infty} c_n x^n = -\frac{x^3(1+x)(x^4+x^3+x^2-1)}{(x^4+2x^3+x^2-1)(x^4+x^2-1)}.$$

In particular, $\lim_{n \rightarrow \infty} \sqrt[n]{c_n} = \frac{1+\sqrt{5}}{2}$.

The proof is a prototype of argument that will occur in several more complex variations later.

Proof. A rational knot is fibered iff all even integers a_i in its iterated fraction expression are ± 2 . This is a well-known fact which seems to have been (algebraically) noted explicitly in this form first by Lines and Weber [LW], although it is also a consequence of the (much older) result of Murasugi

[Mu4], as we shall briefly argue. A geometric proof can be also given, for example using the method of [Ga].

The diagram of the closure of a rational tangle with all integers even is the Murasugi sum of connected sums of Hopf bands with $a_i/2$ full twists. Thus from [Mu4] the multiplicativity of the leading coefficient $\max \text{cf} \Delta$ of the Alexander polynomial under Murasugi sum implies

$$\max \text{cf} \Delta_{C(a_1, \dots, a_{2g})} = \pm 2^{-2g} \prod_{i=1}^{2g} a_i. \quad (2)$$

If the knot is fibered, $\max \text{cf} \Delta = \pm 1$, and hence all $a_i = \pm 2$. Contrarily, if all $a_i = \pm 2$, the knot has a surface which is a plumbing of Hopf bands with one full twist each, and hence a fiber surface.

In the case a rational knot is fibered, each $a_i \neq \pm 2$, except the first one, according to lemma 3.1, contributes one to the crossing number of the knot, if it follows a ± 2 of the different sign, and two otherwise. Thus, by ignoring the contribution of the first ± 2 , we are left by counting compositions into parts 1 and 2 of $n - 2$ of odd length up to transposition. (A composition of a certain number is writing it as a sum of numbers, whose order is relevant.)

To pass from this to the generating function of the theorem is a matter of some combinatorial calculation. One uses the generating function

$$f_1(x) = \frac{x + x^2}{1 - (x + x^2)^2},$$

for the number of odd length compositions into parts 1 and 2, and

$$f_2(x) = \frac{x + x^2}{1 - x^2 - x^4}$$

for the number of palindromic ones.

If we fix the first number $a_1 = 2$ up to mirroring, the notations define the same knot iff they differ by transposition and possible negation (so as the initial term to become positive).

In this situation, f_1 counts all knots we like by their notations twice, except the ones with palindromic and antipalindromic notations. These are enumerated exactly by f_2 . To see this, one needs to remark that the sequence of 1's and 2's contributing from each a_i to the crossing number, with the initial 2 coming from a_1 omitted, is palindromic iff the Conway notation made up of the a_i (without the initial one a_1 omitted) is palindromic or antipalindromic.

Thus the generating function we seek is simply $(f_1 + f_2)/2$. □

Remark 3.1 One can, of course, give using partial fraction decomposition an explicit formula for the c_n in terms of (negative powers of) the zeros of the denominator polynomial of the generating function, from which the limit property (that is, the justification to write above 'lim' rather than 'lim sup') follows, but the resulting expression should be less pleasant, so we waive on its derivation.

Remark 3.2 One could, in a similar way, show that the number of rational knots K with $\max \text{cf} \Delta_K$ being up to sign a fixed natural number n give the Taylor coefficients of a rational function. The complexity of this function will roughly depend on the complexity of the prime decomposition of n . This relies on the fact that for a_1, \dots, a_{2g} even (and non-zero), we have the relation (2).

The fact that we count compositions into parts 1 and 2 already suggests the relation to Fibonacci numbers. Now comes the enumeration result where they appear explicitly.

Theorem 3.2 The number of rational fibered achiral knots of n crossings is $F_{n/2-2}$ for n even (and 0 for n odd).

Proof. It is known, see [Sh], that a rational knot is achiral iff its Conway notation with all integers even is palindromic. Then it follows directly from the lemma that the crossing number must be even (this follows more generally for alternating knots from [Ka, Mu, Th]). Considering only the first half of the (palindromic) sequence, we see that again we count compositions into parts 1 and 2, this time of $1/2(n-4)$, but neither the restriction on the number of parts (genus -1), nor the factoring out of transpositions are necessary. Thus the result follows. \square

The Fibonacci numbers also occur when considering rational fibered unknotting number one knots.

Theorem 3.3 The number of rational fibered unknotting number one knots of n crossings is $2F_{\lfloor n/2-3 \rfloor}$ for $n \geq 6$.

As most of the results before, theorem 3.3 was also suggested by computer, which calculated the various sequences above up to 26 crossings, and confirmed for this sequence equality with the doubled Fibonacci numbers. The completion of the proof depends on the truth of the ‘‘fibered’’ Bleiler conjecture. This is motivation enough to come back to this problem in more detail towards the end of the paper.

Proof of theorem 3.3. Consider the diagrams $C(\pm 2, x, \pm 2, -\bar{x})$, x being a sequence of ± 2 ’s and $-\bar{x}$ its negated transposed. Clearly such knots unknotted by switching a crossing counted by the ± 2 between x and $-\bar{x}$. It is also easy to see that if a diagram $C(b_1, \dots, b_{2g})$ with all b_i even and non-zero is to be unknotted by one crossing change, it must be of this form. The only possible cancellations near a zero entry are of the form

$$(\dots, a, 0, b, \dots) = (\dots, a + b, \dots), \quad (3)$$

and when only non-zero entries remain, the notation does not represent the unknot.

Thus from now on consider Conway notations of the form

$$C(a_0, a_1, \dots, a_k, \pm 2, -a_k, \dots, -a_1), \quad (4)$$

with all $a_i = \pm 2$. Our concern will be to count such notations by crossing number, as given in the lemma 3.1.

Now, for a given knot, the Conway notation with all numbers even is unique up to negating all numbers and transposition. In order to avoid duplicate counting, we must take care what notations still fit into the form (4) after some of these transformations.

Clearly, negating all numbers preserves the form (4), but to get disposed of this transformation, we can simply declare that we count only forms with $a_0 > 0$.

Then we must find out which sequences (4) remain of this form after transposition. For such sequences one sees that the first and last entries determine the rest of the sequence. Since we restricted ourselves only to sequences with $a_0 > 0$, we see that demanding $a_0 = -a_1 = 2$ forces the sequence to become palindromic, and hence it is not counted twice. (This sequence then corresponds to the knots with notation $(n11n)$ given in corollary 2.2.)

Now we can apply the previous arguments. Again one counts compositions into parts 1 and 2 coming from the subsequence $x = (a_1, \dots, a_k)$, and the equality of the numbers one obtains for n and $n+1$ if n is even comes from the switch of signs in x together with the sign of the middle ± 2 . Switching just the sign of the middle ± 2 , fixing x , accounts for the factor 2.

We proved so far that there are *at least* as many knots as we claimed in the formulation of theorem 3.3. To remove that ‘at least’, one needs that any fibered rational knot of unknotting number one should realize its unknotting number in a rational diagram of all Conway coefficients even. This was conjectured for arbitrary rational knots by Bleiler [Bl], but disproved by Kanenobu and Murakami [KM], quoting the counterexample 8_{14} (which, however, is not fibered). Thus, the confirmation of Bleiler’s conjecture for fibered rational knots and unknotting number one is equivalent to establishing equality in (and completing the proof of) the above theorem. As noted, the statement we require was proved in [St5], but another and more generalized proof (which will also lead to generalizations of this theorem) will be given in §7.2. \square

Remark 3.3 To describe the fibered rational knots which are both of unknotting number one and achiral, one uses corollary 2.2. The knots in the first family there are fibered (they are closed alternating 3-braids), while those in the second family are not. To see latter fact, the reader may convince himself, that the crossings counted by the initial and terminal ‘3’ in the Conway notation correspond to reverse(ly oriented) half-twists:



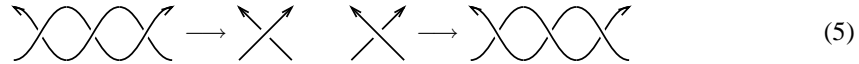
It follows from the description of $\max cf \Delta$ on alternating diagrams given in [Cr] that an alternating diagram with ≥ 3 reverse half-twist crossings always has $|\max cf \Delta| > 1$, and hence never represents a fibered link.

4 Positive rational knots

Positive knots (see [Cr, CM, St, Yo] for example) have been around for a while in knot theory, but apparently no special attention was given to the rational ones among them. We start by a description of such knots, again using the expression with all Conway coefficients even.

Lemma 4.1 If a_1, \dots, a_{2g} are even integers, then the rational knot $C(a_1, \dots, a_{2g})$ is positive, iff all a_i alternate in sign, i.e. $\#\{1 \leq i < 2g : a_i a_{i+1} < 0\} = 2g - 1$.

Proof. If a rational (or alternating) knot is positive, then by [St3, N] so is any of its alternating diagrams, and hence by [Mu3], $\sigma = 2g$ (where σ is the *signature* and g the genus). If all a_i alternate in sign, then $C(a_1, \dots, a_{2g})$ is a positive diagram. However, if some a_i has the wrong sign, then $C(a_1, \dots, a_{2g})$ can be obtained from a positive diagram by undoing positive/creating negative reverse twists.



Any of these moves does not augment σ . Moreover, as in this process at least once some $a_i = 0$, giving a knot of smaller genus, and as $\sigma \leq 2g$, σ strictly decreases. Then $C(a_1, \dots, a_{2g})$ has $\sigma < 2g$, and the knot is not positive. \square

Theorem 4.1 If c_n denotes the number of rational positive knots of n crossings, then these numbers are given by the generating function

$$\sum_{n=1}^{\infty} c_n x^n = \frac{x^3 - 2x^5}{(1 - 3x^2 + x^4)(1 - x^2 - x^4)}.$$

In particular, all positive rational knots have odd crossing number, and $\lim_{n \rightarrow \infty} \sqrt[2n+1]{c_{2n+1}} = \frac{1 + \sqrt{5}}{2}$.

Proof. Since now, by the lemma, if $C(a_1, \dots, a_{2g})$ is positive, all a_i , $i > 1$ contribute $a_i - 1$ to the crossing number of K , by artificially decreasing a_1 by 1, we are left with counting compositions of $n - 1$ into (an even number of) odd parts up to transposition.

Without factoring out transpositions, the number of compositions of n of this type is given by the generating function

$$\frac{1}{1 - \left(\frac{x}{1-x^2}\right)^2}.$$

Table 1: The number of rational knots of $n \leq 26$ crossings with some combinations of the properties achirality, unknotting number one, positivity, and fiberedness (the combination is indicated by joining the initials of the properties considered). The last line contains the number of rational knots of zero signature, whose determination will be explained later.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
f	1	1	1	2	3	4	7	10	16	25	40	62	101	159	257	410	663	1062	1719	2764	4472	7209	11664	18828
fa		1		1		2		3		5		8		13		21		34		55		89		144
u	1	1	1	3	3	6	7	15	15	30	31	63	63	126	127	255	255	510	511	1023	1023	2046	2047	4095
au		1		1		1		2		1		1		2		1		1		2		1		1
fu	1	1		2	2	2	2	4	4	6	6	10	10	16	16	26	26	42	42	68	68	110	110	178
p	1		2		5		12		30		76		195		504		1309		3410		8900		23256	
σ_0		1		3	2	9	6	29	30	99	112	351	450	1275	1734	4707	6762	17577	26208	66197	101862	250953	395804	956385

The number of palindromic compositions is the same as the number of compositions of $n/2$ into (a now not necessarily even number of) odd parts, whose generating function is

$$\frac{1}{1 - \frac{x^2}{1-x^4}}.$$

Thus, accounting for the unknot, we have

$$\sum_{n=1}^{\infty} c_n x^n = -x + \frac{x}{2} \left[\frac{1}{1 - \left(\frac{x}{1-x^2}\right)^2} + \frac{1}{1 - \frac{x^2}{1-x^4}} \right],$$

whence the result. \square

The theorem roughly suggests that there should be approximately qualitatively equally many positive and fibered rational knots up to a given crossing number. This should be contrasted to the distribution of their iterated fractions: while $\{p/q : S(p,q) \text{ is positive}\}$ should be dense in $\mathbf{R} \setminus (-1, 1)$, the closure of $\{p/q : S(p,q) \text{ is fibered}\}$ will have dense *complement* (possibly even zero Lebesgue measure¹).

Of course, that positive knots have odd crossing number is not true even for prime alternating knots, as shows 8_{15} .

Corollary 4.1 The only fibered positive rational knots are the $(2, n)$ -torus knots for n odd.

Proof. Combining the lemma with the fiberedness property, we obtain a notation $C((2, -2)^g)$, which belongs to the $(2, 2g + 1)$ torus knot. \square

This shows that almost all fibered positive knots are not rational. In [Mu2], Murasugi mainly settled the problem of non-alternation (so in particular non-rationality) for closed positive braids. However, he needs the technical assumption that such knots have positive braid representation of minimal strand number, and moreover, not all fibered positive knots are closures of positive braids, as shows the example 10_{161} discussed in [St, example 4.2]. In [St3], we generalize corollary 4.1 to alternating knots and links.

Table 1 summarizes some of the numbers discussed above.

The previous arguments can be applied to several similar enumeration problems. We discuss in some detail how to obtain the number of rational knots of given genus and/or given signature.

5 Genus

For the genus, one can prove

Theorem 5.1 If $c_{n,g}$ is the number of rational knots of n crossings and genus g , then

$$f(x, z) = \sum_{g=1}^{\infty} \sum_{n=3}^{\infty} c_{n,g} x^n z^g = - \frac{x^3 z (-1 + x^3 z + x^4 z + x^2 (1 + z))}{(1 + x) (1 + x^2) (-1 + 2x + x^2 (-1 + z)) (-1 + x^2 (1 + z))} \quad (6)$$

is a rational function in x and z .

¹According to a remark of A. Sikora, not any complement of an open dense subset must have zero measure. In fact, there are open dense subsets in \mathbf{R} of arbitrarily small positive measure!

This is a similar, but slightly stronger, analogue of a result of [St2], where we showed that

$$f_g(x) = \sum_{n=3}^{\infty} c'_{n,g} x^n$$

is a rational function in x for fixed g , with $c'_{n,g}$ being the number of *alternating* knots of n crossings and genus g . However, the dependence of f_g on g is too complicated to let expect any nice (in particular, rational) expression for the two-variable function (6) in the alternating case.

Proof of theorem 5.1. To prove the assertion for the genus, use that it is one half of the number of entries in the even Conway notation. Let $w = (a_1, \dots, a_{2g})$ be the sequence of these entries. Assume $a_1 > 0$ to factor out one of the ambiguities. Define as before a transformation of w to a sequence \hat{w} of positive integers by $\hat{w} = (b_1, b_2, \dots, b_{2g})$, such that

$$b_i = \begin{cases} |a_i| & \text{if } i = 1 \text{ or } a_{i-1}a_i > 0 \\ |a_i| - 1 & \text{otherwise} \end{cases}.$$

Every sequence of positive integers with the first one even has a unique preimage under $\hat{\cdot}$. We are interested in counting those sequences w such that the sum of entries of \hat{w} is n . Since $\hat{\cdot}$ is injective, this is the same as counting compositions of n into $2g$ (positive integer) parts, the first one being even. If $a_{n,g}$ is the number of such compositions, then

$$g(x, z) = \sum_{g=1}^{\infty} \sum_{n=1}^{\infty} a_{n,g} x^n z^g = \frac{x}{1+x} \left(\frac{1}{1 - \frac{zx^2}{(1-x)^2}} - 1 \right).$$

We count now every sequence w once, but still there are different sequences giving the same knot, coming from the ambiguity of reversing the notation. Namely, this always happens except if the sequence w is palindromic ($w = \bar{w}$) or anti-palindromic ($w = -\bar{w}$). Let y be the first half of w (of length g). Then \hat{y} has a sum of entries either $n/2$ if w is palindromic, or $(n+1)/2$, if w is anti-palindromic.

Thus for given n , only palindromic or only anti-palindromic sequences w occur, and their number is the same as the number of compositions of $\lceil \frac{n}{2} \rceil$ of length g with the first integer being even.

If $b_{n,g}$ is the number of compositions of n of length g with the first integer being even, then

$$h(x, z) = \sum_{g=1}^{\infty} \sum_{n=1}^{\infty} b_{n,g} x^n z^g = \frac{x}{1+x} \left(\frac{1}{1 - \frac{zx}{1-x}} - 1 \right).$$

To replace n by $\lceil \frac{n}{2} \rceil$, one has to divide by x , replace x by x^2 , and multiply by $x + x^2$.

$$h_1(x, z) = \frac{x + x^2}{1 + x^2} \left(\frac{1}{1 - \frac{zx^2}{1 - x^2}} - 1 \right).$$

Then $f(x, z) = 1/2(g(x, z) + h_1(x, z))$. □

Remark 5.1 Since the even-degree- x part of h_1 counts the palindromic sequences w , which correspond exactly to the achiral knots, $1/2(h_1(x, z) + h_1(-x, z))$ gives the function enumerating achiral knots by genus.

Using the result of Murasugi [Mu2, Theorem B (2)], one can obtain a similar formula for counting by crossing number and braid index:

Theorem 5.2 If $c_{n,b}$ is the number of rational knots of n crossings and braid index b , then

$$\sum_{b=2}^{\infty} \sum_{n=3}^{\infty} c_{n,b} x^n z^b = - \frac{x^3 z^2 (-1 - xz + 2x^4 z^2 + x^5 z^3 + x^2 (1+z) + x^3 z (2+z))}{(1+x) (-1+x+2x^2 z) (-1+x^2+2x^4 z^2)}.$$

Proof. The proof is analogous and largely omitted. (Take, however, care of the different convention for building iterated fractions.) We remark only that instead of the previous function g we must take

$$\frac{xz}{1+xz} \left(\frac{1}{1 - \left[\left(1 + \frac{1}{xz}\right) \left(\frac{1}{1-x^2 z} - 1\right) \right]^2} - 1 \right),$$

and instead of h_1

$$\frac{xz(1+xz)}{1+x^2 z^2} \left(\frac{1}{1 - \left(1 + \frac{1}{x^2 z^2}\right) \left(\frac{1}{1-x^4 z^2} - 1\right)} - 1 \right). \quad \square$$

One can also count by genus and braid index *without* incorporating the crossing number, as one observes that for given genus and braid index there are only finitely many rational knots (a fact which can be proved in larger generality). The discussion so far should explain sufficiently how to proceed, so that we leave this task to an interested reader.

6 Signature

Another variation of the enumeration problem (for which an analogue for alternating knots, if it exists, is even harder to prove) is to count rational knots by signature σ .

One has the following formula for the signature (see [HNK, p. 71]):

Lemma 6.1

$$\sigma(C(a_1, \dots, a_{2g})) = \sum_{i=1}^{2g} (-1)^{i-1} \text{sgn}(a_i), \quad (7)$$

for $a_i \neq 0$ all even.

This formula shows a close relationship between signature and genus. Thus in this case we must again take care of the genus, and so this is a refinement of the enumeration by genus. Set in the sequel for simplicity $\chi' = 1 - \chi = 2g$.

Since now σ depends on mirroring and takes negative values, we must be careful about what and how exactly we are going to count.

There are several options how to avoid the chirality and the negative value problems.

- 1) It appears suggestive to count chiral pairs twice, since both knots give distinct contributions, and there is no natural way to distinguish one of them. Then we must deal with negative values of σ , since it is desirable to avoid negative powers in the generating series. There are also two options:

- a) we count knots by $|\sigma| \in [0, \chi']$, or
 - b) we count knots by $\sigma + \chi' \in [0, 2\chi']$.
- 2) Alternatively, but less naturally, one can declare to count in a chiral pair only the knot with $\sigma > 0$, and if both knots have $\sigma = 0$, to take any one of both, since their contribution is the same. This has the advantage of also eliminating the $\sigma < 0$ problem.

We will thus for the rest of §6 specify according to what convention we count rational knots by signature, and in particular whether we count chiral pairs once or twice.

6.1 σ with mirroring

First we will deal with the version 1b). It can be approached most naturally and leads to the “simplest” generating series. It fits into the picture we described throughout the preceding discussion.

Theorem 6.1 Let G_1 be the function in 3 variables x, y and z which counts in its Taylor coefficient of $x^m y^l z^k$ the number of rational knots of crossing number m with $1 - \chi = l$ and $1 - \chi + \sigma = k$, such that (unlike so far in the paper) *both* knots in a chiral pair are counted. Then G_1 is a certain rational function (shown in full form in figure 3).

After our proof we will indicate how to proceed with enumeration version 2), whose function can be expressed from the one of version 1b) by means of a certain complex integral (and thus is no longer rational). The function for enumeration version 1a) is obtained similarly, and so we do not present it here.

Proof. Let us start as before. Consider again a sequence w of even integers $w = (a_1, \dots, a_{2g})$ with $a_1 > 0$, and the associated sequence \hat{w} . The formula for σ in the lemma can be read as follows in terms of \hat{w} : subdivide \hat{w} into subsequences starting with an even integer, followed by some (possibly empty) sequence of odd integers. Each such subsequence contributes its length with alternating sign to the signature. Call a subsequence σ -positive or σ -negative dependingly on the sign of its contribution to σ .

Let

$$\hat{F}(x, y, z) = yz \frac{x^2}{1-x^2} \left(\frac{1}{1-y \frac{xz}{1-x^2}} \right).$$

By the previous arguments we see that

$$F_1(x, y) = \hat{F}(x, y, 1)$$

counts a single σ -negative group of entries by χ' in (powers of) y and crossing number in x (here $\chi' + \sigma = 0$). Similarly

$$F_2(x, y, z) = \hat{F}(x, y, z^2)$$

counts a single σ -positive group of entries by χ' in y , crossing number in x and $\chi' + \sigma$ in z .

Now \hat{w} is made up of an arbitrary number of interchangingly positive and negative subsequences, starting with a positive one. Thus to count \hat{w} we consider

$$\mathring{F}(x, y, z) = \left(1 + \frac{1}{F_2} \right) \frac{F_1 F_2}{1 - F_1 F_2},$$

which counts an arbitrary sequence of σ -positive/negative groups by χ' in y and crossing number in x . This function now contains odd powers of y (=values of χ'). They are discarded by setting

$$F(x, y, z) = \frac{\mathring{F}(x, y, z) + \mathring{F}(x, -y, z)}{2},$$

Figure 3

$$\begin{aligned}
G_1(x,y,z) &= -x^3 y^2 \left(-1 - z^4 + x^8 z^2 (-1 + y^2 z^2)^2 (1 + y^2 z^2) - x (1 + z^2 + z^4) + x^6 z^2 \left(1 + 2y^6 z^6 + 2y^2 (1 + z^4) - y^4 z^2 (2 + 3z^2 + 2z^4) \right) \right) + \\
&+ x^7 z^2 \left(1 + y^6 z^6 + y^2 (1 + z^2 + z^4) - y^4 (z^2 + 3z^4 + z^6) \right) + x^2 \left(-z^2 + y^2 (1 + 4z^4 + z^8) \right) + x^3 \left(-z^2 + y^2 (1 + z^2 + 3z^4 + z^6 + z^8) \right) + \\
&+ x^5 \left(1 + z^2 + z^4 - y^4 z^4 (2 + z^2 + 2z^4) + y^2 (1 + 2z^2 + 2z^6 + z^8) \right) + x^4 \left(1 - z^2 + z^4 - 3y^4 (z^4 + z^8) + y^2 (1 + 2z^2 + z^4 + 2z^6 + z^8) \right) \times \\
&\times \frac{1}{(1+x)(1+x^2)(1-xy(1+z^2)+x^2(-1+y^2z^2))(1+xy(1+z^2)+x^2(-1+y^2z^2))(1-x^2y^2(1+z^4)+x^4(-1+y^4z^4))} \\
\\
f_0(x) &= \frac{-x}{2(1+x)(1+x^2)\sqrt{(-1+4x^2)(-1+4x^4)}} \left(-\sqrt{1-4x^2} + 2x^5\sqrt{1-4x^2} - \sqrt{1-4x^4} + 2\sqrt{(-1+4x^2)(-1+4x^4)} + \right. \\
&+ 2x^4 \left(\sqrt{1-4x^2} + \sqrt{1-4x^4} \right) - x \left(\sqrt{1-4x^2} + \sqrt{1-4x^4} - 2\sqrt{(-1+4x^2)(-1+4x^4)} \right) - \\
&\left. -x^3 \left(\sqrt{1-4x^2} + \sqrt{1-4x^4} - 2\sqrt{(-1+4x^2)(-1+4x^4)} \right) + x^2 \left(-\sqrt{1-4x^2} + \sqrt{1-4x^4} + 2\sqrt{(-1+4x^2)(-1+4x^4)} \right) \right)
\end{aligned}$$

which selects all knots ($1 - \chi$ even), and counts knots without factoring by palindromic ambiguity.

As before any knot, whose w is not palindromic or antipalindromic, is counted twice. However, here “counted twice” might have meant that actually the knot and its mirror image have been counted, thus contributing to two different coefficients in the power series.

Keeping in mind this point, we turn to care about palindromic sequences.

- 1) Consider the antipalindromic case. w is automatically of even length. Let w' be the first half of w . To simplify notation, let

$$c(w) = |\hat{w}|_1 = \sum b_i, \quad \chi'(w) = \text{length of } w, \quad \sigma(w) = \sum (-1)^{i-1} \text{sgn}(a_i).$$

We remarked in the genus enumeration that

$$c(w') = \frac{c(w) + 1}{2} \quad \text{and} \quad \chi'(w') = \frac{\chi'(w)}{2}.$$

It remains to observe that also

$$\sigma(w') = \frac{\sigma(w)}{2},$$

which easily follows from the definition.

Thus antipalindromic cases are counted by

$$F_0(x, y, z) = \frac{1}{x} \hat{F}(x^2, y^2, z^2).$$

- 2) In the palindromic case we may have χ' odd. However, it is easy to see that $\chi'(w)$ is odd if and only if $c(w)$ is so, so that working only with even powers of x will ensure that we count only knots. Assuming $\chi'(w)$ is even and letting w' be the first half of w , we have

$$c(w') = \frac{c(w)}{2}, \quad \chi(w') = \frac{\chi'(w)}{2} \quad \text{and} \quad \sigma(w) = 0,$$

so that

$$(\chi' + \sigma)(w) = \chi'(w) = 2\chi'(w').$$

Then we obtain F_3 enumerating palindromic cases from \hat{F} by replacing y with $y^2 z^2$ and z by 1, as $\sigma(w')$ has no contribution to $\sigma(w)$:

$$F_3(x, y, z) = \hat{F}(x^2, y^2 z^2, 1).$$

Let

$$G(x, y, z) = \frac{F(x, y, z) + F_0(x, y, z) + F_3(x, y, z)}{2}.$$

Now the coefficient of $x^k y^{2g} z^l$ + the coefficient of $x^k y^{2g} z^{2g-l}$ in $G(x, y, z)$ counts the number of rational knots with crossing number k , genus g and $2g + \sigma = l$, where for each chiral pair either only one knot is recorded, or both are recorded with factor $1/2$ (the coefficients of G lie only in $\mathbf{Z} \cup \mathbf{Z} + 1/2$!). F_3 counts the achiral ones.

To count for each chiral pair both knots, we set

$$G_1(x, y, z) = G(x, y, z) + G(x, yz^2, 1/z) - F_3(x, y, z),$$

which counts both knots in chiral pairs by χ' and $\chi' + \sigma$ (the variable substitution in the second term accounts for $y^l z^{l+k} \rightarrow y^l z^{l-k}$). Thus G_1 is the function we sought. \square

Remark 6.1 One has the (σ -forgetting) identity

$$G_1(x, y, 1) = 2f(x, y^2) - \frac{h_1(-x, y^2) + h_1(x, y^2)}{2},$$

with f being the 2-variable function in theorem 5.1, and h_1 the one occurring in its proof. See remark 5.1. Also, it is easy to see from the proof of lemma 4.1, that $G_1(x, 1, 0)$ enumerates negative rational knots by crossing number. Since they correspond bijectively to positive knots, $G_1(x, 1, 0)$ must coincide with the function we obtained in theorem 4.1. Both identities are easily verified.

6.2 $|\sigma|$ without mirroring

In G_1 a knot and its mirror image are represented by two coefficients, for $\pm|\sigma|$, i.e. for monomials $y^{2g}z^{2g\pm|\sigma|}$, which accounts for the symmetry of G_1 under $(y, z) \rightarrow (yz^2, 1/z)$. We can eliminate this redundancy and count rational knots according to version 2). Then we have

Proposition 6.1 Let J be the function in x, y and z which counts in its coefficient of $x^m y^l z^k$ rational knots of crossing number m with $1 - \chi = l$ and $|\sigma| = k$, such that again only one knot in a chiral pair is counted. Then J is a certain closedly expressible function (too complicated to display).

Proof. To obtain J from G_1 , basically we want to “cut off” terms in G_1 of monomials $y^l z^k$ with $k < l = 2g$ (so far $[G_1]_{y^l z^k x^m} \neq 0$ for $0 \leq k \leq 2l$), and substitute $y^l z^k \rightarrow y^l z^{k-l}$. We must care about the chiral knots with $\sigma = 0$. Thus we consider

$$G_2(x, y, z) = G_1(x, y, z) + F_3(x, y, z),$$

and must multiply the coefficients in G_2 of $x^m y^l z^k$ by

$$\begin{cases} 0 & k < l \\ 1/2 & k = l \\ 1 & k > l \end{cases},$$

and make the variable substitution $y \rightarrow y/z$.

If $H = \sum a_i x^i$ and $G = \sum b_i x^i$ converge in a complex neighborhood of 0, then for any $\alpha \in [0, 1]$ and $|x|$ small

$$\sum a_i b_i x^i = \int_0^1 G(x^\alpha e^{2\pi i t}) H(x^{1-\alpha} e^{-2\pi i t}) dt. \quad (8)$$

This formula is justified under the assumption of absolute convergence and integrability of the limit function. The values of x , for which this happens usually depends on α , but it is only important that it contains a set with a convergence point. Then, if the integral can be solved in closed form for these x , by the uniqueness of the holomorphic extension it also holds for all x for which the series on the left converges.

With this formula (under the convergence and integrability assumption, which can be achieved with $\alpha = 0$ for $|y|, |z| < 1$ and $|x| < 1/2$), the function $J(x, y, z)$ we seek can be expressed by an integral

$$J(x, y, z) = \frac{1}{2\pi} \int_0^{2\pi} G_2(x, y e^{-is}, e^{is}) \left(\frac{1}{1 - z e^{-is}} - \frac{1}{2} \right) ds. \quad (9)$$

This integral is, regrettably, too hard to solve pleasantly even with the help of a computer, not from the point of view of method, but of the structural complexity of the expressions to handle. As we stated the proposition only qualitatively, we mostly avoid the quotation of exact calculation results.

The integral was evaluated as follows. First, one uses standard substitution $t = \tan s/2$, with which it turns into a rational integral

$$\int_{-\infty}^{\infty} G_2 \left(x, y \frac{1-t^2-2it}{1+t^2}, \frac{1-t^2+2it}{1+t^2} \right) \left(\frac{1+t^2}{1+t^2-z(1-t^2-2it)} - \frac{1}{2} \right) \frac{dt}{\pi(1+t^2)}.$$

This integral can be solved by calculating the residues of the (meromorphic) integrand in the upper half-plane. One integrates along a region given by the interval $[-R, R]$ together with the half-arc of radius R around the complex origin in the $\{\Im m > 0\}$ half-plane. Since the integrand has degree -2 in t , the half-arc contribution vanishes for $R \rightarrow \infty$.

Expand the integrand as a rational function $N(x, y, z, t)/D(x, y, z, t)$ of x, y, z, t . The calculation of the discriminant of the (smallest) denominator polynomial $D(x, y, z, t) = D(t)$, regarded as a polynomial in t , shows that this discriminant has a non-trivial expansion around $(x, y, z) = (0, 0, 0)$ (even if it vanishes in this point). Thus for generic x, y, z of small norm, D will have only single zeros. These zeros are explicitly calculable since $D(t)$ decomposes into quadratic factors in t and t^2 . Since the solutions depend continuously on x, y, z , to decide which zeros t_0 are relevant, one calculates them for $(x, y, z) = (0, 0, 0)$. The residues are then given by $N(t_0)/D'(t_0)$.

The result can be obtained with MATHEMATICA™ [Wo] after some time. It occupied almost 300 lines. Such an expression is difficult to handle even with the computer. For example, while the result should have real coefficients, I could not make MATHEMATICA eliminate the complex units out of it. Nonetheless, substituting small real values for x, y and z showed that $J(x, y, z)$ is indeed real. After hand manipulation I obtained an expression without occurrences of i , and MATHEMATICA simplified it to about 250 lines. As a check, expanding $J(x, 1, 1)$ and $J(x, 1, 0)$ as power series in x reveals – as expected – the numbers of all resp. $\sigma = 0$ rational knots. \square

By applying the same integration to a symmetrized version of G_2 , one can also (theoretically) obtain a similar expression for problem 1a).

Remark 6.2 Of course, one could try to solve the integral in (9) directly by residues, without substitution, but it turned out that, when using MATHEMATICA, manual “intervention” was necessary (at least for me) at an earlier stage. Clearly I tried to avoid this (as long as possible) with regard to the difficulty of the expressions.

Remark 6.3 The very useful formula (8) seems natural with harmonic analysis in mind, but I could not find, or get referred to, an occurrence of it in combinatorial literature. M. Bousquet-Mélou pointed out to me that this product of series is called the Hadamard product. It has been intensively studied from the point of view of showing closure properties of certain families of power series under it (see e.g. [Lp]). A subsequent electronic search for this term led at least to one reference [Br], where the integral expression is given explicitly. Thus it appears known in analysis, even if not popularly. I have no access to that paper and to the history of the formula, but at least it was discovered independently by myself. (It occurs also in my previous paper [St8].)

7 Further applications

7.1 Applications of the signature and genus enumeration

We give another result, in whose proof the integration method is again applied, and leads to a(n at least electronically) feasible calculation with a manageably presentable result. (It can be considered as a special case of proposition 6.1, up to the different handling of mirror images.)

Corollary 7.1 If c_n denotes the number of rational knots of n crossings with signature 0 (see last line of table 1), such that chiral pairs are counted twice, then these numbers have a generating function

$$f_0(x) = \sum_{n=1}^{\infty} c_n x^n = x^4 + 3x^6 + 2x^7 + 9x^8 + 6x^9 + 29x^{10} + \dots,$$

which can be expressed in closed form (see figure 3). Also $\lim_{n \rightarrow \infty} \sqrt[n]{c_n} = 2$.

Proof. The generating function we seek can now be expressed as

$$\frac{1}{2\pi} \int_0^{2\pi} G_1(x, e^{it}, e^{-it}) dt,$$

which certainly converges at least for $|x| < 1/2$. If G_1 is a rational function, as in our situation, such an integral can always be solved. Most generally, with the standard substitution $z = \tan^t/2$, it turns into a rational integral

$$\int_{-\infty}^{\infty} G_1\left(x, \frac{1-z^2+2iz}{1+z^2}, \frac{1-z^2-2iz}{1+z^2}\right) \frac{dz}{\pi(1+z^2)},$$

which can be solved by the residue method or by partial fraction decomposition. The result was obtained with MATHEMATICA in a few minutes. \square

Remark 7.1 Using the Darboux method (see [Wi, §5.3]), one can determine a more precise asymptotic behaviour of the numbers c_n , which is a bit more interesting since their generating function is not rational. Using the multi-singularity version of Darboux' theorem [Wi, theorem 5.3.2] attributed to Szegö, and Stirling's formula, one obtains that the leading term in the asymptotic expansion of c_n is $\frac{2^{n-1}}{3\sqrt{2\pi n}}$. (The next order term contains an oscillating contribution given by a constant multiple of $(-2)^n n^{-3/2}$.)

We also remark that we can now easily obtain statistical data about the distribution of genera and signatures among rational knots. We give only the (asymptotic) expectation values; dispersion and the other moments can be determined similarly.

Proposition 7.1 The average genus of a rational knot of n crossings behaves as $n \rightarrow \infty$, up to lower order terms, like $\frac{n}{4}$. The average absolute signature $|\sigma|$ behaves like $\sqrt{\frac{2n}{\pi}}$.

Proof. The average genus of a rational knot of n crossings is given by

$$\tilde{g}(n) := \frac{\sum_{K \in C_n} g(K)}{|C_n|}, \quad \text{with} \quad C_n := \{K \text{ rational}, c(K) = n\}. \quad (10)$$

Since achiral knots drop exponentially compared to all knots, it is unimportant for the asymptotics whether we consider knots up to mirroring or not in C_n . For convenience, we will assume for the average genus calculation that we distinguish mirror images, while for the average absolute signature that we do not.

The behaviour of numerator and denominator in (10) are found by partial fraction decomposition. For the denominator one considers $G_1(x, 1, 1)$, and the relevant term one obtains is $\frac{1}{12(1-2x)}$ (which is basically the Ernst-Summers result). For the numerator one applies the same procedure to $\frac{\partial G_1}{\partial y}(x, 1, 1)$, and finds that $\frac{1}{(1+2x)^2}$ does not occur and that the coefficient of $\frac{1}{(1-2x)^2}$ is $1/24$. Then note that G_1 counted in the powers of y the double genus.

Now consider the average signature (obviously defined). One calculates $\frac{\partial J}{\partial z}(x, 1, 1)$. The term whose denominator has zeros of smallest norm is

$$S(x) = \frac{(1-x)x^3}{(1+x)(1-2x)^{3/2}\sqrt{1+2x}}.$$

The dominating term thus comes (expectedly) from the zero $x = 1/2$. By the Darboux-Szegö theorem, the leading contribution of this zero is given by

$$2^n \binom{n+1/2}{n} \cdot (S(x) \cdot (1-2x)^{3/2}) \Big|_{x=1/2}. \quad (11)$$

The right factor evaluates to $\frac{1}{24\sqrt{2}}$, and Stirling's formula yields

$$\binom{n+1/2}{n} = \frac{\Gamma(n+3/2)}{\Gamma(n+1)\Gamma(3/2)} \asymp \frac{\sqrt{n}}{\Gamma(3/2)},$$

with $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$ and $a_n \asymp b_n$ meaning $a_n/b_n \rightarrow 1$. Then (11) gives

$$2^n \cdot \frac{\sqrt{n}}{12\sqrt{2\pi}},$$

which, divided by the asymptotical behaviour $2^{n-3}/3$ of the total number of rational knots up to mirroring, leads to the stated asymptotics. \square

Remark 7.2 The generating function $\sum_n \tilde{g}(n)x^n$ of the mean genus \tilde{g} (and also mean $|\sigma|$) itself seems difficult to express.

Remark 7.3 We have for simplicity omitted the following asymptotical terms, but their contribution is $O(1/n)$ compared to the one of the leading term, so that latter alone does not necessarily give a good approximation. For example, by expanding $\frac{\partial J}{\partial z}(x, 1, 1)$ as a power series, one finds that the sum of $|\sigma|$ over 1000-crossing rational knots is about 1.12×10^{301} . Only these first 3 digits are approximated correctly from the leading term given in the proposition (when multiplied by the total number of knots).

7.2 Unknotting number one and the Bleiler conjecture revisited

Now we return to the enumeration of rational knots of unknotting number one (with the convention of not distinguishing mirror images). We promised to give a proof of corollary 2.1. For this we consider again the Conway notation with even numbers, and describe such notations occurring for unknotting number one knots.

In [St4, §3.1], we described exactly arithmetically which knots $S(p, q)$ give counterexamples to the Bleiler conjecture – this occurs iff at least one of the four pairs $(p, \pm q^{\pm 1})$ can be written as $(2mn \pm 1, 2n^2)$ with $m > n > 1$ coprime, but no one can be done so such that additionally one of m and n is even. The main point here is to describe the even-integer notations for these counterexamples.

Proposition 7.2 Let K be an unknotting number one rational knot, and $C(a_1, \dots, a_k)$ its Conway notation with non-zero even integers. Then (a_1, \dots, a_k) is up to transposition of (at least) one of the following forms:

- 1) $(a, a_1, \dots, a_l, \pm 2, -a_l, \dots, -a_1)$ with $l \geq 0$ or
- 2) $(a, a_1, \dots, a_l, \pm 2, a'_l, -a_{l-1}, \dots, -a_1)$ with $l \geq 1$, such that $|a_l + a'_l| = 2$, and the sign of the absolutely larger one of a_l and a'_l is opposite to the one of the ± 2 in between.

Also, each such sequence is realized by an unknotting number one rational knot.

Proof. We use the argument in [KM, proof of theorem 1, (ii) \Rightarrow (iii)]. Take a rational unknotting number one knot $K = S(2mn \pm 1, 2n^2)$ with $(m, n) = 1$. If $n = 1$ we have a twist knot, which is of form 1. Thus let $n > 1$. Then $m > n$. Kanenobu and Murakami write $m = an + t$, and now we can choose $a \neq 0$ to be even, possibly having $t < 0$. Then express n/t as a continued fraction. If one of n and t is even, then one can choose the continued fraction expression to be only of even integers a_1, \dots, a_l , and by the argument of Kanenobu and Murakami obtains that the form 1 can be chosen so

that indeed all numbers are even. Contrarily, every form 1 clearly gives an unknotting number one knot.

Now consider the case that both n and t are odd. Then one can write n/t as a continued fraction, such that all integers $[[a_1, \dots, a_l]]$ are even except a_l , which is odd. One can also assume that for $l > 1$ we have $(a_{l-1}, a_l) \neq (\pm 2, \mp 1)$, and that if $l = 1$, then $|a_1| = n > 1$. Then use the transformation for $a > 0$

$$(\dots, a, 2, -a, \dots) \longrightarrow (\dots, a+1, -2, 1-a, \dots) \quad (12)$$

together with its negated and transposed versions. After an application of this transformation one obtains a notation of the form 2. Also, none of the neighbors of the middle ± 2 might have become zero under (12), because we excluded the sequences $(\dots, \pm 2, \mp 1)$ and (± 1) . Thus no collapsing occurs, according to the rule (3).

Finally, it is again easy to see that each sequence of the form 2 can be realized. \square

Now we can prove corollary 2.1.

Proof of corollary 2.1. First exclude all twist knots from the consideration. These are the knots whose notation is of length 2. They are counted clearly by

$$\frac{x^3}{1-x}.$$

Now we count the notations of type 1 and 2 by crossing number. Such notations are unique up to transposition and negation. To fix the negation ambiguity, we assume $a > 0$.

By similar arguments as before, and using lemma 3.1, one can find that the generating function of the (remaining, non-twist-knot) notations of type 1 by crossing number is

$$\frac{2x^6}{(1-x^2-2x^4)(1-x)}.$$

To enumerate type 2 sequences, just note that such a sequence of a crossing number n knot bijectively corresponds to a (non-twist-knot) sequence of a crossing number $n-2$ of type 1. Simply raise in latter sequence the absolute value of one of the neighbors of the middle ± 2 by 2. The neighbor is determined by having the opposite sign to the ± 2 . Thus type 2 sequences are counted by

$$\frac{2x^8}{(1-x^2-2x^4)(1-x)}.$$

Now we must care about which sequences w are counted several times up to transposition and possible negation. Clearly w cannot be at the same time of type 1 and of type 2. Similarly if both w and $\pm \bar{w}$ are of type 1, or both are of type 2, it is easy to see that w is itself (anti)palindromic, so that it is not generated twice.

Finally, we must care about the case that one of w and $\pm \bar{w}$ is of type 1, and the other one is of type 2. Then one indeed obtains a series of duplications, namely for the sequences of the form $(4-2)^k - 22(-42)^{k-1}$, and $(2-4)^k 22(-24)^k$ with $k \geq 1$. These forms give one knot, in crossing numbers $8+4r$, $r \geq 0$.

Thus the function we seek is

$$\frac{x^3}{1-x} + \frac{2x^6+2x^8}{(1-x^2-2x^4)(1-x)} - \frac{x^8}{1-x^4} = \frac{x^3}{1-x} + \frac{2x^6}{(1-x)(1-2x^2)} - \frac{x^8}{1-x^4},$$

which is what we claimed. \square

The proof also gives the following consequence:

Proposition 7.3 If c_n is the number of rational unknotting number one knots of n crossings (chiral pairs counted once), that do *not* provide counterexamples to the Bleiler conjecture (that is, unknot by one crossing change in their rational diagrams with all Conway coefficients even), then

$$\begin{aligned} \sum_{n=1}^{\infty} c_n x^n &= \frac{x^3 - x^5 + 2x^6 - 2x^7}{(1 - x^2 - 2x^4)(1 - x)} \\ &= x^3 + x^4 + x^5 + 3x^6 + 3x^7 + 5x^8 + 5x^9 + 11x^{10} + 11x^{11} + \dots \end{aligned}$$

This formula shows that asymptotically $1/3$ of the n crossing unknotting number one knots do not have the property conjectured by Bleiler.

Proof. This is simply obtained by counting only the twist knots and the (remaining) ones of type 1. \square

As another consequence we obtain the proof of a weaker form of Bleiler's conjecture. This form was suggested by, but is nonetheless still more general than the one proved in [St5].

Corollary 7.2 Any unknotting number one counterexample to Bleiler's conjecture has even leading coefficient $\max \text{cf} \Delta$ of the Alexander polynomial. In particular, Bleiler's conjecture holds for unknotting number one fibered rational knots.

Proof. Use (2) and the observation that in type 2, at least one of a_i and a'_i is divisible by 4. \square

The more general version of the fibered Bleiler conjecture also extends theorem 3.3 to odd values of $\max \text{cf} \Delta_K$. We can, however, obtain a formula even in some cases where the Bleiler conjecture fails, because we understand well the exceptions. From the proof of theorem 3.3, and proposition 7.2, the following can be obtained easily; we leave the proof to the reader.

Proposition 7.4 Let p be a square-free positive integer. Then the number of rational unknotting number one knots K with $\max \text{cf} \Delta_K = \pm p$ and crossing number n is given by

$$2 \left(F_{\lfloor n/2-2-p \rfloor} + \sum_{r: r(r+1)|p} F_{\lfloor n/2-1-2r-p/(r+r^2) \rfloor} \right) + \left\{ \begin{array}{l} -1 \text{ if } (n, p) = (8, 2) \\ 1 \text{ if } n \in \{1+2p, 2+2p\} \\ 0 \text{ otherwise} \end{array} \right\}.$$

(In this formula we assume that $r > 0$ and that $F_k = 0$ if $k < 0$.) \square

In particular, for square-free odd p and $n \geq 4 + 2p$ we obtain $2F_{\lfloor n/2-2-p \rfloor}$, and for $p = 2$ and $n \geq 9$ we have $4F_{\lfloor n/2-4 \rfloor}$. When $n \geq 4 + 2p$, one can use the recursive behaviour to rewrite the formula also for any other square-free p to contain only two (mutually index-shifted and bulkily coefficiented) Fibonacci sequences. For the remaining, non-square-free values of p one should still obtain rational generating functions enumerating the corresponding knots, but these functions will be much less pleasant. (Their shape will depend on the prime decomposition of the greatest integer whose square divides p .)

Unfortunately, a similarly nice Fibonacci number version is not possible for achiral unknotting number one knots K of higher $|\max \text{cf} \Delta_K|$, as for each achiral rational knot K the formula (2) shows that $\pm \max \text{cf} \Delta_K$ is a square. (In [St7], we show that this is more generally true for alternating knots, a result obtained also by Weber and Quach [VW].)

7.3 Counting lens spaces

We conclude our counting results with an application to the enumeration of lens spaces. In [St7] we gave the number of different lens spaces of fundamental group \mathbf{Z}_p . This is equivalent to counting rational knots by determinant.

Theorem 7.1 ([St7]) Let $p \geq 3$ be odd. When considering the lens space $L(p, q)$ and its mirror image $L(p, -q) = L(p, p - q)$ as equivalent, the number of different lens spaces with fundamental group \mathbf{Z}_p is

$$\frac{1}{4} \left\{ \phi(p) + r_2^0(p) + 2^{\omega(p)} \right\}, \quad (13)$$

with $r_2^0(p)$ being given by

$$r_2^0(p) = \left| \{ (a, b) \in \mathbf{N}^2 : (a, b) = 1, a^2 + b^2 = p \} \right|,$$

$\omega(p)$ denoting the number of different prime divisors of p and $\phi(p) = |\mathbf{Z}_p^*|$ being Euler's totient function.

When distinguishing between $L(p, q)$ and $L(p, -q)$ (if they are orientation-reversingly inequivalent), the number of such lens spaces is

$$\frac{1}{2} \left\{ \phi(p) + 2^{\omega(p)} \right\}.$$

We can now determine the number of lens spaces which can be obtained by a $p/\pm 2$ surgery along a knot K .

Theorem 7.2 Let $p \geq 5$ be odd. Then the number c_p of different lens spaces with fundamental group \mathbf{Z}_p , which are obtainable by a $p/\pm 2$ surgery along a knot K , is given by

$$c_p = 2^{\omega((p+1)/2)-1} + 2^{\omega((p-1)/2)-1} + \left\{ \begin{array}{l} -2 \text{ if } p = p_s \text{ for some } s \geq 0 \\ -1 \text{ otherwise} \end{array} \right\}. \quad (14)$$

In this formula $\omega(n)$ denotes as before the number of different prime divisors of n , and

$$p_s = \frac{1}{4} \left((58 - 41\sqrt{2})(3 - 2\sqrt{2})^s + (58 + 41\sqrt{2})(3 + 2\sqrt{2})^s \right). \quad (15)$$

In this enumeration we consider the lens space $L(p, q)$ and its mirror image $L(p, p - q)$ as equivalent. If we distinguish them, the number is

$$2c_p - \left| \{p\} \cap \mathcal{A} \right| - \left| \{p\} \cap \mathcal{S} \right|,$$

with $\mathcal{A} := \{2n^2 + 2n + 1 : n \geq 1\}$, $\mathcal{S} := \{q_s : s \geq 0\}$, and

$$q_s = \frac{1}{3} \left((97 - 56\sqrt{3})(2 - \sqrt{3})^{2s} + (97 + 56\sqrt{3})(2 + \sqrt{3})^{2s} + 1 \right). \quad (16)$$

Remark 7.4 The numbers p_s and q_s can be given alternatively in terms of their generating functions

$$\sum_{s=0}^{\infty} p_s x^s = \frac{29 - 5x}{1 - 6x + x^2} = 29 + 169x + 985x^2 + 5741x^3 + 33461x^4 + 195025x^5 + \dots$$

and

$$\begin{aligned} \sum_{s=0}^{\infty} q_s x^s &= \frac{65 - 74x + 5x^2}{(1-x)(1-14x+x^2)} \\ &= 65 + 901x + 12545x^2 + 174725x^3 + 2433601x^4 + 33895685x^5 + \dots, \end{aligned}$$

or by their initial values and linear recursions

$$p_s = 6p_{s-1} - p_{s-2} \quad (s \geq 2) \quad \text{and} \quad q_s = 15(q_{s-1} - q_{s-2}) + q_{s-3} \quad (s \geq 3).$$

The q_s do not seem to have been so far of any particular attention, but the sequence of p_s is listed in [Sl] as A001653, with several references.

Proof of theorem 7.2. Let us first prove (14). We know from the arguments of [KM], which rely on the results of Culler–Gordon–Luecke–Shalen [CGLS] and Moser [Mo], that a lens space $L(p, q)$ is obtainable by $p/\pm 2$ surgery along a knot K if and only if $S(p, q)$ has unknotting number one. Thus what we claim is equivalent to counting unknotting number one rational knots (up to mirroring) by determinant.

It is easy to see, and we remarked it already in [St7], that, when counting the Schubert notations $S(p, 2n^2)$ with $p = 2mn \pm 1$, the first two terms in the formula for c_p just come from the ways of writing $(p \pm 1)/2 = m_{\pm}n_{\pm}$ with $(m_{\pm}, n_{\pm}) = 1$ up to interchange of m_{\pm}, n_{\pm} . We also remarked that the twist knot with determinant p is counted twice, as occurring in both representations. The problem was which other duplications occur.

Whenever $n > 1$, clearly n determines m , and hence t . Moreover, it is easy to see from the proof of proposition 7.2 that both n and t can be recovered from the forms of both types. They are just the numerator and denominator of the continued fractions of a_1, \dots, a_l , possibly first undoing the modification of a_l in type 2. Thus the duplications we sought occur exactly if the even integer Conway notation, up to reversion and negation, can be put into these two types of proposition 7.2 in a different way. But we know now what sequences these are: we found they belong to one of the two series $(4 - 2)^k - 22(-42)^{k-1}$ and $(2 - 4)^k 22(-24)^k$ for $k \geq 1$. Also, for any of these Conway notations exactly two different representations occur.

It is now a matter of a simple (even if somewhat tedious) calculation to show that the determinants of corresponding knots are the p_s . The easiest way is to note that negated inversion and addition of an integer correspond to the action of $SL(2, \mathbf{Z})$ on the upper half-plane $\{\Im m > 0\}$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{ax+b}{cx+d}.$$

Thus the map

$$[[\dots, x]] \mapsto [[4, -2, \dots, x]]$$

can be described by the action of an $SL(2, \mathbf{Z})$ -matrix. (Note that prepending a single integer to the iterated fraction, in our convention, rather than that of [Mu2], cannot be described by such an action because of the sign switch. However, when prepending two integers, the two sign changes cancel at the cost of negating the first number prepended.) This matrix has two distinct Eigenvalues $\lambda_{1,2} = 3 \pm \sqrt{8}$. Thus for any of the two series the determinants are given by

$$a\lambda_1^{2k} + b\lambda_1^k + c + d\lambda_2^k + e\lambda_2^{2k},$$

and the coefficients can be determined from the first five values. Then to verify (15), one needs to check it only for $s \leq 9$. (Either series are obtained by specifying the parity of s .)

When distinguishing $L(p, q)$ and $L(p, -q)$, one needs to take account of achiral unknotting number one rational knots. We classified these knots in corollary 2.2 into two series. (Possibly one can prove the corollary also from the even-integer notation, but it does not seem worthwhile to get into this now.) The determinants of the first series are obvious, while those of the second series q_s are found similarly to p_s . \square

Remark 7.5 One can see that for the doubly counted knots of determinant p_s in the derivation of (14), one of $2m+m_{\pm}^{\pm 1}$ is a square root of -1 in $\mathbf{Z}_{p_s}^*$. Thus, like the F_{2k} , none of the p_s has a divisor of the form $4r+3$. (This also follows from the descriptions of the p_s in Sloane’s manual.)

Remark 7.6 In corollary 2.2, the notation (1111) was artificially excluded from the second series by writing $(3(12)^k 1^4 (21)^k 3)$ instead of $((12)^k 1^4 (21)^k)$, in order to avoid mentioning the figure-8-knot twice. Except eventually for its determinant 5, it is not clear whether another determinant can be realized by knots in both series simultaneously, i.e. for some $s \geq 0$ and $n \geq 1$ we have

$$q_s = 2n^2 + 2n + 1. \tag{17}$$

This problem falls into the class of polynomial-exponential equations, which have been for a long time intensively studied and connected to deep work in number theory (see e.g. [Ev]). It is known that, under certain regularity properties (that our example enjoys), the number of solutions is finite. Apply for instance theorem 3 of [NP] with $G_m = 3q_m - 1$ (which form a binary recurrence with $A = 14$ and $B = 1$) and $P(x) = 6x^2 + 6x + 2$. While several particular examples have been studied in detail and some of them solved completely (see *loc. cit.* in [NP]), ours is apparently not among them, and good general bounds on the number or size of solutions are very hard to obtain.

At least we have

Proposition 7.5 Assume $2n^2 + 2n + 1 \in S \cap \mathcal{N} \neq \emptyset$ (with S and \mathcal{N} defined as in theorem 7.2). Then

i) $x = 2n + 1$ is an integer point on the elliptic curve

$$y^2 = 3x^4 + 2x^2 - 5 = (x^2 - 1)(3x^2 + 5), \quad (18)$$

with $|x| > 3$. ($x = \pm 1, \pm 3$ are obvious points.)

ii) $|S \cap \mathcal{N}| \leq 2^{20222} - 2 \approx 2.68 \times 10^{6087}$, and

$$10^{114,000} \leq \min S \cap \mathcal{N} \leq \max S \cap \mathcal{N} \leq e^{e^{4640516}}. \quad (19)$$

Proof. Consider first part ii). After the most recent work of Schlickewei and Schmidt [SS, SS2, SS3], the best estimate for the number of solutions (s, n) of (17) one finds is from theorem 2.2(a) of [SS3] applied on $3q_s - 1$ (with $d = t = 2$). This gives at most 2^{20224} integer solutions (s, n) . Since we have the solutions $(-1, 1)$, $(-2, 0)$, and $n \mapsto -1 - n$ and/or $q \mapsto -4 - q$ preserve solutions, we obtain at most $2^{20222} - 2$ solutions with $s, n \geq 0$.

Using MATHEMATICA, I verified that no solution of (17) occurs for $0 \leq s \leq 10^5$. This establishes the left inequality in (19) by evaluating the logarithm of the dominating root $\log_{10}(2 + \sqrt{3}) \approx 0.572$. To obtain the right inequality, first we transform the problem to consider integer points on the elliptic curve (18).

Since the bases $2 \pm \sqrt{3}$ appear with even exponents in (16), $3q_s - 1$ must be an index-2-subsequence of a simpler binary linear recurrence. This recurrence is found to be

$$\tilde{q}_0 = 2, \quad \tilde{q}_1 = 4, \quad \tilde{q}_s = 4\tilde{q}_{s-1} - \tilde{q}_{s-2},$$

and then $3q_s - 1 = \tilde{q}_{4+2s}$. Define

$$\tilde{r}_0 = 0, \quad \tilde{r}_1 = 1, \quad \tilde{r}_s = 4\tilde{r}_{s-1} - \tilde{r}_{s-2}.$$

Then $(2 + \sqrt{3})^s = \frac{1}{2}\tilde{q}_s + \tilde{r}_s\sqrt{3}$. Also $\frac{1}{4}\tilde{q}_s^2 - 3\tilde{r}_s^2 = 1$, because $2 + \sqrt{3}$ is a unit of $\mathbf{Z}[\sqrt{3}]$ and has norm 1. Now if

$$\tilde{q}_s = 6n^2 + 6n + 2,$$

then $2\tilde{q}_s = 3(2n + 1)^2 + 1 = 3x^2 + 1$, and so

$$(3x^2 + 1)^2 - 48\tilde{r}_s^2 = 16,$$

which yields (18) with $y = 4\tilde{r}_s$. This proves part i).

By Baker's work [B] the norm $\max(|x|, |y|)$ of an integer solution (x, y) of (18) is at most

$$e^{e^{4640516}}.$$

This leads to the upper bound inequality in (19), since $y = 4\tilde{r}_s \geq \tilde{q}_s$ for $s \geq 1$. \square

Remark 7.7 There have been recently several substantial improvements of Baker’s result (see e.g. Voutier [V]). However, all these bounds depend on constants which are (effectively computable but) not explicitly given. Even if so done, the estimates still seem too large to close the gap in (19). Nonetheless, many special hyperelliptic equations like (18) can be, and have been, solved completely; this usually requires though, besides use of general results and computer calculation, a fair bit of number-theoretically (for me too) advanced extra arguments. I decided to consult Yu. Bilu about (18); his collaborator G. Hanrot informed me then that, using Magma, he computed that $x = \pm 1, \pm 3$ are indeed the only solutions.

8 Unimodular matrices and the “fibered” Bleiler conjecture

Finally, we make some remarks on how the above discussion on rational knots can be transferred into a completely arithmetic setting using iterated fractions. As explained, this is historically motivated by the result of [St5]. In particular, we will see that certain partial cases of the “fibered” Bleiler conjecture follow by purely arithmetic arguments.

Consider $M_{k,l}$ for $k, l \in \mathbf{Z}$ to be the 2×2 matrix

$$\begin{pmatrix} \operatorname{sgn}(k) + 4|kl| & 2|k| \operatorname{sgn}(l) \\ 2|l| & \operatorname{sgn}(l) \end{pmatrix} \in \Gamma_{\pm}(2).$$

Here $\Gamma_{\pm}(2)$ denotes the subgroup of 2×2 matrices in $PGL(2, \mathbf{Z})$ (that is, matrices of determinant ± 1) with even lower left entry², and we adopt the convention that $\operatorname{sgn}(0) = 1$. Let \mathcal{M}_Z for $Z \subset \mathbf{Z} \times \mathbf{Z}$ be the submonoid (not subgroup) of $\Gamma_{\pm}(2)$ generated by $\{M_{k,l} : (k, l) \in Z\}$, and $\mathcal{M}_Z v = \{Mv : M \in \mathcal{M}_Z\}$ be its “orbit” on some vector $v \in \mathbf{R}^2$.

Then knot theory allows to prove some properties of such orbits.

Proposition 8.1 $\mathcal{M}_{\mathbf{N} \times (\mathbf{Z} \setminus \mathbf{N}_+)}$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ does not contain a vector of the form $\begin{pmatrix} 2mn \pm 1 \\ 2n^2 \end{pmatrix}$ for any coprime integers $m > n > 1$ (and sign choice ‘ \pm ’).

Proof. When identifying p/q for $p > q \geq 1$, $(p, q) = 1$ with $\begin{pmatrix} p \\ q \end{pmatrix}$, then multiplication with $M_{k,l}$ is just the prepending of $2k, 2l$ to the iterated fraction. Then the statement is just a translation of the fact, proved in [St2] for arbitrary knots, that rational positive knots which are not twist knots do not have unknotting number one.

Clearly the cases of twist knots are of the given form with $n = 1$, thus we need to pose $n > 1$. Now, because of the ambiguity $S(p, q) = S(p, \pm q^{-1})$ (where the additive and multiplicative inversions are meant in \mathbf{Z}_p^*), one needs to take care that for $p = 2mn \pm 1$ the even one of the numbers $\frac{p \pm 1}{2}$ is not of the form $2n^2$, that is, $4n^2 \not\equiv \pm 1 \pmod{2mn \pm 1}$. However, for $(m, n) = 1$ and $m > n \geq 1$ such a congruence holds only if $n = 1$ (and $m \leq 3$), which gets irrelevant once one poses $n > 1$. \square

A more delicate statement can be made in $\mathcal{M}_{\mathbf{N}_+ \times (\mathbf{Z} \setminus \{0\})}$.

Proposition 8.2 Assume $M_{k_1, l_1} \cdots M_{k_g, l_g} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2mn \pm 1 \\ 2n^2 \end{pmatrix}$ with $m > n \geq 1$ coprime, $k_i \in \mathbf{N}_+$ and $l_i \in \mathbf{Z} \setminus \{0\}$. Then

i) $n = g = 1$ or $g = 2$, and

²We could avoid the use of ‘sgn’ and ‘|.’ and define $M_{k,l} = \begin{pmatrix} 1+4kl & 2k \\ 2l & 1 \end{pmatrix}$ to be strictly unimodular, but it is more convenient here to normalize the matrix so as it to preserve the set of integer vectors $\left\{ \begin{pmatrix} p \\ q \end{pmatrix} : p > q \geq 1, (p, q) = 1 \right\}$.

- ii) at most one l_i is negative, and one is exactly if $2mn \pm 1 \equiv 3 \pmod{4}$ (with the same sign choice as above).

Proof. For i), consider the forms of proposition 7.2 for the unknotting number one knot $K = S(2mn \pm 1, 2n^2)$. It is easy to see that the only forms in which all entries of one of the index parities can be chosen to have all the same sign are form 1 for $l \leq 1$ and form 2 for $l = 1$; and $g = l + 1$. If $g = 1$ (in form 1), then K is a twist knot, so that by the argument in the proof of proposition 8.1, $n = 1$.

For ii) compute $\sigma(C(2k_1, 2l_1, \dots, 2k_g, 2l_g))$ using lemma 6.1, and show that it is given by $2\#\{i : l_i < 0\}$. Then use the results of [Mu3] that for any knot K , $u(K) \geq |\sigma(K)/2|$ and that $|\Delta_K(-1)| - \sigma(K) \equiv 1 \pmod{4}$. \square

While for a number theorist such statements, although possibly not obvious, may be of insufficient importance, the actual reason for considering rational knots in this light is because it may hopefully make the problem of the Bleiler conjecture for fibered rational knots more arithmetically approachable. In particular, it can be described by a slight modification of the above propositions.

Proposition 8.3 $\mathcal{M}_{\{-1,1\} \times 2} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \not\cong \left(\begin{smallmatrix} 2mn \pm 1 \\ 2n^2 \end{smallmatrix} \right)$ for any coprime *odd* integers $m > n > 1$ and sign choice ‘ \pm ’.

Proof. This is a slightly reworded version of Bleiler’s conjecture for fibered rational knots. To explain this, we prove first that

$$\mathcal{M}_{\{-1,1\} \times 2} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) = \left\{ \begin{smallmatrix} p \\ q \end{smallmatrix} : 2 \mid q, p > q > 1, (p, q) = 1, \text{ and } S(p, q) \text{ is fibered} \right\} \cup \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}. \quad (20)$$

The fact that $p > q > 1$ are coprime and q is even can be verified by simple arithmetic by virtue of being preserved by multiplication with any of the $M_{k,l}$. Thus we should explain the fiberedness property.

$S(p, q)$ is fibered for q even iff the iterated fraction of even integers expressing p/q (which is of even length) contains only ± 2 . When identifying $\begin{pmatrix} p \\ q \end{pmatrix}$ with p/q , then for some $(\varepsilon_1, \varepsilon_2) \in \{-1, 1\} \times 2$ the prepending of the two numbers $2\varepsilon_{1,2}$ to the iterated fraction expression is equivalent to multiplication with $M_{\varepsilon_1, \varepsilon_2}$ up to change of sign in one of p and q . This discrepancy can be dealt with by negating all subsequent entries in the indices of the M ’s to be multiplied with, which passes the discrepancy through until it is cancelled with a subsequent sign change, or until the end, where it gets obsolete. This establishes (20).

Now if $(m, n) = 1$ with $m > n > 1$ odd, then $K = S(2mn \pm 1, 2n^2)$ is of unknotting number one, and the proof of proposition 7.2 shows that its even-integer notation is of type 2. Then we observed that K is not fibered, so that the claim we want to show follows from (20). \square

Prior to its proof, a computer calculation verified proposition 8.3 for word length ≤ 16 , thus in particular Bleiler’s conjecture for rational fibered knots K of crossing number at most 35 (note that the word length in the matrices $M_{k,l}$ is just $g(K)$). It also showed that vectors of the stated form with both m and n odd, but *not* necessarily coprime occur only for the expected word length 9. This suggests stronger to conjecture that such vectors can not be obtained except for word lengths being an odd square.

Note also, that because of the reversal of the iterated fraction expression with even integers preserves its form for fibered knots, $\mathcal{M}_{\{-1,1\} \times 2} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$ contains with $\begin{pmatrix} p \\ q \end{pmatrix}$ also $\begin{pmatrix} p \\ \pm q^{-1} \end{pmatrix}$ (where negation and inversion of q are meant in \mathbf{Z}_p^* and the sign is chosen so as the number to be even).

One can also try to deduce the above statements in purely arithmetic terms. As suggested to me by D. Hejhal, a naive approach is to look at congruences in the $M_{k,l}$ ’s. Although this unlikely will lead

to a complete recovering of corollary 7.2 or proposition 8.3, we record at the end two properties of genus and determinant that indeed come up this way, namely from considering $p \bmod 8$ and $q \bmod 4$. Working with these congruences, and using (2), allows, as before, to weaken the fiberedness assumption on K to $\max \text{cf} \Delta_K$ being odd, because each (odd) factor $a_i/2$ of $\max \text{cf} \Delta_K$ can be made to ± 1 by adding a multiple of 4, and this does not affect $a_i \bmod 8$.

Proposition 8.4 If K is an unknotting number one counterexample to the Bleiler conjecture with $\max \text{cf} \Delta_K$ odd, then

- i) the genus $g(K)$ of K is also odd,
- ii) $3 \mid g(K) \iff |\Delta_K(-1)| \equiv \pm 1 \pmod 8$ and $3 \nmid g(K) \iff |\Delta_K(-1)| \equiv \pm 3 \pmod 8$. □

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