Polymatrix and Generalized Polynacci Numbers

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Abstract

We consider m-th order linear recurrences that can be thought of as generalizations of the Lucas sequence. We exploit some interplay with matrices that again can be considered generalizations of the Fibonacci matrix. We introduce the definition of reflected sequence and inverted sequence and we establish some relationship between the coefficients of the Cayley-Hamilton equation for these matrices and the introduced sequences.

1 Antefacts

Let us define the $m \times m$ matrix $\mathbf{A}_m$ as a matrix with the first column of all ones, as well as the first upper diagonal, while all the other elements are equal to zero, that is

$$\begin{cases} a_{i1}^{(m)} & = & 1 & \text{for } i = 1, \ldots, m \\ a_{i,i+1}^{(m)} & = & 1 & \text{for } i = 1, \ldots, m-1 \\ a_{ij}^{(m)} & = & 0 & \text{otherwise.} \end{cases}$$

We can write in partitioned form

$$\mathbf{A}_m = \begin{bmatrix} \mathbf{A}_{m-1} & \mathbf{e}_1 \\ \mathbf{e}_2' & 0 \end{bmatrix}, \tag{1}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are $m \times 1$ vectors

$$\mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{e}_2 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

Note that the inverse of $\mathbf{A}_m$ is a matrix $\mathbf{B}_m$ such that

$$\begin{cases} b_{1m} & = & 1 & \\ b_{im} & = & -1 & \text{for } i = 2, \ldots, m \\ b_{i,i-1} & = & 1 & \text{for } i = 2, \ldots, m \\ b_{ij} & = & 0 & \text{otherwise.} \end{cases}$$

If we evaluate the determinant developping according the elements of the last row we see that the only non-zero summand corresponds to element in position $(m, 1)$ while the corresponding minor is a $m - 1 \times m - 1$ identity matrix: it follows

$$|\mathbf{A}_m| = (-1)^{m+1}.$$

The characteristic polynomial of $\mathbf{A}_m$ is

$$g(x) = x^m - x^{m-1} - x^{m-2} - \cdots - x - 1. \tag{2}$$

Now, just to fix notation, given n numbers

$$\{a_i\}_1^n = \{a_1, \, a_2, \, \ldots, \, a_n\}$$

let $S_k^n \langle a_i \rangle$ denote the symmetric functions, that is

$$S_k^n \langle a_i \rangle = \sum_{\substack{1 \le i_1, \ldots, i_k \le n \\ i_1 < i_2 < \cdots < i_k}} a_{i_1} a_{i_2} \cdots a_{i_k}, \qquad 1 \le k \le n.$$

The Cayley-Hamilton equation for $\mathbf{A}_m$ is

$$\mathbf{A}_m^m - \mathbf{A}_m^{m-1} - \cdots - \mathbf{A}_m - \mathbf{I} = \mathbf{0}.$$

Using relationships among coefficients of the Cayley-Hamilton equation and the eigenvalues $\{r_i, \, i = 1, \ldots, m\}$ we have

$$S_i^m(r_i) = (-1)^{i+1}, \qquad i = 1, \ldots, m.$$

Note that the maximal real root approaches 2 for m going to infinity (see [6]).
From the characteristic polynomial we can define the m-th order recurrence

$$U_n^{(m)} = U_{n-1}^{(m)} + U_{n-2}^{(m)} + \cdots + U_{n-m}^{(m)}, \quad n \geq m. \tag{3}$$

With $m = 2$ we have the Fibonacci-Lucas sequence, with $m = 3$ the Tribonacci sequence, with $m = 4$ the Tetranacci sequence, and so on. For this reason we might call this matrix a Polynacci matrix (Polymatrix).
Note that (see [1])

$$\sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} \cdots \sum_{i_{m-1}=0}^{n-i_1-\cdots-i_{m-2}} r_1^{i_1} r_2^{i_2} \times \cdots \times r_{m-1}^{i_{m-1}} r_m^{n-i_1-\cdots-i_{m-1}} = V_{n+1},$$

where

$$V_n = V_{n-1} + V_{n-2} + \cdots + V_{n-m+1} + V_{n-m},$$

with initial conditions

$$V_0 = V_1 = V_2 = \cdots = V_{m-2} = V_{m-1} = 1.$$

2 Generalized Polynacci Sequences

Now we want to choose the initial conditions of $U_n^{(m)}$ in such a way that, for any m, there holds the Binet form

$$U_n^{(m)} = r_1^n + r_2^n + \cdots + r_m^n.$$

In this way we obtain generalized Polynacci sequences. The term *generalized* stems from the fact that the Tribonacci numbers ($m = 3$) so defined bears with the Tribonacci with initial conditions 0, 1, 1 (see [4]) the same resemblance as the Lucas sequence does with the Fibonacci sequence.
It follows

$$U_n^{(m)} = \mathrm{tr}(\mathbf{A}_m^n).$$

Now we are going to establish the following result

$$\mathrm{tr}(\mathbf{A}_m^k) = \mathrm{tr}(\mathbf{A}_{m-1}^k), \qquad k < m, \tag{4}$$

where $\mathrm{tr}(\cdot)$ is the trace operator, so that

$$\mathrm{tr}(\mathbf{A}_m^k) = \mathrm{tr}(\mathbf{A}_k^k). \tag{5}$$

We assume $k > 0$, since $\mathrm{tr}(\mathbf{A}_m^0) = m$. We need the following result, which can be established with tedious calculations,

$$\mathbf{e}_2'\mathbf{A}_m^i\mathbf{e}_1 = 0, \qquad i = 0,\, 1,\, \ldots,\, m-2. \tag{6}$$

Using repeated matrix multiplication on the partitioned form (Equation 1) of $\mathbf{A}_m$ the asked for trace is the sum of the trace of the resulting first diagonal blok plus the resulting scalar which is the second diagonal block. Starting analyzing this scalar we see that, using Equation 6, for $h \le k$, this term in $\mathbf{A}_m^h$ is $\mathbf{e}_2'\mathbf{A}_{m-1}^{h-2}\mathbf{e}_1$. Since $h \le k$ and $k < m$ it follows $h-2 < m-2$ so that using again Equation 6 with m replaced by $m-1$ we have $\mathbf{e}_2'\mathbf{A}_{m-1}^{h-2}\mathbf{e}_1 = 0,\ \forall\, h$.
As for the first diagonal block the resulting expression in $\mathbf{A}_m^k$ is a rather messy sum one term of which is $\mathbf{A}_{m-1}^k$. But since we are interested just in the trace, using the fact that for a matrix $\mathbf{B}$ and vectors $\mathbf{u}$ and $\mathbf{z}$ we have $\mathrm{tr}(\mathbf{Auz}') = \mathrm{tr}(\mathbf{z}'\mathbf{Au})$, we get that the trace of this block is $\mathrm{tr}(\mathbf{A}_{m-1}^k)$ plus the sum of terms of the form

$$c_i \prod_{j=1}^{\lceil \frac{k}{2} \rceil - 1} \mathbf{e}_2'\mathbf{A}_{m-1}^{2j+\alpha}\mathbf{e}_1, \tag{7}$$

where

$$\alpha = 2\left\{ \left\lfloor \frac{k}{2} \right\rfloor - \frac{k}{2} \right\},$$

that is $\alpha = 0$ if k is even, and $\alpha = -1$ if k is odd, and c_i are constant. Since the greatest power of $\mathbf{A}_{m-1}$ in Equation 7 is $k-2$ and $k-2 < m-2$ again invoking Equation 6 all these terms turn out to be equal to zero. From this we obtain Equation 4.
It follows that we can determine the initial conditions recursively from Lucas initial conditions ($m = 2$), obtaining, for example, for generalized Tribonacci (A001644 in [5]) 3,1,3; for generalized Tetranacci (A073817 in [5]) 4,1,3,7; for generalized Pentanacci (A074048 in [5]) 5,1,3,7,15; for generalized Hexanacci (A074584 in [5]) 6,1,3,7,15,31.
But we can also obtain a general formula. As we know $U_0^{(m)} = m$, $U_1^{(m)} = 1$. We are going to prove that

$$U_i^{(m)} = 2^i - 1, \quad i = 2,\, \ldots,\, m-1. \tag{8}$$

This will be done easily by induction on m. This is true for $m = 2$ since $U_2^{(2)}$ is L_2 which is $3 = 2^2 - 1$. Assume that the claim holds for m: then we have to show that

$$U_i^{(m+1)} = 2^i - 1, \quad i = 2, \ldots, m.$$

Now, for $i = 2, \ldots, m-1$,

$$\begin{aligned} U_i^{(m+1)} &= \operatorname{tr}(\mathbf{A}_{m+1}^i) \\ &= \operatorname{tr}(\mathbf{A}_m^i) \\ &= U_i^(m) \\ &= 2^i - 1, \end{aligned}$$

because of Equation 4. For $i = m$ we have

$$\begin{aligned} U_m^{(m+1)} &= \operatorname{tr}(\mathbf{A}_{m+1}^m) \\ &= \operatorname{tr}(\mathbf{A}_m^m) \\ &= U_m^{(m)}. \end{aligned}$$

But $U_m^{(m)}$ is the first determined by the recurrence, so

$$\begin{aligned} U_m^{(m)} &= \sum_{i=0}^{m-1} U_i^{(m)} \\ &= m + 1 + \sum_{i=2}^{m-1} (2^i - 1) \\ &= 2^m - 1, \end{aligned}$$

since

$$\sum_{i=2}^{m-1} (2^i - 1) = 2^m - m - 2,$$

and so the claim is proved.

Now we are going to derive a closed form of the ordinary generating function (ogf). So let $G(x)$ be the ogf of $U_n^{(m)}$

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where $a_i = U_i^{(m)}$. Multiply both sides by x^i, $i = 1, \ldots, m$. Then

$$\begin{aligned} G(x)(1 - x - x^2 - x^3 - \ldots - x^m) &= a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 \\ &+ (a_3 - a_2 - a_1 - a_0)x^3 + \cdots + (a_m - a_{m-1} - \cdots - a_1 - a_0)x^m + \cdots \end{aligned}$$

Because of the recurrence relationship all the coefficients of x^j, $j \geq m$ are equal to zero. Now insert the initial conditions a_i, $i = 0, \ldots, m-1$ and we are left with

$$G(x)(1 - x - x^2 - x^3 - \ldots - x^m) = m + (1-m)x + (2-m)x^2 + \cdots + \\ + \left(2^{m-1} - 1 - \sum_{i=2}^{m-2}(2^i - 1) - 1 - m\right)x^{m-1}.$$

The last summand turns out equal to -1. Then we can conclude

$$G(x) = \frac{m - (m-1)x - (m-2)x^2 - \cdots - x^{m-1}}{1 - x - x^2 - x^3 - \cdots - x^m}. \tag{9}$$

Note that we can allow for negative subscripts, following and generalizing [3]. If we have a general m-th order linear recurrence

$$U_n^{(m)} = s_1 U_{n-1}^{(m)} + s_2 U_{n-2}^{(m)} + \cdots + s_m U_{n-m}^{(m)}, \quad n \geq m \tag{10}$$

then we can define

$$U_{-n}^{(m)} = -\frac{s_{m-1}}{s_m} U_{-(n-1)}^{(m)} - \frac{s_{m-2}}{s_m} U_{-(n-2)}^{(m)} - \cdots - \frac{s_1}{s_m} U_{-(n-m+1)}^{(m)} + \frac{1}{s_m} U_{-(n-m)}^{(m)}. \tag{11}$$

3 Reflected Sequences

The reflected polynomial ([2, p. 339]) of $g(x)$ is

$$g^R(x) = -x^m - x^{m-1} - x^{m-2} - \cdots - x + 1. \tag{12}$$

The roots are the reciprocals of the roots of $g(x)$, that is $\left\{\frac{1}{r_i}\right\}$, and so $g^R(x)$ is the characteristic polynomial of matrix $\mathbf{B}_m$.

Definition 1 *The* **reflected** *recurrence $\tilde{U}_n^{(m)}$ of recurrence 3 is the recurrence with characteristic polynomial which is the reflected characteristic polynomial and with initial conditions such that the coefficients of the respective Binet forms are the same.*

It follows

$$\tilde{U}_n^{(m)} = -\tilde{U}_{n-1}^{(m)} - \tilde{U}_{n-2}^{(m)} - \cdots - \tilde{U}_{n-m+1}^{(m)} + \tilde{U}_{n-m}^{(m)}, \quad n \geq m,$$

and

$$\tilde{U}_n^{(m)} = \frac{1}{r_1^n} + \frac{1}{r_2^n} + \cdots + \frac{1}{r_m^n} = \mathrm{tr}(\mathbf{B}_m^n).$$

Now we are going to evaluate $\mathrm{tr}(\mathbf{B}_m^n)$, for $n = 0, 1, \ldots m-1$ so that we get the required initial conditions. Of course $\mathrm{tr}(\mathbf{B}_m^0) = m$. We know what is the expression for $\mathbf{B}_m$, so $\mathrm{tr}(\mathbf{B}_m^1) = -1$. Now consider what happens when we perform matrix multiplication, starting from $\mathbf{B}_m\mathbf{B}_m$: when we postmultiply by $\mathbf{B}_m$ then the columns of the matrix to the left are shifted to the left by one place and the last column is the linear combination of the columns of the matrix to the left with coefficients $\{1, -1, -1, \ldots, -1\}$. It is easy to see that for $\mathbf{B}_m\mathbf{B}_m$ the last column is $\{-1, 2, 0, \ldots, 0\}$. Then all diagonal elements are zero except for the element in the next-to-last column which is -1, so that the trace is -1. If we go ahead and consider $\mathbf{B}_m^3$ we see that the last column is $\{0, -1, 2, 0, \ldots, 0\}$ and the only non zero diagonal element appears in column $m-2$ and it is equal to -1, so the trace is -1. Repeating the same reasoning we come to $\mathbf{B}_m^{m-1}$: here the last column is $\{0, \ldots, 0, -1, 2, 0\}$ and the only non zero diagonal element appear in column 2 and it is equal to -1, so the trace is -1. So we have proved that the asked for initial conditions are

$$\tilde{U}_0^{(m)} = m,\ \tilde{U}_1^{(m)} = \tilde{U}_2^{(m)} = \cdots = \tilde{U}_{m-1}^{(m)} = -1.$$

Of course either using the recurrence or going on with the multiplication process it turns out that $\tilde{U}_m^{(m)} = 2m-1$. If for example $m = 4$ we get

$$\tilde{U}_0^{(4)} = 4,\ \tilde{U}_1^{(4)} = -1,\ \tilde{U}_2^{(4)} = -1,\ \tilde{U}_3^{(4)} = -1,$$

while

$$U_0^{(4)} = 4,\ U_1^{(4)} = 1,\ U_2^{(4)} = 3,\ U_3^{(4)} = 7.$$

Now we are going to derive a closed form of the ordinary generating function (ogf) for the reflected recurrence. So let $\tilde{G}(x)$ be the ogf of $\tilde{U}n^{(m)}$

$$\tilde{G}(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

where $a_i = \tilde{U}_i^{(m)}$. Multiply both sides by x^i, $i = 1, \ldots, m$. Then

$$\begin{aligned}\tilde{G}(x)(1 + x + x^2 + - \ldots + x^{m-1} - x^m) &= a_0 + (a_1 + a_0)x + (a_2 + a_1 + a_0)x^2 \\ &+ (a_3 + a_2 + a_1 + a_0)x^3 + \cdots + (a_m + a_{m-1} + \cdots + a_1 - a_0)x^m + \cdots\end{aligned}$$

Because of the recurrence relationship all the coefficients of x^j, $j \geq m$ are equal to zero. Now insert the initial conditions a_i, $i = 0, \ldots, m-1$ and we are left with

$$\tilde{G}(x)(1 + x + x^2 + \ldots + x^{m-1} - x^m) = m + (m-1)x + (m-2)x^2 + \cdots + (2m - 1 - 1 - \cdots - 1 - m)x^{m-1}.$$

The last summand turns out equal to 1. Then we can conclude

$$\tilde{G}(x) = \frac{m + (m-1)x + (m-2)x^2 + \cdots + x^{m-1}}{1 + x + x^2 + \cdots + x^{m-1} - x^m}. \tag{13}$$

Note that for generalized Polynacci sequences Equation 11 becomes

$$U_{-n}^{(m)} = -U_{-(n-1)}^{(m)} - U_{-(n-2)}^{(m)} - \cdots - U_{-(n-m+1)}^{(m)} + U_{-(n-m)}^{(m)}. \tag{14}$$

From Equation 8 we get easily

$$\begin{aligned} U_{-1}^{(m)} &= -U_0^{(m)} - U_1^{(m)} - \cdots - U_{m-2}^{(m)} + U_{m-1}^{(m)} \\ &= -1. \end{aligned}$$

In the same way

$$\begin{aligned} U_{-2}^{(m)} &= -U_{-1}^{(m)} - U_0^{(m)} - \cdots - U_{m-3}^{(m)} + U_{m-2}^{(m)} \\ &= -1, \end{aligned}$$

and in general

$$U_{-(i-1)} = -1 \qquad i = 2, \ldots, m.$$

From this it follows that

$$\tilde{U}_n^{(m)} = U_{-n}^{(m)}. \tag{15}$$

Reflected Tribonacci is A073145, reflected Tetranacci is A074058, reflected Pentanacci is A074062 in [5].

4 Inverted Sequences

Another related sequence $\hat{U}_n^{(m)}$ is obtained in the following way. Define its generating function $\hat{G}(x)$ as

$$\hat{G}(x) = \frac{1}{x} G\left(\frac{1}{x}\right).$$

Then we get

$$\hat{G}(x) = \frac{1 + 2x + 3x^2 + \cdots + (m-1)x^{m-2} - mx^{m-1}}{1 + x + x^2 + \cdots + x^{m-1} - x^m}.$$

Note that the numerator is the derivative of the denominator and also that the denominator is the same as in $\tilde{G}(x)$. Using the Rational Expansion Theorem for Distinct Roots in [2, p. 340] we get easily the closed form

$$\begin{aligned} \hat{U}_n^{(m)} &= -\frac{1}{r_1^{n+1}} - \frac{1}{r_2^{n+1}} - \cdots - \frac{1}{r_m^{n+1}} \\ &= -\tilde{U}_{n+1}^{(m)}. \end{aligned} \tag{16}$$

The recurrence is then

$$\hat{U}_n^{(m)} = -\hat{U}_{n-1}^{(m)} - \hat{U}_{n-2}^{(m)} - \cdots - \tilde{U}_{n-m+1}^{(m)} + \hat{U}_{n-m}^{(m)},$$

with

$$\hat{U}_0^{(m)} = 1,\ \hat{U}_1^{(n)} = 1,\ \ldots,\ \hat{U}_{m-1}^{(m)} = 1 - 2m.$$

$\hat{U}_n^{(3)}$ is sequence A075298 in [5].

5 Cayley-Hamilton Equation

Now consider the Cayley-Hamilton equation for $\mathbf{A}_m^n$

$$\mathbf{A}_m^{nm} - c_1^{(n)}\mathbf{A}_m^{n(m-1)} + \cdots + (-1)^i c_i^{(n)}\mathbf{A}_m^{n(m-i)} + \cdots + (-1)^m c_m^{(n)}\mathbf{I} = \mathbf{0},$$

where $c_i^{(n)}$ is the sum of the determinants of the principal minors of order i of $\mathbf{A}_m^n$. Immediately we have

$$c_1^{(n)} = \mathrm{tr}(\mathbf{A}_m^n) = U_n^{(m)},$$
$$c_m^{(n)} = |\mathbf{A}_m|^n = (-1)^{(m+1)n}.$$

Note that if m is odd $c_m^{(n)} = 1$, while if m is even $c_m^{(n)} = (-1)^n$.
Then

$$\begin{aligned} \frac{1}{r_1^n} + \frac{1}{r_2^n} + \cdots + \frac{1}{r_m^n} &= \frac{S_{m-1}^m \langle r_i^n \rangle}{\prod_{i=1}^m r_i^n} \\ &= \frac{S_{m-1}^m \langle r_i^n \rangle}{(-1)^{(m+1)n}} \\ &= \mathrm{tr}(\mathbf{B}_m^n) \\ &= \tilde{U}_n^{(m)}. \end{aligned}$$

From this it follows

$$c_{m-1}^{(n)} = (-1)^{(m+1)n} \tilde{U}_n^{(m)}, \tag{17}$$

that is $c_{m-1}^{(n)} = \tilde{U}_n^{(m)}$ if m is odd, and $c_{m-1}^{(n)} = (-1)^n \tilde{U}_n^{(m)}$ if m is even. Note that if the generating function of $\tilde{U}_n^{(m)}$ is $\tilde{G}(x)$ then the generating function of $(-1)^n \tilde{U}_n^{(m)}$ is $\tilde{G}(-x)$.

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