

LATTICE PATH MATROIDS: ENUMERATIVE ASPECTS AND TUTTE POLYNOMIALS

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ABSTRACT. Fix two lattice paths P and Q from $(0,0)$ to (m,r) that use East and North steps with P never going above Q . We show that the lattice paths that go from $(0,0)$ to (m,r) and that remain in the region bounded by P and Q can be identified with the bases of a particular type of transversal matroid, which we call a lattice path matroid. We consider a variety of enumerative aspects of these matroids and we study three important matroid invariants, namely the Tutte polynomial and, for special types of lattice path matroids, the characteristic polynomial and the β invariant. In particular, we show that the Tutte polynomial is the generating function for two basic lattice path statistics and we show that certain sequences of lattice path matroids give rise to sequences of Tutte polynomials for which there are relatively simple generating functions. We show that Tutte polynomials of lattice path matroids can be computed in polynomial time. Also, we obtain a new result about lattice paths from an analysis of the β invariant of certain lattice path matroids.

1. INTRODUCTION

This paper develops a new connection between matroid theory and enumerative combinatorics: with every pair of lattice paths P and Q that have common endpoints we associate a matroid in such a way that the bases of the matroid correspond to the paths that remain in the region bounded by P and Q . These matroids, which we call lattice path matroids, appear to have a wealth of interesting and striking properties. In this paper we focus on the enumerative aspects of lattice path matroids, including the study of important matroid invariants like the Tutte and the characteristic polynomials. Structural aspects of lattice path matroids and their relation with other families of matroids will be the subject of a forthcoming paper [3].

Lattice path matroids provide a bridge between matroid theory and the theory of lattice paths that, as we demonstrate here and in [3], can lead to a mutually enriching relationship between the two subjects. One example starts with the path interpretation we give for each coefficient of the Tutte polynomial of a lattice path matroid. Computing the Tutte polynomial of an arbitrary matroid is known to be $\#P$ -complete; the same is true even within special classes such as graphic or transversal matroids. However, by using the path interpretation of the coefficients in the case of lattice path matroids, we show that the Tutte polynomial of such a

Date: July 10, 2004.

1991 Mathematics Subject Classification. Primary: 05A15, 05B35.

Key words and phrases. Matroid, transversal matroid, Tutte polynomial, characteristic polynomial, Möbius function, β invariant, broken circuit complex, nbc-set, basis activities, lattice path, Dyck path, Catalan number.

matroid can be computed in polynomial time. On the lattice path side, as we illustrate in Section 8, this interpretation of the coefficients along with easily computed examples of the Tutte polynomial can suggest new theorems about lattice paths.

Relatively little matroid theory is required to understand this paper and what is needed is sketched in the first part of Section 2. We follow the conventional notation for matroid theory as found in [11]. A few topics of matroid theory of a more specialized nature (the Tutte polynomial, the broken circuit complex, the characteristic polynomial, and the β invariant) are presented in the sections in which they play a role. The last part of Section 2 outlines the basic facts on lattice path enumeration that we use.

Lattice path matroids, the main topic of this paper, are defined in Section 3, where we also identify their bases with lattice paths (Theorem 3.3). We introduce special classes of lattice path matroids, among which are the Catalan matroids and, more generally, the k -Catalan matroids, for which the numbers of bases are the Catalan numbers and the k -Catalan numbers. We also treat some basic enumerative results for lattice path matroids and prove several structural properties of these matroids that play a role in enumerative problems that are addressed later in the paper. Counting connected lattice path matroids on a given number of elements is the topic of Section 4.

The next four sections consider matroid invariants in the case of lattice path matroids. Section 5 gives a lattice path interpretation of each coefficient of the Tutte polynomial of a lattice path matroid (Theorem 5.4) as well as generating functions for the Tutte polynomials of the sequence of k -Catalan matroids (Theorem 5.6) and, from that, a formula for each coefficient of each of these Tutte polynomials (Theorem 5.7). In Section 6 we give an algorithm for computing the Tutte polynomial of any lattice path matroid in polynomial time; we provide a second method of computation that applies for certain classes of lattice path matroid and which, although limited in scope, is particularly simple to implement on a computer. In Section 7 we show that the broken circuit complex of a lattice path matroid is the independence complex of another lattice path matroid and we develop formulas for the coefficients of the characteristic polynomial for special classes of lattice path matroids. Section 8 shows that k times the Catalan number C_{kn-1} counts lattice paths of a special type (Theorem 8.3); the key to discovering this result was looking at a particular coefficient (the β invariant) of the Tutte polynomials of certain lattice path matroids.

The final section connects lattice path matroids with a problem of current interest in enumerative combinatorics, namely, the $(k+l, l)$ -tennis ball problem.

We use the following common notation: $[n]$ denotes the set $\{1, 2, \dots, n\}$ and $[m, n]$ denotes the set $\{m, n+1, \dots, n\}$. We follow the convention in matroid theory of writing $X \cup e$ and $X - e$ in place of $X \cup \{e\}$ and $X - \{e\}$.

2. BACKGROUND

In this section we introduce the concepts of matroid theory that are needed in this paper. For a thorough introduction to the subject we refer the reader to Oxley [11]; the proofs we omit in this section can be found there, mostly in Chapters 1 and 2. We conclude this section with the necessary background on the enumerative theory of lattice paths.

Definition 2.1. A matroid is a pair $(E(M), \mathcal{B}(M))$ consisting of a finite set $E(M)$ and a collection $\mathcal{B}(M)$ of subsets of $E(M)$ that satisfy the following conditions:

- (B1) $\mathcal{B}(M) \neq \emptyset$,
- (B2) $\mathcal{B}(M)$ is an antichain, that is, no set in $\mathcal{B}(M)$ properly contains another set in $\mathcal{B}(M)$, and
- (B3) for each pair of distinct sets B, B' in $\mathcal{B}(M)$ and for each element $x \in B - B'$, there is an element $y \in B' - B$ such that $(B - x) \cup y$ is in $\mathcal{B}(M)$.

The set $E(M)$ is the *ground set* of M and the sets in $\mathcal{B}(M)$ are the *bases* of M . Subsets of bases are *independent sets*; the collection of independent sets of M is denoted $\mathcal{I}(M)$. Sets that are not independent are *dependent*. A *circuit* is a minimal dependent set. If $\{x\}$ is a circuit, then x is a *loop*. Thus, no basis of M can contain a loop. An element that is contained in every basis is an *isthmus*.

It is easy to show that all bases of M have the same cardinality. More generally, for any subset A of $E(M)$ all maximal independent subsets of A have the same cardinality; $r(A)$, the *rank* of A , denotes this common cardinality. If several matroids are under consideration, we may use $r_M(A)$ to avoid ambiguity. In place of $r(E(M))$, we write $r(M)$.

The *closure* of a set $A \subseteq E(M)$ is defined as

$$\text{cl}(A) = \{x \in E(M) : r(A \cup x) = r(A)\}.$$

A set F is a *flat* if $\text{cl}(F) = F$. The flats of a matroid, ordered by inclusion, form a geometric lattice.

It is well-known that matroids can be characterized in terms of each of the following objects: the independent sets, the dependent sets, the circuits, the rank function, the closure operator, and the flats (see Sections 1.1–1.4 of [11]).

EXAMPLE. A matroid of rank r is a *uniform matroid* if all r -element subsets of the ground set are bases. There is, up to isomorphism, exactly one uniform matroid of rank r on an m -element set; this matroid is denoted $U_{r,m}$.

One fundamental concept in matroid theory is duality. Given a matroid M , its *dual matroid* M^* is the matroid on $E(M)$ whose set of bases is given by

$$\mathcal{B}(M^*) = \{E(M) - B : B \in \mathcal{B}(M)\}.$$

A matroid is *self-dual* if it is isomorphic to its dual; a matroid is *identically self-dual* if it is equal to its dual. For example, the dual of the uniform matroid $U_{r,m}$ is the uniform matroid $U_{m-r,m}$. The matroid $U_{r,2r}$ is identically self-dual. For any matroid M , the element x is a loop of M if and only if x is an isthmus of the dual M^* .

This paper investigates a special class of transversal matroids. Let $\mathcal{A} = (A_j : j \in J)$ be a set system, that is, a multiset of subsets of a finite set S . A *transversal* (or system of distinct representatives) of \mathcal{A} is a set $\{x_j : j \in J\}$ of $|J|$ distinct elements such that $x_j \in A_j$ for all j in J . A *partial transversal* of \mathcal{A} is a transversal of a set system of the form $(A_k : k \in K)$ with K a subset of J . The following theorem is a fundamental result due to Edmonds and Fulkerson.

Theorem 2.2. *The partial transversals of a set system $\mathcal{A} = (A_j : j \in J)$ are the independent sets of a matroid on S .*

A *transversal matroid* is a matroid whose independent sets are the partial transversals of some set system $\mathcal{A} = (A_j : j \in J)$; we say that \mathcal{A} is a *presentation* of the

transversal matroid. The bases of a transversal matroids are the maximal partial transversals of \mathcal{A} . For more on transversal matroids see [11, Section 1.6].

Given two matroids M_1, M_2 on disjoint ground sets, their *direct sum* is the matroid $M_1 \oplus M_2$ with ground set $E(M_1) \cup E(M_2)$ whose collection of bases is

$$\mathcal{B}(M_1 \oplus M_2) = \{B_1 \cup B_2 : B_1 \in \mathcal{B}(M_1) \text{ and } B_2 \in \mathcal{B}(M_2)\}.$$

It is easy to check that the lattice of flats of $M_1 \oplus M_2$ is isomorphic to the direct product (or cartesian product) of the lattice of flats of M_1 and that of M_2 . A matroid M is *connected* if it is not a direct sum of two nonempty matroids. Note that connected matroids with at least two elements have neither loops nor isthmuses.

We say that the matroid $M \oplus U_{1,1}$ is formed by *adding an isthmus* to M . In the case that the ground set of the uniform matroid $U_{1,1}$ is e , we shorten this notation to $M \oplus e$ if there is no danger of ambiguity. Of course, e is an isthmus of $M \oplus e$.

There is a well developed theory of extending matroids by single elements [11, Section 7.2]. The case that is relevant to this paper is that of free extension, which consists of adding an element to the matroid as independently as possible without increasing the rank. Precisely, the *free extension* $M + e$ of M is the matroid on $E(M) \cup e$ whose collection of independent sets is given as follows:

$$\mathcal{I}(M + e) = \mathcal{I}(M) \cup \{I \cup e : I \in \mathcal{I}(M) \text{ and } |I| < r(M)\}.$$

The bases of $M + e$, where M has rank r , are the bases of M together with the sets of the form $I \cup e$, where I is an $(r-1)$ -element independent set of M . Equivalently, the rank function of $M + e$ is given by the following equations: for X a subset of $E(M)$,

$$r_{M+e}(X) = r_M(X)$$

and

$$r_{M+e}(X \cup e) = \begin{cases} r_M(X) + 1, & \text{if } r_M(X) < r(M); \\ r(M), & \text{otherwise.} \end{cases}$$

The particular matroids of interest in this paper arise from lattice paths, to which we now turn. We consider two kinds of lattice paths, both of which are in the plane. Most of the lattice paths we consider use steps $E = (1, 0)$ and $N = (0, 1)$; in several cases it is more convenient to use lattice paths with steps $U = (1, 1)$ and $D = (1, -1)$. The letters are abbreviations of East, North, Up, and Down. We will often treat lattice paths as words in the alphabets $\{E, N\}$ or $\{U, D\}$, and we will use the notation α^n to denote the concatenation of n letters, or strings of letters, α . If $P = s_1 s_2 \dots s_n$ is a lattice path, then its *reversal* is defined as $P^\rho = s_n s_{n-1} \dots s_1$. The *length* of a lattice path $P = s_1 s_2 \dots s_n$ is n , the number of steps in P .

Here we recall the facts we need about the enumeration of lattice paths; the proofs of the following lemmas can be found in Sections 3 to 5 of the first chapter of [10]. The most basic enumerative results about lattice paths are those in the following lemma.

Lemma 2.3. *For a fixed positive integer k , the number of lattice paths from $(0, 0)$ to (kn, n) that use steps E and N and that never pass above the line $y = x/k$ is the n -th k -Catalan number*

$$C_n^k = \frac{1}{kn+1} \binom{(k+1)n}{n}.$$

In particular, the number of paths from $(0, 0)$ to (n, n) that never pass above the line $y = x$ is the n -th Catalan number

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

We also use the following result, which generalizes Lemma 2.3. For $k = 1$ the numbers displayed in Lemma 2.4 are called the *ballot numbers*.

Lemma 2.4. *For $m \geq kn \geq 0$, the number of lattice paths from $(0, 0)$ to (m, n) with steps E and N that never go above the line $y = x/k$ is*

$$\frac{m - kn + 1}{m + n + 1} \binom{m + n + 1}{n}.$$

The next lemma treats paths in the alphabet $\{U, D\}$; the first assertion, which concerns what are usually called Dyck paths, is equivalent to the second part of Lemma 2.3 by the obvious identification of the alphabets.

Lemma 2.5. (i) *The number of paths from $(0, 0)$ to $(2n, 0)$ with steps U and D that never pass below the x -axis is the n -th Catalan number C_n .*
(ii) *The number of paths of n steps in the alphabet $\{U, D\}$ that start at $(0, 0)$ and never pass below the x -axis (not necessarily ending on the x -axis) is $\binom{n}{\lfloor n/2 \rfloor}$.*

The following result will be used to count certain types of lattice paths.

Lemma 2.6. *Let*

$$C(z) = \sum_{n \geq 0} \frac{1}{kn + 1} \binom{(k+1)n}{n} z^n$$

be the generating function for the k -Catalan numbers. The coefficient of z^t in $C(z)^j$ is

$$\frac{j}{t} \binom{(k+1)t + j - 1}{t - 1}.$$

3. LATTICE PATH MATROIDS

In this section we define lattice path matroids as well as several important subclasses. Later sections of this paper develop much of the enumerative theory for lattice path matroids in general and this theory is pushed much further for certain special families of lattice path matroids.

Definition 3.1. *Let $P = p_1 p_2 \dots p_{m+r}$ and $Q = q_1 q_2 \dots q_{m+r}$ be two lattice paths from $(0, 0)$ to (m, r) with P never going above Q . Let $\{p_{u_1}, \dots, p_{u_r}\}$ be the set of North steps of P with $u_1 < u_2 < \dots < u_r$; similarly, let $\{q_{l_1}, \dots, q_{l_r}\}$ be the set of North steps of Q with $l_1 < l_2 < \dots < l_r$. Let N_i be the interval $[l_i, u_i]$ of integers. Let $M[P, Q]$ be the transversal matroid that has ground set $[m+r]$ and presentation $(N_i : i \in [r])$; the pair (P, Q) is a presentation of $M[P, Q]$. A lattice path matroid is a matroid M that is isomorphic to $M[P, Q]$ for some such pair of lattice paths P and Q .*

Several examples of lattice path matroids are given after Theorem 3.3, which identifies the bases of these matroids in terms of lattice paths. To avoid needless repetition, throughout the rest of the paper we assume that the lattice paths P and Q are as in Definition 3.1.

We think of $1, 2, \dots, m+r$ as the first step, the second step, etc. Observe that the set N_i contains the steps that can be the i -th North step in a lattice path from $(0, 0)$ to (m, r) that remains in the region bounded by P and Q . When thought of as arising from the particular presentation using bounding paths P and Q , the elements of the matroid are ordered in their natural order, i.e., $1 < 2 < \dots < m+r$; we will frequently use this order throughout the paper. However, this order is not inherent in the matroid structure; the elements of a lattice path matroid typically can be linearly ordered in many ways so as to correspond to steps in lattice paths. (This point will be addressed in greater detail in [3].)

We associate a lattice path $P(X)$ with each subset X of the ground set of a lattice path matroid as specified in the next definition.

Definition 3.2. *Let X be a subset of the ground set $[m+r]$ of the lattice path matroid $M[P, Q]$. The lattice path $P(X)$ is the word*

$$s_1 s_2 \dots s_{m+r}$$

in the alphabet $\{E, N\}$ where

$$s_i = \begin{cases} N, & \text{if } i \in X; \\ E, & \text{otherwise.} \end{cases}$$

Thus, the path $P(X)$ is formed by taking the elements of $M[P, Q]$ in the natural linear order and replacing each by a North step if the element is in X and by an East step if the element is not in X .

The fundamental connection between the transversal matroid $M[P, Q]$ and the lattice paths that stay in the region bounded by P and Q is the following theorem which says that the bases of $M[P, Q]$ can be identified with such lattice paths.

Theorem 3.3. *A subset B of $[m+r]$ with $|B| = r$ is a basis of $M[P, Q]$ if and only if the associated lattice path $P(B)$ stays in the region bounded by P and Q .*

Proof. Let B be $\{b_1, \dots, b_r\}$ with $b_1 < b_2 < \dots < b_r$ in the natural order. Suppose first that B is a basis of $M[P, Q]$, that is, a transversal of $(N_i : i \in [r])$. The conclusion will follow if we prove that b_i is in N_i . Assume, to the contrary, $b_i \notin N_i$. Since either $b_i < l_i$ or $b_i > u_i$, we obtain the following contradictions: in the first case, the set $\{b_1, b_2, \dots, b_i\}$ must be a transversal of $(N_1, N_2, \dots, N_{i-1})$; in the second, $\{b_i, b_{i+1}, \dots, b_r\}$ must be a transversal of $(N_{i+1}, N_{i+2}, \dots, N_r)$.

Conversely, if the lattice path $P(B)$ goes neither below P nor above Q , then for every i we have that b_i , the i -th North step of $P(B)$, satisfies $l_i \leq b_i \leq u_i$, and hence that B is a transversal of $(N_i : i \in [r])$. \square

Corollary 3.4. *The number of basis of $M[P, Q]$ is the number of lattice paths from $(0, 0)$ to (m, r) that go neither below P nor above Q .*

Figure 1 illustrates Theorem 3.3. In this example we have $N_1 = \{2, 3, 4\}$, $N_2 = \{4, 5\}$, and $N_3 = \{6\}$. There are five bases of this transversal matroid. Note that 1 is a loop and 6 is an isthmus.

EXAMPLE. For the lattice paths $P = E^m N^r$ and $Q = N^r E^m$, every r -subset of $[m+r]$ is a basis of $M[P, Q]$. Thus, the uniform matroid $U_{r, m+r}$ is a lattice path matroid.

Recall that the bases of the dual M^* of a matroid M are the set complements of the bases of M with respect to the ground set $E(M)$. Thus, for a lattice path

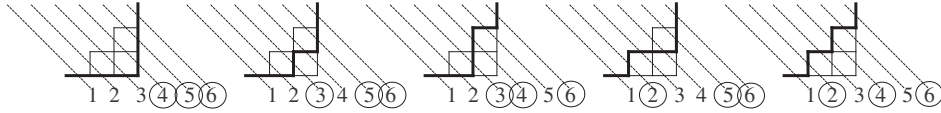


FIGURE 1. The bases $\{4, 5, 6\}$, $\{3, 5, 6\}$, $\{3, 4, 6\}$, $\{2, 5, 6\}$, and $\{2, 4, 6\}$ of a lattice path matroid represented as the North steps of lattice paths.

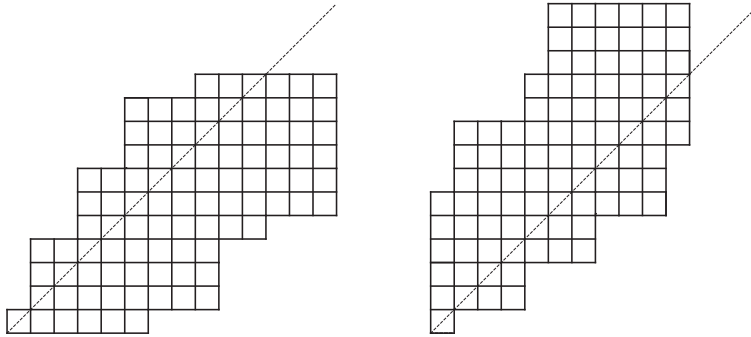


FIGURE 2. Presentations of a lattice path matroid and its dual.

matroid M , the bases of the dual matroid correspond to the East steps in lattice paths. Reflecting a lattice path presentation of M about the line $y = x$ shows that the dual M^* is also a lattice path matroid. (See Figure 2.) This justifies the following theorem.

Theorem 3.5. *The class of lattice path matroids is closed under matroid duality.*

Note that a 180° rotation of the region bounded by P and Q , translated to start at $(0, 0)$, yields the same matroid although the labels on the elements are reversed. Thus the lattice path matroids $M[P, Q]$ and $M[Q^\rho, P^\rho]$ are isomorphic. It follows, for example, that the lattice path matroid in Figure 1 is self-dual; note that this matroid is not identically self-dual since the loop 1 and the isthmus 6 in the matroid are, respectively, an isthmus and a loop in the dual.

Figure 3 illustrates the next result. The proof is immediate from Theorem 3.3 and the definition of direct sums.

Theorem 3.6. *The class of lattice path matroids is closed under direct sums. Furthermore, the lattice path matroid $M[P, Q]$ is connected if and only if the bounding lattice paths P and Q meet only at $(0, 0)$ and (m, r) .*

We now turn to a special class of lattice path matroids, the generalized Catalan matroids, as well as to various subclasses that exhibit a structure that is simpler than that of typical lattice path matroids. Later sections of this paper will give special attention to these classes since the simpler structure allows us to obtain more detailed enumerative results.

Definition 3.7. *A lattice path matroid M is a generalized Catalan matroid if there is a presentation (P, Q) of M with $P = E^m N^r$. In this case we simplify the*

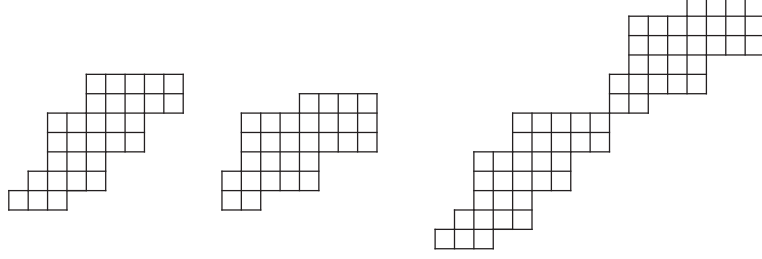


FIGURE 3. Presentations of two lattice path matroids and their direct sum.

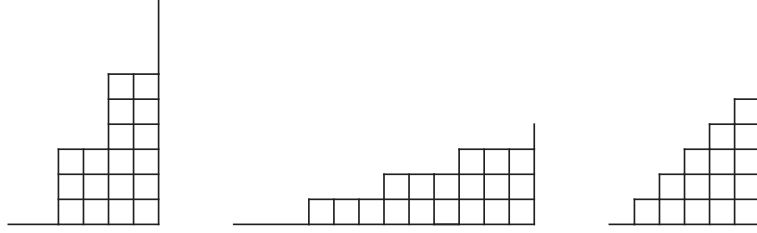


FIGURE 4. Presentations of the rank nine matroid $M_3^{2,3}$, the 3-Catalan matroid M_4^3 of rank four, and the rank six Catalan matroid M_6 .

notation $M[P, Q]$ to $M[Q]$. If in addition the upper path Q is $(E^k N^l)^n$ for some positive integers k, l , and n , we say that M is the (k, l) -Catalan matroid $M_n^{k,l}$. In place of $M_n^{k,l}$ we write M_n^k ; such matroids are called k -Catalan matroids. In turn, we simplify the notation M_n^1 to M_n ; such matroids are called Catalan matroids.

Figure 4 gives presentations of a $(2, 3)$ -Catalan matroid, a 3-Catalan matroid, and a Catalan matroid. These matroids have, respectively, two loops and three isthmuses, three loops and one isthmus, and a single loop and isthmus.

Note that (k, l) -Catalan matroids have isthmuses and loops; specifically, the elements $1, \dots, k$ are the loops and $(k+l)n-l+1, (k+l)n-l+2, \dots, (k+l)n$ are the isthmuses of $M_n^{k,l}$. Also, observe that for the k -Catalan matroid M_n^k , Theorem 3.3 can be restated by saying that an n -element subset B of $[(k+1)n]$ is a basis of M_n^k if and only if its associated lattice path $P(B)$ does not go above the line $y = x/k$.

We next note an immediate consequence of Corollary 3.4 and Lemma 2.3. As we will see in Section 9, there is no known formula that leads to a similar result for (k, l) -Catalan matroids.

Corollary 3.8. *The number of bases of the k -Catalan matroid M_n^k is the k -Catalan number C_n^k . In particular, the number of bases of the Catalan matroid M_n is the Catalan number C_n .*

The comments before and after Theorem 3.5, including that about 180° rotations of presentations, give the following result.

Theorem 3.9. *The dual of the (k, l) -Catalan matroid $M_n^{k,l}$ is the (l, k) -Catalan matroid $M_n^{l,k}$. Thus, the matroid $M_n^{k,k}$, and in particular the Catalan matroid M_n , is self-dual but not identically self-dual.*

We turn to lattice path descriptions of circuits and independent sets in generalized Catalan matroids. Recall from Definition 3.2 that we associate a lattice path $P(X)$ with each subset X of the ground set $[m+r]$ of the lattice path matroid $M[P, Q]$. Of course, only sets of r elements give paths that end at (m, r) .

Theorem 3.10. *A subset C of $[m+r]$ is a circuit of the generalized Catalan matroid $M[Q]$ if and only if for the largest element i of C , the i -th step of $P(C)$ is the only North step of $P(C)$ above Q .*

Proof. First assume that for the largest element i of C , the i -th step of $P(C)$ is the only North step of $P(C)$ above Q . It is clear that for any superset X of C , the i -th step of $P(X)$ is also above Q . Thus, C is not contained in any basis and so is dependent. Note that for any element c in C , the lattice path $P(C-c)$ has no steps above Q ; also, the path that follows $P(C-c)$ to the line $x=m$ and then goes directly North to (m, r) is a lattice path that never goes above Q and so corresponds to a basis that contains $C-c$, specifically, the basis $(C-c) \cup Y$ where Y contains the last $r - (|C| - 1)$ elements in $[m+r]$. Thus, every proper subset of C is independent. Therefore, C is a circuit.

Conversely, assume that C is a circuit. By the same type of argument as in the second half of the last paragraph, it is clear that $P(C)$ must have at least one North step that goes above Q ; since C is a minimal dependent set, it is clear that this step must correspond to the greatest element of C . \square

Corollary 3.11. *The number of i -element circuits in the Catalan matroid M_n is the Catalan number C_{i-1} .*

Proof. From the last theorem, it follows that the lattice path $P(C)$ associated with an i -element circuit C can be decomposed as follows: a lattice path from $(0, 0)$ to $(i-1, i-1)$ that does not go above the line $y=x$, followed by one North step above the line $y=x$, followed by only East steps. Conversely, any such path corresponds to an i -element circuit. From this the result follows. \square

From Theorem 3.10 we also get the following result.

Corollary 3.12. *The independent sets in the generalized Catalan matroid $M[Q]$ are precisely the subsets X of $[m+r]$ such that the associated lattice path $P(X)$ never goes above the bounding lattice path Q .*

From this result it follows that for k -Catalan matroids, the paths that correspond to independent sets of a given size are precisely those given by Lemma 2.4.

Corollary 3.13. *The number of independent sets of size i in the k -Catalan matroid M_n^k is*

$$\frac{(k+1)(n-i)+1}{(k+1)n+1} \binom{(k+1)n+1}{i}.$$

Generalized Catalan matroids have previously appeared in the matroid theory literature [17]; they were also studied, from a very different perspective, in [12] and they were recently rediscovered in yet another context in [1]. It can be shown that generalized Catalan matroids are exactly the minors of Catalan matroids [3].

We conclude this section with yet another perspective by showing that generalized Catalan matroids are precisely the matroids that can be constructed from the empty matroid by repeatedly adding isthmuses and taking free extensions. Theorem 3.14 can be generalized to all lattice path matroids; the generalization uses more matroid theory, in particular, more general types of extensions than free extensions, so this result will appear in [3]. We present the special case here since in the last part of Section 6 we will use this result to give simple and efficient algebraic rules to compute the Tutte polynomial of any generalized Catalan matroid.

Theorem 3.14. *Let $Q = q_1q_2 \dots q_{m+r}$ be a word of length $m+r$ in the alphabet $\{E, N\}$. Let M^0 be the empty matroid and define*

$$M^i = \begin{cases} M^{i-1} + i, & \text{if } q_i = E; \\ M^{i-1} \oplus i, & \text{if } q_i = N. \end{cases}$$

Then M^{m+r} and the generalized Catalan matroid $M[Q]$ are equal.

Proof. Let Q_i be the initial segment $q_1q_2 \dots q_i$ of Q , let R_i be the region determined by the bounding paths of $M[Q_i]$, and let the paths that correspond to bases of $M[Q_i]$ go from $(0,0)$ to (m_i, r_i) . We prove the equality $M^i = M[Q_i]$ by induction on i . Both M^0 and $M[Q_0]$ are the empty matroid. Assume $M^{i-1} = M[Q_{i-1}]$. Assume first that q_i is N , so i is an isthmus of M^i . Thus we need to show that the bases of $M[Q_i]$ are precisely the sets of the form $B \cup i$ where B is a basis of $M[Q_{i-1}]$, which is clear from Theorem 3.3 since the bounding paths for $M[Q_i]$ have a common last (i -th) North step. Now suppose that q_i is E . Note the equality $r_i = r_{i-1}$. Lattice paths in the region R_i from $(0,0)$ to (m_i, r_i) are of two types: those in which the final step is North, and so correspond to sets of the form $I \cup i$ where I is an independent set of size $r_{i-1} - 1$ in $M[Q_{i-1}]$; those in which the final step is East, and so correspond to bases of $M[Q_{i-1}]$. From this and the basis formulation of free extensions, the equality $M^i = M[Q_i]$ follows. \square

4. ENUMERATION OF LATTICE PATH MATROIDS

In this section we give a formula for the number of connected lattice matroids on a given number of elements up to isomorphism; to make the final result slightly more compact, we let the number of elements be $n+1$. The proof has two main ingredients, the first of which is the following result from [3]. (Recall that P^ρ denotes the reversal $s_{n+1}s_n \dots s_1$ of a lattice path $P = s_1s_2 \dots s_ns_{n+1}$.)

Lemma 4.1. *Two connected lattice path matroids $M[P, Q]$ and $M[P', Q']$ are isomorphic if and only if either $P' = P$ and $Q' = Q$, or $P' = Q^\rho$ and $Q' = P^\rho$.*

The second main ingredient is the following bijection, going back at least to Pólya, between the pairs of lattice paths of length $n+1$ that intersect only at their endpoints and the Dyck paths of length $2n$. (See, for example, [8].) A pair (P, Q) of nonintersecting lattice paths from $(0,0)$ to (m, r) can be viewed as the special type of polyomino that in [8] is called a parallelogram polyomino. Associate two sequences (a_1, \dots, a_m) and (b_1, \dots, b_{m-1}) of integers with such a polyomino: a_i is the number of cells of the i -th column of the polyomino (columns are scanned from left to right) and $b_i + 1$ is the number of cells of column i that are adjacent to cells of column $i+1$. Since the paths are nonintersecting, each b_i is nonnegative. Now associate to (P, Q) the Dyck path π having m peaks at heights a_1, \dots, a_m and $m-1$ valleys at heights b_1, \dots, b_{m-1} . Figure 5 shows a polyomino and its associated Dyck

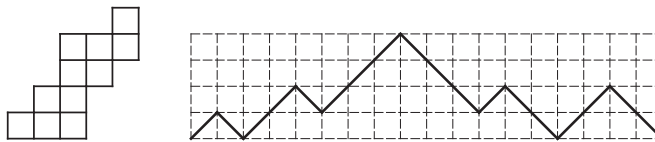


FIGURE 5. A parallelogram polyomino and its associated Dyck path.

path; the corresponding sequences for this polyomino are $(1, 2, 4, 2, 2)$ and $(0, 1, 1, 0)$. It can be checked that the correspondence $(P, Q) \mapsto \pi$ is indeed a bijection. Hence the number of such pairs (P, Q) of lattice paths of length $n + 1$ is the Catalan number C_n .

Note that C_n is not the number of connected lattice path matroids on $n + 1$ elements since different pairs of paths can give the same matroid. According to Lemma 4.1, this happens only for a pair (P, Q) and its reversal (Q^ρ, P^ρ) , so we need to find the number of pairs (P, Q) for which $(P, Q) = (Q^\rho, P^\rho)$. It is immediate to check that $(P, Q) = (Q^\rho, P^\rho)$ if and only if the corresponding Dyck path π is symmetric with respect to its center or, in other words, is equal to its reversal. Since a symmetric Dyck path of length $2n$ is determined by its first n steps, the number of such paths is given in part (ii) of Lemma 2.5. From the number C_n we obtained in the last paragraph we must subtract half the number of nonsymmetric Dyck paths, thus giving the following result.

Theorem 4.2. *The number of connected lattice path matroids on $n + 1$ elements up to isomorphism is*

$$C_n - \frac{1}{2} \left(C_n - \binom{n}{\lceil n/2 \rceil} \right) = \frac{1}{2} C_n + \frac{1}{2} \binom{n}{\lceil n/2 \rceil}.$$

This number is asymptotically of order $O(4^n)$. Since it is known that the number of transversal matroids on n elements grows like c^{n^2} for some constant c (see [4]), it follows that the class of lattice path matroids is rather small with respect to the class of all transversal matroids.

We remark that the total number of lattice path matroids (connected or not) on k elements is the number of multisets of connected lattice path matroids, the sum of whose cardinalities is k . A generating function for these numbers can be derived using standard tools; however, the result does not seem to admit a compact form so we omit it.

5. TUTTE POLYNOMIALS

The Tutte polynomial is one of the most widely studied matroid invariants. From the Tutte polynomial one obtains, as special evaluations, many other important polynomials, such as the chromatic and flow polynomials of a graph, the weight enumerator of a linear code, and the Jones polynomial of an alternating knot. (See [6, 18] for many of the numerous occurrences of this polynomial in combinatorics, in other branches of mathematics, and in other sciences.) In this section, after reviewing the definition of the Tutte polynomial, we show that for lattice path matroids this polynomial is the generating function for two basic lattice path statistics. We use this lattice path interpretation of the Tutte polynomial to give a

formula for the generating function $\sum_{n \geq 0} t(M_n^k; x, y) z^n$ for the sequence of Tutte polynomials $t(M_n^k; x, y)$ of the k -Catalan matroids. Using this generating function, we then derive a formula for each coefficient of the Tutte polynomial $t(M_n^k; x, y)$.

The Tutte polynomial $t(M; x, y)$ of a matroid M is most briefly defined as follows:

$$(1) \quad t(M; x, y) = \sum_{A \subseteq E(M)} (x-1)^{r(M)-r(A)} (y-1)^{|A|-r(A)}.$$

However, for our work the formulation in terms of internal and external activities, which we review below, will prove most useful. For a proof of the equivalence of these definitions (and that the formulation in terms of activities is well-defined), see, for example [2].

Fix a linear order $<$ on $E(M)$ and let B be a basis of M . An element $e \notin B$ is *externally active with respect to B* if there is no element y in B with $y < e$ for which $(B-y) \cup e$ is a basis. An element $b \in B$ is *internally active with respect to B* if there is no element y in $E(M) - B$ with $y < b$ for which $(B-b) \cup y$ is a basis. The *internal (external) activity* of a basis is the number of elements that are internally (externally) active with respect to that basis. We denote the activities of a basis B by $i(B)$ and $e(B)$. Note that $i(B)$ and $e(B)$ depend not only on B but also on the order $<$. The following lemma is well-known and easy to prove.

Lemma 5.1. *Let the elements of a matroid M and its dual M^* be ordered with the same linear ordering. An element e is internally active with respect to the basis B of M if and only if e is externally active with respect to the basis $E(M) - B$ of M^* .*

The Tutte polynomial, as defined in equation (1), can alternatively be expressed as follows:

$$(2) \quad t(M; x, y) = \sum_{B \in \mathcal{B}(M)} x^{i(B)} y^{e(B)}.$$

In particular, although $i(B)$ and $e(B)$, for a particular basis B , depend on the order $<$, the multiset of pairs $(i(B), e(B))$, as B ranges over the bases of M , does not depend on the order. Thus, the coefficient of $x^i y^j$ in $t(M; x, y)$ is the number of bases of M with internal activity i and external activity j .

The crux of understanding the Tutte polynomial of a lattice path matroid is describing internal and external activities of bases in terms of the associated lattice paths; this is what we turn to now. Recall that if the bounding lattice paths P and Q go from $(0, 0)$ to (m, r) , then the lattice path matroid $M[P, Q]$ has ground set $[m+r]$; the elements in $[m+r]$ represent the first step, the second step, and so on. We use the natural linear order on $[m+r]$, that is, $1 < 2 < \dots < m+r$. We start with a lemma that is an immediate corollary of Theorem 3.3.

Lemma 5.2. *Assume that $\{b_1, b_2, \dots, b_r\}$ is a basis of a lattice path matroid with $b_1 < b_2 < \dots < b_r$. Then b_i is in the set N_i of potential i -th North steps.*

The following theorem describes externally active elements for bases of lattice path matroids.

Theorem 5.3. *Assume that $B = \{b_1, b_2, \dots, b_r\}$ is a basis of a lattice path matroid $M[P, Q]$ with $b_1 < b_2 < \dots < b_r$. Assume that x is not in B ; say $b_i < x < b_{i+1}$. There is a j with $j \leq i$ and with $(B - b_j) \cup x$ a basis of $M[P, Q]$ if and only if x is*

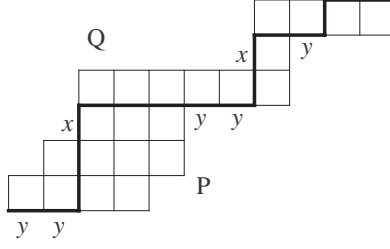


FIGURE 6. The lattice path corresponding to a basis with internal activity 2 and external activity 5, which contributes x^2y^5 to the Tutte polynomial.

in N_i . Equivalently, x is externally active in B if and only if the x -th step of the lattice path that corresponds to B is an East step of the lower bounding path P .

Proof. If x is in N_i , then clearly $(B - b_i) \cup x$ is a transversal of the set system N_1, N_2, \dots, N_r and so is a basis of $M[P, Q]$. Conversely, if $(B - b_j) \cup x$ is a basis of $M[P, Q]$ for some j with $j \leq i$, then since x is the i -th element in this basis, Lemma 5.2 implies that x is in N_i . The equivalent formulation of external activity follows immediately by interpreting the first assertion in terms of lattice paths. \square

By the last theorem, Lemma 5.1, and the lattice path interpretation of matroid duality, we get the following result.

Theorem 5.4. *Let B be a basis of the lattice path matroid $M[P, Q]$ and let $P(B)$ be the lattice path associated with B . Then $i(B)$ is the number of times $P(B)$ meets the upper path Q in a North step and $e(B)$ is the number of times $P(B)$ meets the lower path P in an East step.*

Theorem 5.4 is illustrated in Figure 6. It is worth noting the following simpler formulation in the case of k -Catalan matroids.

Corollary 5.5. *Let B be a basis of a k -Catalan matroid and let $P(B)$ be the associated lattice path. Then $i(B)$ is the number of times $P(B)$ returns to the line $y = x/k$ and $e(B)$ is j where $(j, 0)$ is the last point on the x -axis in $P(B)$.*

This lattice path interpretation of basis activities is one of the keys for obtaining the following generating function for the sequence of Tutte polynomials of the k -Catalan matroids.

Theorem 5.6. *Let*

$$C = C(z) = \sum_{n \geq 0} \frac{1}{kn + 1} \binom{(k+1)n}{n} z^n$$

be the generating function for the k -Catalan numbers. The generating function for the Tutte polynomials of the k -Catalan matroids is

$$(3) \quad \sum_{n \geq 0} t(M_n^k; x, y) z^n = 1 + \left(\frac{xyz^k}{1 - z \sum_{l=1}^k y^l C^{k-l+1}} \right) \frac{1}{1 - xzC^k}.$$

Proof. From our lattice path interpretation of bases of the k -Catalan matroid M_n^k , we are concerned with lattice paths that

- (i) go from $(0, 0)$ to (kn, n) and
- (ii) do not go above the line $y = x/k$.

We consider two special types of such lattice paths. Let d_{jn} be the number of such lattice paths that, in addition, have the following two properties:

- (iii) the last point of the path that is on the x -axis is the point $(j, 0)$ and
- (iv) the path returns to the line $y = x/k$ exactly once.

By property (iv), we have $d_{j0} = 0$ for all j . Let $D = D(y, z) = \sum_{n, j > 0} d_{jn} y^j z^n$. Let e_{in} be the number of lattice paths that satisfy properties (i)–(ii) and the following property:

- (iii') the path returns to the line $y = x/k$ exactly i times.

Here the term e_{00} is 1. Let $E = E(x, z) = \sum_{n, i \geq 0} e_{in} x^i z^n$. By the lattice path interpretation of bases and activities, we have

$$(4) \quad \sum_{n \geq 0} t(M_n^k; x, y) z^n = 1 + x D(y, z) E(x, z).$$

Equation (3) follows immediately from equation (4) and the following two equations, the justifications of which are the focus of the rest of the proof.

$$(5) \quad E(x, z) = \frac{1}{1 - xzC(z)^k}$$

$$(6) \quad D(y, z) = \frac{zy^k}{1 - z \sum_{l=1}^k y^l C(z)^{k-l+1}}$$

To prove equations (5) and (6), it will be convenient to use the notation l_s for the line $y = (x - s)/k$.

Equation (5) is immediate once we prove that the generating function $\sum e_{1n} z^n$ for the number of paths that return exactly once to the line $y = x/k$ is $zC(z)^k$. To see this, consider the following decomposition of such a path P^* . By considering the last point of P^* that is on the line l_1 , and then the last point of P^* that is on l_2 , and so on up to l_k , it follows that the path P^* can be decomposed uniquely as a sequence $P^* = EP_1EP_2 \cdots EP_{k-1}EP_kN$, where P_i is a path beginning and ending on the line l_i and never going above this line.

We turn to equation (6). Let P^* be a path that returns to the line $y = x/k$ exactly once. If P^* consists of kn East steps followed by n North steps, then P^* contributes $z^n y^{kn}$ to $D(y, z)$; all such paths contribute $\sum_{i \geq 1} z^i y^{ki}$, that is $zy^k/(1 - zy^k)$, to $D(y, z)$. Assume path P^* is not of this type. Let i be the minimum value of s such that P^* intersects l_s in a point neither on the line $y = 0$ nor on $x = kn$. Let t be $\lceil \frac{i}{k} \rceil$. Since P^* contains the point $(kn, n - t)$, it follows that P^* can be decomposed uniquely as follows:

$$P^* = E^i P' P_i EP_{i+1} E \cdots EP_{tk} N^t,$$

where P' is a non-empty path that begins and ends on the line l_i and that returns only once to this line, and P_s is a path that begins and ends on the line l_s and does not go above this line. There are $kt - i + 1$ paths among $P_i, P_{i+1}, \dots, P_{tk}$ and such paths are enumerated by $C(z)$. If i is kt , then the path $P_i EP_{i+1} E \cdots EP_{tk}$ reduces to P_i . In this case if the path P_i were trivial, then the path P^* would intersect

the line l_i only in the lines $y = 0$ and $x = kn$, which contradicts the choice of i . Therefore, when i is kt we have to guarantee that P_i is nontrivial. Hence, we get

$$\begin{aligned} D &= \frac{zy^k}{1-zy^k} + \sum_{\substack{i \geq 1, \\ kt-i+1 \neq 1}} y^i D z^t C^{kt-i+1} + \sum_{\substack{i \geq 1, \\ kt-i+1 = 1}} y^i D z^t (C-1) \\ &= \frac{zy^k}{1-zy^k} + D \sum_{i \geq 1} y^i z^t C^{kt-i+1} - D \sum_{\substack{i \geq 1, \\ kt-i+1 = 1}} y^i z^t. \end{aligned}$$

Since $kt - i + 1$ is 1 if and only if i is kt , the last term above is $Dzy^k/(1-zy^k)$. To simplify the rest, note that $\sum_{i \geq 1} y^i z^t C^{kt-i+1}$ is

$$\begin{array}{ccccccc} yzC^k & + & y^2zC^{k-1} & + & \cdots & + & y^kzC & + \\ y^{k+1}z^2C^k & + & y^{k+2}z^2C^{k-1} & + & \cdots & + & y^{2k}z^2C & + \\ y^{2k+1}z^3C^k & + & y^{2k+2}z^3C^{k-1} & + & \cdots & + & y^{3k}z^3C & + \\ \vdots & & \vdots & & \cdots & & \vdots & \end{array}$$

which, by adding the columns, gives

$$\frac{z}{1-zy^k} \sum_{l=1}^k y^l C^{k-l+1}.$$

Thus,

$$D = \frac{zy^k}{1-zy^k} + \frac{Dz}{1-zy^k} \sum_{l=1}^k y^l C^{k-l+1} - \frac{Dzy^k}{1-zy^k}.$$

Solving for D gives equation (6), thereby completing the proof of the theorem. \square

By extracting the coefficients of the expression in Theorem 5.6 we find a formula for the coefficients of the Tutte polynomial of a k -Catalan matroid. To write this formula more compactly, let us denote by $S(m, s, k)$ the number of solutions to the equation

$$l_1 + \cdots + l_s = m$$

such that $1 \leq l_i \leq k$ for all i with $1 \leq i \leq s$. Set $S(0, 0, k) = 1$. It will be useful to note that

$$S(m, s, 1) = \begin{cases} 1, & \text{if } m = s; \\ 0, & \text{otherwise.} \end{cases}$$

An elementary inclusion-exclusion argument gives

$$S(m, s, k) = \sum_{i=0}^s (-1)^i \binom{s}{i} \binom{m - ki - 1}{s - i}.$$

Theorem 5.7. *The coefficient of $x^i y^j$ in the Tutte polynomial $t(M_n^k; x, y)$ of the k -Catalan matroid M_n^k is*

$$\sum_{s=0}^m S(m, s, k) \binom{(k+1)(n-1) - i - m}{n - s - i - 1} \frac{s(k+1) - m + k(i-1)}{n - s - i},$$

where $m = j - k$. Equivalently, this is the number of lattice paths that

- (i) go from $(0, 0)$ to (kn, n) ,
- (ii) use steps $(1, 0)$ and $(0, 1)$,
- (iii) do not go above the line $y = x/k$,

- (iv) have as their last point on the x -axis the point $(j, 0)$, and
- (v) return to the line $y = x/k$ exactly i times.

Proof. We need to extract the coefficient of $x^i y^j z^n$ in equation (3). We start by extracting the coefficient of y^{j-k} in

$$\frac{1}{1 - z \sum_{l=1}^k y^l C^{k-l+1}} = \sum_{s \geq 0} \left(z \sum_{l=1}^k y^l C^{k-l+1} \right)^s.$$

Let $m = j - k$. From basic algebra the coefficient of y^m in $(z \sum_{l=1}^k y^l C^{k-l+1})^s$ is $z^s S(m, s, k) C^{s(k+1)-m}$. From this it follows that the coefficient of $x^i y^j$ in the right-hand side of equation (3) is

$$z^i C^{k(i-1)} \left(\sum_{s=0}^m z^s S(m, s, k) C^{s(k+1)-m} \right).$$

To conclude the proof, we have to extract the coefficient of z^n in the above expression; this is done using Lemma 2.6. \square

It is an open problem to obtain explicit expressions for the Tutte polynomials of the matroids $M_n^{k,l}$ for values of k and l not covered by the previous theorem, namely $k > 1$ and $l > 1$. The first unsolved case is $k = l = 2$. The sequence 1, 6, 53, 554, 6362, 77580, ... that gives the number of bases of $M_n^{2,2}$ also arises in the enumeration of certain types of planar trees, and in that context Lou Shapiro gave a nice expression for the corresponding generating function (see entry A066357 in [14]). This sequence also appears in [9]; indeed, as we show in Section 9, there is a simple connection between the number of bases in certain lattice path matroids and the problem considered in [9].

We single out a corollary of Theorem 5.7 that shows a very rare property possessed by the Tutte polynomials of the Catalan matroids M_n .

Corollary 5.8. *For $n > 1$, the Tutte polynomial of the Catalan matroid M_n is*

$$\sum_{i,j>0} \frac{i+j-2}{n-1} \binom{2n-i-j-1}{n-i-j+1} x^i y^j.$$

In particular, the coefficient of $x^i y^j$ in the Tutte polynomial $t(M_n; x, y)$ of the Catalan matroid M_n depends only on n and the sum $i + j$.

We close this section with some simple observations. A well-known corollary of Lemma 5.1 is that the Tutte polynomial of a matroid and its dual are related by the following equation:

$$t(M^*; x, y) = t(M; y, x).$$

From this and Theorem 3.9 we get the following corollary.

Corollary 5.9. *The Tutte polynomial of the (k, l) - and (l, k) -Catalan matroids are related as follows:*

$$t(M_n^{k,l}; x, y) = t(M_n^{l,k}; y, x).$$

Thus, the Tutte polynomial of the (k, k) -Catalan matroid $M_n^{k,k}$, and in particular the Catalan matroid M_n , is a symmetric function in x and y .

6. COMPUTING THE TUTTE POLYNOMIAL OF LATTICE PATH MATROIDS

There is no known polynomial-time algorithm for computing the Tutte polynomial of an arbitrary matroid, or even its evaluations at certain points in the plane [18]. There are many evaluations of the Tutte polynomial that are particularly significant; for instance, it follows from equation (2) that $t(M; 1, 1)$ is the number of bases of M . Since the bases of a lattice path matroid correspond to paths that stay in a given region and the number of such paths is given by a determinant (see Theorem 1 in Section 2.2 of [10]), the number of bases of a lattice path matroid can be computed in polynomial time. It turns out that other evaluations like $t(M; 1, 0)$ and $t(M; 0, 1)$ can also be expressed as determinants. This led us to suspect that the Tutte polynomial of a lattice path matroid could be computed in polynomial time. In this section, we show that this is indeed the case: we give such a polynomial-time algorithm. Also, we give a second technique for computing the Tutte polynomial in the case of generalized Catalan matroids (this second technique, although more limited in scope, is particularly simple to implement using standard mathematical software). The results in this section stand in striking contrast to those in [7], where it is shown that for fixed x and y with $(x-1)(y-1) \neq 1$, the problem of computing $t(M; x, y)$ for a transversal matroid M is #P-complete.

By Theorem 5.4, for a lattice path matroid $M = M[P, Q]$, the Tutte polynomial $t(M; x, y)$ is the generating function

$$\sum_{B \in \mathcal{B}(M)} x^{i(B)} y^{e(B)}$$

where $i(B)$ is the number of North steps that the lattice path $P(B)$ corresponding to B shares with the upper bounding path Q and $e(B)$ is the number of East steps that $P(B)$ shares with the lower bounding path P . Any lattice path can be viewed as a sequence of shorter lattice paths. This perspective gives the following algorithm for computing the Tutte polynomial of the lattice path matroid $M = M[P, Q]$ where P and Q go from $(0, 0)$ to (m, r) . With each lattice point (i, j) in the region R bounded by P and Q , associate the polynomial

$$f(i, j) = \sum_{P'} x^{i(P')} y^{e(P')}$$

where the sum ranges over the lattice paths P' that go from $(0, 0)$ to (i, j) and stay in the region R , and where, as for $t(M; x, y)$, the exponent $i(P')$ is the number of North steps that P' shares with Q and $e(P')$ is the number of East steps that P' shares with P . In particular, $f(m, r) = t(M; x, y)$. Note that for a point (i, j) in R other than $(0, 0)$, at least one of $(i-1, j)$ or $(i, j-1)$ is in R ; furthermore, only $(i-1, j)$ is in R if and only if the step from $(i-1, j)$ to (i, j) is an East step of P , and, similarly, only $(i, j-1)$ is in R if and only if the step from $(i, j-1)$ to (i, j) is a North step of Q . The following rules for computing $f(i, j)$ are evident from these observations and the definition of $f(i, j)$.

- (a) $f(0, 0) = 1$.
- (b) If the lattice points (i, j) , $(i-1, j)$ and $(i, j-1)$ are all in the region R , then $f(i, j) = f(i-1, j) + f(i, j-1)$.
- (c) If the lattice points (i, j) and $(i-1, j)$ are in R but $(i, j-1)$ is not in R , then $f(i, j) = y f(i-1, j)$.

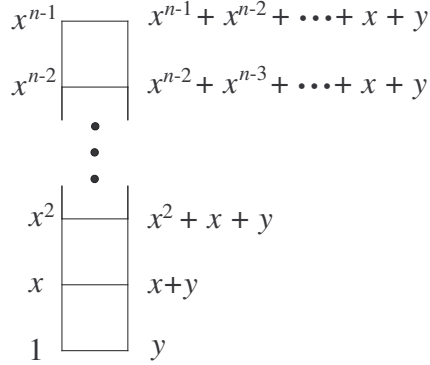


FIGURE 7. The recursive computation of the Tutte polynomial of an n -element circuit via lattice path statistics.

- (d) If the lattice points (i, j) and $(i, j - 1)$ are in R but $(i - 1, j)$ is not in R , then $f(i, j) = x f(i, j - 1)$.

This algorithm is illustrated in Figure 7 where we apply it to compute the Tutte polynomial of an n -element circuit.

If x and y are set to 1, the algorithm above reduces to a well-known technique for counting lattice paths. This is consistent with the general theory of Tutte polynomials; as noted above, $t(M; 1, 1)$ is the number of bases of M .

The recurrence above requires at most $(r + 1)(m + 1)$ steps to compute the Tutte polynomial of a lattice path matroid whose bounding paths go from $(0, 0)$ to (m, r) . Thus, we have the following corollary.

Corollary 6.1. *The Tutte polynomial of a lattice path matroid can be computed in polynomial time.*

We remark that the recurrence expressed in (a)–(d) above is essentially the deletion-contraction rule for Tutte polynomials, along with the corresponding rules for loops and isthmuses (see, e.g., [6] for this perspective on the Tutte polynomial). This follows by considering the lattice path interpretations of deletion and contraction, which are given in [3]. We also remark that while the deletion-contraction rule for computing $t(M; x, y)$ generally gives rise to a binary tree with $2^{|E(M)|}$ leaves, for lattice path matroids there are relatively few isomorphism types for minors, and the geometry of lattice paths automatically collects minors of the same isomorphism type. To make this more specific, let R be the region bounded by the lattice paths P and Q of the lattice path matroid $M = M[P, Q]$. As can be seen from the description of minors in [3], each minor whose ground set is an initial segment $[k]$ of $[m + r]$ can be viewed as having as bases the lattice paths in R from $(0, 0)$ to some specific point in R of the form $(i, k - i)$. It follows that the number of possible minors of $M = M[P, Q]$ that arise when computing $t(M; x, y)$, rather than being exponential, is bounded above by $(r + 1)(m + 1)$.

By Theorem 3.14, generalized Catalan matroids are formed from the empty matroid by iterating the operations of taking free extensions and direct sums with the uniform matroid $U_{1,1}$. The following rule is well-known and easy to check: for

any matroid M ,

$$(7) \quad t(M \oplus U_{1,1}; x, y) = x t(M; x, y).$$

For free extensions, we have the following result, which is easy to prove using formula (1) and the rank function of the free extension. (This formula is equivalent to the expression for the Tutte polynomial of a free extension given in Proposition 4.2 of [5].)

Theorem 6.2. *The Tutte polynomial of the free extension $M + e$ of M is given by the formula*

$$(8) \quad t(M + e; x, y) = \frac{x}{x-1} t(M; x, y) + \left(y + \frac{x}{x-1} \right) t(M; 1, y).$$

Formulas (7) and (8) can be used, for instance, to compute Tutte polynomials of (k, l) -Catalan matroids very quickly. It is through such computations that we were lead, for instance, to Theorem 8.3.

7. THE BROKEN CIRCUIT COMPLEX AND THE CHARACTERISTIC POLYNOMIAL

In this section we study two related objects for lattice path matroids, the broken circuit complex and the characteristic polynomial. The second of these is an invariant of the matroid but the first depends on a linear ordering of the elements. We show that under the natural ordering of the elements, the broken circuit complex of any loopless lattice path matroid has a property that is not shared by the broken circuit complexes of arbitrary matroids, namely, the broken circuit complex of a lattice path matroid is the independence complex of another matroid, indeed, of a lattice path matroid. Our study of the characteristic polynomial is more specialized; we focus on the characteristic polynomial $\chi(\widehat{M}_n^k; \lambda)$ of the matroid \widehat{M}_n^k obtained from the k -Catalan matroid M_n^k by omitting the loops. Our results on the broken circuit complex lead to a lattice path interpretation of each coefficient of $\chi(\widehat{M}_n^k; \lambda)$ from which we obtain a formula for these coefficients. We start by outlining the necessary background on broken circuit complexes; for an extensive account, see [2].

Given a matroid M and a linear order $<$ on the ground set $E(M)$, a *broken circuit* of the resulting ordered matroid is a set of the form $C - x$ where C is a circuit of M and x is the least element of C relative to the linear ordering. A subset of $E(M)$ is an *nbc-set* if it contains no broken circuit. Clearly subsets of nbc-sets are nbc-sets. Thus, $E(M)$ and the collection of nbc-sets of M form a simplicial complex, the *broken circuit complex of M relative to $<$* , which is denoted $BC_{<}(M)$. Different orderings of $E(M)$ can produce nonisomorphic broken circuit complexes (see, e.g., [2, Example 7.4.4]). The facets of $BC_{<}(M)$ are the *nbc-basis*, that is, the basis of M that are nbc-sets. The following characterization of nbc-bases is well-known and easy to prove.

Lemma 7.1. *The nbc-basis of M are the bases of M of external activity zero.*

Note that nbc-sets contain no circuits and so are independent. Thus, the broken circuit complex $BC_{<}(M)$ of M is contained in the *independence complex of M* , that is, the complex with ground set $E(M)$ in which the faces are the independent sets of M . Note also that if M has loops, then the empty set is a broken circuit, so

M has no nbc-sets. Thus, throughout this section we consider only matroids with no loops.

As in Section 5, we use the natural ordering on the points of lattice path matroids. The examples in [2] show that the broken circuit complex need not be the independence complex of another matroid. In contrast, Theorem 7.2 shows that the broken circuit complex of a lattice path matroid without loops is the independence complex of another lattice path matroid.

Theorem 7.2. *With the natural order, the broken circuit complex of a lattice path matroid $M[P, Q]$ with no loops is the independence complex of the lattice path matroid $M[P', Q]$ where $NP = P'N$.*

Proof. Since a subset of $E(M[P, Q])$ is an nbc-set of $M[P, Q]$ if and only if it is contained in an nbc-basis of $M[P, Q]$, it suffices to show that the nbc-bases of $M[P, Q]$ are precisely the bases of $M[P', Q]$. By Lemma 7.1 and Theorem 5.4, the nbc-bases of $M[P, Q]$ correspond to the lattice paths in the region bounded by P and Q that share no East step with P . Thus, the nbc-bases of $M[P, Q]$ correspond to the lattice paths in the region bounded by P' and Q where the East steps of P' occur exactly one unit above those of P . This condition on P' is captured by the equality $NP = P'N$. \square

Corollary 7.3. *Let $M[Q]$ be a generalized Catalan matroid with no loops. A subset X of the ground set of $M[Q]$ is an nbc-set if and only if $X \cup 1$ is independent in $M[Q]$. In particular, all independent sets of $M[Q]$ that contain 1 are nbc-sets and the nbc-basis of M are exactly the bases of $M[Q]$ that contain 1.*

We now turn to the characteristic polynomial, which plays an important role in many enumeration problems in matroid theory (see [13, 19]) and which can be defined in a variety of ways. As mentioned above, the isomorphism type of the broken circuit complex of a matroid M can depend on the ordering of the points. However, it can be shown that the number of nbc-sets of each size is an invariant of the matroid; these numbers are the coefficients of the characteristic polynomial. Specifically, the *characteristic polynomial* $\chi(M; \lambda)$ of a matroid M is

$$(9) \quad \chi(M; \lambda) = \sum_{i=0}^{r(M)} (-1)^i \mathbf{nbc}(M; i) \lambda^{r(M)-i},$$

where $\mathbf{nbc}(M; i)$ is the number of nbc-sets of size i . Thus, $(-1)^{r(M)} \chi(M; -\lambda)$ is the face enumerator of the broken circuit complex of M . (Equation (9) applies even if the matroid M has loops, in which case $\chi(M; \lambda)$ is 0.) Alternatively, $\chi(M; \lambda)$ can be expressed in terms of the Tutte polynomial as follows:

$$\chi(M; \lambda) = (-1)^{r(M)} t(M; 1 - \lambda, 0) = \sum_{A \subseteq E(M)} (-1)^{|A|} \lambda^{r(M)-r(A)}.$$

The characteristic polynomial can also be expressed in the following way in terms of the Möbius function of the lattice of flats:

$$\chi(M; \lambda) = \sum_{\substack{\text{flats } F \\ \text{of } M}} \mu(\emptyset, F) \lambda^{r(M)-r(F)}.$$

(See, e.g., [2, Theorem 7.4.6], for details.) In particular, the absolute value of the constant term of $\chi(M; \lambda)$ is both the number of nbc-bases of M and the absolute

value of the Möbius function $\mu(M)$. This and Theorem 7.2 give the following corollary.

Corollary 7.4. *The absolute value of the Möbius function $\mu(M[P, Q])$ of a loopless lattice path matroid is the number of bases of the lattice path matroid $M[P', Q]$ where $NP = P'N$, or, equivalently, of the lattice path matroid $M[P^*, Q^*]$ where $P = P^*N$ and $Q = NQ^*$.*

Our interest is in the characteristic polynomial of a specific type of lattice path matroid. Recall that the elements $1, 2, \dots, k$ are loops of the k -Catalan matroid M_n^k . Thus, the characteristic polynomial of M_n^k is zero. This motivates considering the *loopless Catalan matroid* \widehat{M}_n , which we define to be $M[(NE)^{n-1}N]$, and more generally the *loopless k -Catalan matroid* \widehat{M}_n^k , which we define to be $M[(NE^k)^{n-1}N]$. Thus, these matroids are formed from almost the same bounding paths as those for the Catalan and k -Catalan matroids except that the initial East steps that give loops have been omitted.

We start with the following consequence of Corollary 7.4.

Corollary 7.5. *The number of nbc-bases of \widehat{M}_n^k , that is, $|\mu(\widehat{M}_n^k)|$, is the k -Catalan number C_{n-1}^k . In particular, $|\mu(\widehat{M}_n)| = C_{n-1}$.*

Proof. From the second part of Corollary 7.4, we have that $|\mu(\widehat{M}_n^k)|$ is the number of bases of M_{n-1}^k , which is C_{n-1}^k . \square

By combining Corollary 3.12 and Corollary 7.3, we get the following characterization of the nbc-sets of size i of \widehat{M}_n^k in terms of lattice paths.

Corollary 7.6. *Via the map $X \mapsto P(X)$, the nbc-sets of size i , for $0 \leq i \leq n$, in the loopless k -Catalan matroid \widehat{M}_n^k correspond bijectively to the following two types of lattice paths.*

- (i) *Lattice paths from $(0, 1)$ to $((k+1)(n-1) - i + 1, i)$ that do not go above the line $y = \frac{1}{k}x + 1$.*
- (ii) *Lattice paths from $(0, 1)$ to $((k+1)(n-1) - i, i + 1)$ that do not go above the line $y = \frac{1}{k}x + 1$.*

By using this characterization of nbc-sets we obtain the following expression for each coefficient of the characteristic polynomial.

Theorem 7.7. *The absolute value of the coefficient of λ^{n-i} in the characteristic polynomial of the loopless k -Catalan matroid \widehat{M}_n^k is given by the formula*

$$\mathbf{nbc}(\widehat{M}_n^k; i) = \begin{cases} 1, & \text{if } i = 0; \\ \frac{(k+1)(n-i-1) + 2}{(k+1)(n-1) + 2} \binom{(k+1)(n-1) + 2}{i}, & \text{if } 1 \leq i \leq n-1; \\ C_{n-1}^k, & \text{if } i = n. \end{cases}$$

Proof. Since \widehat{M}_n^k is loopless, the empty set is an nbc-set; from this the case $i = 0$ follows. The case $i = n$ has been treated in Corollary 7.5. For i with $1 \leq i \leq n-1$, we have to count the number of paths as in Corollary 7.6. This is equivalent to counting the following:

- (i) *lattice paths from $(0, 0)$ to $((k+1)(n-1) - i + 1, i - 1)$ that do not go above the line $y = x/k$, and*

- (ii) lattice paths from $(0, 0)$ to $((k+1)(n-1) - i, i)$ that do not go above the line $y = x/k$.

Observe that the sum of the number of paths described in items (i) and (ii) is the number of paths from $(0, 0)$ to $((k+1)(n-1) - i + 1, i)$ that do not go above the line $y = x/k$. The formula follows then from Lemma 2.4. \square

From the formula in Theorem 7.7 and appropriate manipulation, we see that the linear term in the characteristic polynomial of \widehat{M}_n is also a Catalan number. Note that, however, for the loopless k -Catalan matroid the linear term of the characteristic polynomial is not the corresponding k -Catalan number.

Corollary 7.8. *The linear term in $\chi(\widehat{M}_n, \lambda)$ is C_n .*

8. THE β INVARIANT

The β invariant $\beta(M)$ of a matroid M , which was introduced by Crapo, can be defined in several ways; see [19, Section 3] for a variety of perspectives on the β invariant, as well as its applications to connectivity and series-parallel networks. We use the following definition. It can be shown that for any matroid M , the coefficients of x and y in the Tutte polynomial $t(M; x, y)$ are the same; this coefficient is $\beta(M)$. Since loops are externally active with respect to every basis, no basis of a matroid M with loops will have external activity zero, so $\beta(M)$ is zero; dually, if M has isthmuses, then $\beta(M)$ is zero. Therefore, in this section we focus on matroids with neither loops nor isthmuses.

Let $N_n^{k,k}$ be the generalized Catalan matroid whose upper path is $Q = (N^k E^k)^n$. It is clear from the lattice path presentation that $N_n^{k,k}$ is formed from the (k, k) -Catalan matroid $M_{n+1}^{k,k}$ by deleting the k loops and the k isthmuses. The main result of this section is that $\beta(N_n^{k,k})$ is k times the Catalan number C_{kn-1} . This result was suggested by looking at examples of Tutte polynomials of lattice path matroids, but it can be formulated entirely in terms of lattice paths, which is the perspective we use in the proof. Indeed, the result is most striking when viewed in terms of lattice paths.

The β invariant of $N_n^{k,k}$ is the number of bases with internal activity one and external activity zero; let B be such a basis and let $P(B)$ be its associated lattice path. By Theorem 5.4, the first step of $P(B)$ is N , the second is E , and $P(B)$ does not contain any other North step in Q . It is easy to see that such lattice paths $P(B)$ are in 1-1 correspondence with the paths from $(0, 0)$ to $(kn-1, kn-1)$ that do not go above the path $N^{k-1}(E^k N^k)^{n-1} E^{k-1}$. Recall that the number of paths from $(0, 0)$ to $(kn-1, kn-1)$ that do not go above the line $y = x$ is C_{kn-1} . In this section we show that the number of paths that do not go above the path $N^{k-1}(E^k N^k)^{n-1} E^{k-1}$ is k times C_{kn-1} . We start with the case $k = 1$.

Theorem 8.1. *The β invariant of $N_n^{1,1}$ is C_{n-1} .*

Proof. By the discussion above, $\beta(N_n^{1,1})$ is the number of paths from $(0, 0)$ to $(n-1, n-1)$ that do not go above the path $(EN)^{n-1}$, which is C_{n-1} . \square

From here on, we consider only paths that use steps U and D . From the discussion above and the correspondence between the alphabets, we get the following lemma.

Lemma 8.2. *The β invariant of the matroid $N_n^{k,k}$ is the number of paths that*

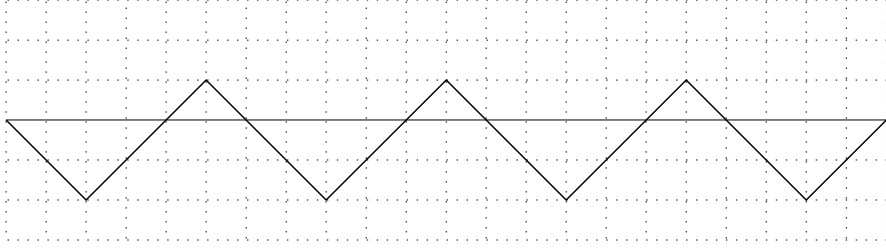


FIGURE 8. The path $D^{k-1}(U^k D^k)^{n-1} U^{k-1}$ for $k = 3$ and $n = 4$.

- (i) go from $(0, 0)$ to $(2(nk - 1), 0)$,
- (ii) use steps U and D , and
- (iii) never go below the path $D^{k-1}(U^k D^k)^{n-1} U^{k-1}$.

A path of the form $D^{k-1}(U^k D^k)^{n-1} U^{k-1}$ is depicted in Figure 8. The next theorem is the main result of this section.

Theorem 8.3. *The number of paths that go from $(0, 0)$ to $(2(nk - 1), 0)$, use steps U and D , and do not go below the path $D^{k-1}(U^k D^k)^{n-1} U^{k-1}$ is $k C_{nk-1}$.*

Before proving the theorem, we mention that if we change the bounding path to $(D^k U^k)^n$, the elegance and brevity of the result seem to disappear; currently there is no known comparably simple answer. Indeed, the path $(D^k U^k)^n$ is connected with an open problem in enumeration that is discussed in the next section. The following corollary is an immediate consequence of Lemma 8.2 and Theorem 8.3.

Corollary 8.4. *The β invariant of the matroid $N_n^{k,k}$ is $k C_{nk-1}$.*

Proof of Theorem 8.3. Let us denote the path $D^{k-1}(U^k D^k)^{n-1} U^{k-1}$ by B . In this proof we consider paths from $(0, 0)$ to $(2kn - 1, -1)$ using steps U and D . When we say that one such path does not go below a given border, we refer to the path with the last step removed. Hence, a Dyck path is a path from $(0, 0)$ to $(2kn - 1, -1)$ that does not go below the line $y = 0$ (except for the last step). A *cyclic permutation* of a path $s_1 s_2 \dots s_l$ is a path $s_i s_{i+1} \dots s_l s_1 \dots s_{i-1}$ for some i with $1 \leq i \leq l$. It is easy to show that all cyclic permutations of a Dyck path from $(0, 0)$ to $(2kn - 1, -1)$ are different; note that this does not hold if we consider Dyck paths ending in a point of the form $(2l, 0)$.

The proof is in the spirit of several results generically known as the Cycle Lemma (see the notes at the end of Chapter 5 of [15]). One such result states that among the $2l + 1$ possible cyclic permutations of a path from $(0, 0)$ to $(2l + 1, -1)$, there is exactly one that is a Dyck path. Moreover, the cyclic permutation that leads to a Dyck path is the one that starts after the leftmost minimum of the path (see [16, Theorem 1.1] for more details on this). To prove the theorem, we show that for every Dyck path from $(0, 0)$ to $(2kn - 1, -1)$, exactly k of its cyclic permutations are paths that do not go below B ; conversely, every path that does not go below B can be obtained as one of these k cyclic permutations of a Dyck path. To describe these permutations we need to introduce some terminology.

It is clear that if a lattice point (x, y) is in a path that begins at $(0, 0)$ and uses steps U and D , then $x + y$ is even. We partition the lattice points whose coordinates have an even sum into k disjoint classes. The point (x, y) is *in class* c

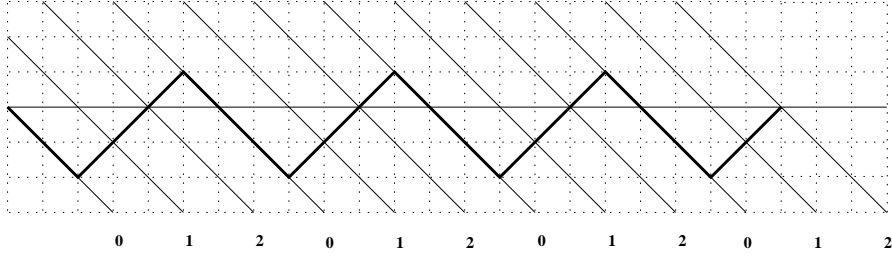


FIGURE 9. The partition of the points (x, y) for which $x + y$ is even ($k = 3$).

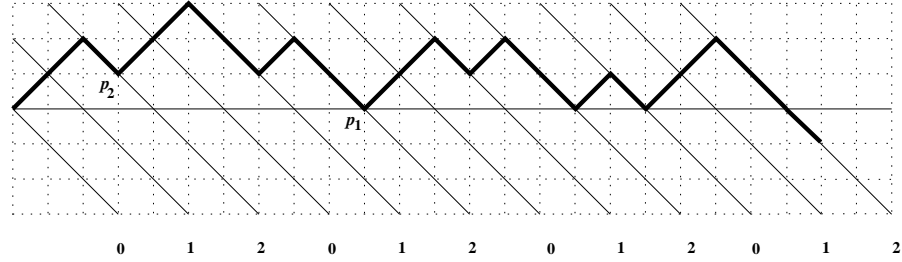


FIGURE 10. A Dyck path and the points p_i of the proof of Theorem 8.3.

with $0 \leq c \leq k - 1$ if $(x + y)/2 \equiv c$ modulo k . As can be seen in Figure 9, each class corresponds to an infinite family of parallel lines.

We say that a point (x, y) has *height* y . It is easy to see that a point in class c is not below the path B if and only if the height of the point is strictly greater than $c - k$. Let $p = (x, y)$ be a point in a path that uses steps U and D ; we say that p is a *down point* if p is the end of a D step. The *cyclic permutation at p* is the permutation that starts in the step that has p as the first point.

Let R be a Dyck path from $(0, 0)$ to $(2kn - 1, -1)$; clearly, R does not go below B . The other $k - 1$ cyclic permutations of R that do not go below B are given by the points p_1, \dots, p_{k-1} that we define next. The point p_1 is the first down point of R that is in class $k - 1$ and has height 0. The point p_2 is the first down point of R that is in class $k - 2$ and has height 0, if such a point exists; otherwise, take the first down point in class $k - 1$ and with height 1. To find the i -th point p_i , among all down points that are in class $k - i + j$ and have height j for $0 \leq j \leq i - 1$, take the ones that have minimum j , and among those take as p_i the one that appears first in R . See Figure 10 for an example.

We next show that the points p_i exist. Since R is a Dyck path, the point $(2(kn - 1), 0)$ is always in R . Moreover, it is a down point and belongs to class $k - 1$; hence there is at least one down point in R in class $k - 1$ with height 0. In general, we prove that if for j with $j < i - 1$ there is no down point in class $k - i + j$ with height j , then there exists a down point in class $k - 1$ with height $i - 1$, and thus we take as p_i the first such point that appears in R . Assume that R contains no point in class $k - i + j$ with height j for $j < i - 1$. Since R

is a Dyck path all points at height 0 are down, so the point $(2(kn - i), 0)$ is not in the path R . The point $(2(kn - i) + 1, 1)$ is in class $k - i + 1$ and has height 1, so by assumption it cannot be down. Then, if it is in R , the previous step must be U , but that forces the point $(2(kn - i), 0)$ to be in R , which is a contradiction. Hence $(2(kn - i) + 1, 1)$ is not in R . In the same way one proves that the points of the form $(2(kn - i) + j, j)$ are not in R for j with $0 \leq j \leq i - 2$. Note that this implies that the path R goes above all these points. Now consider the point $(2(kn - i) + i - 1, i - 1)$, which is in class $k - 1$. If this point is not in R , then the point in R with first coordinate equal to $2(kn - i) + i - 1$ would have height at least $i + 1$; however, from such a point it is impossible to reach the point $(2(kn - 1), 0)$, which is always in the path. Therefore, the point $(2(kn - i) + i - 1, i - 1)$ is in R , and since the point $(2(kn - i) + i - 2, i - 2)$ is not, it has to be a down point. Hence R contains a down point in class $k - 1$ with height $i - 1$, and the existence of p_i is proved.

Now we have to check that $\pi_i(R)$, the cyclic permutation of R at p_i , is a path that does not go below the path B . We split $\pi_i(R)$ into two subpaths R_1 and R_2 such that $R = R_1R_2$ and $\pi_i(R) = R_2R_1$. We prove that there is no point in either part R_1 or R_2 of $\pi_i(R)$ below the path B . Assume that p_i belongs to class $k - i + j$ and has height j for some j with $j \leq i - 1$.

Suppose first there is a point in the subpath R_1 that goes below B and let q be the first such point; this point is a down point and if it is in class c , then its height is $c - k$. Let us move the point q to R , that is, let the point q_R be the point of R that goes to q after the cyclic permutation at p_i . It is easy to check that the point q_R has height $j + 1 + c - k$ and belongs to class $c + j - i + 1$ modulo k . Since R is a Dyck path we have $j + 1 + c - k \geq 0$; from this and the inequality $j \leq i - 1$ it follows that the class of q_R is indeed $c + j - i + 1$. Since $c < k$, we have that $j + 1 + c - k \leq j$. This together with the fact that the point q_R comes before p_i in R contradict the choice of p_i .

Similarly, suppose there is a point in the subpath R_2 of $\pi_i(R)$ that goes below B and let q' be the first such point. As before, the point q' is down and if it is in class c' , then its height is $c' - k$. Let q'_R be the point of R that is mapped to q' by the cyclic permutation at p_i . The point q'_R has height $j + c' - k$; thus since R is a Dyck path, $j + c' - k \geq 0$. The class of q'_R is $k - i + j + c'$ modulo k . By combining the inequalities $j \leq i - 1$, $c' < k$, and $j + c' - k \geq 0$, we get that class of q'_R is $k - i + (j + c' - k)$. Since $j + c' - k < j$, the point q'_R contradicts the choice of p_i . This finishes the proof that the cyclic permutation at p_i is a path that does not go below B .

We now have that every Dyck path from $(0, 0)$ to $(2kn - 1, -1)$ gives rise to k paths that do not go below B , including the Dyck path itself. As noted above, all cyclic permutations of a Dyck path are different and for every path only one cyclic permutation is a Dyck path. Since there are C_{kn-1} Dyck paths, we have that the number of paths as described in the statement of the theorem is at least kC_{kn-1} .

To complete the proof of the equality, we have to show that every path that does not go below B is either a Dyck path R or one of the $k - 1$ cyclic permutations of a Dyck path R at one of the points p_1, p_2, \dots, p_{k-1} defined above. Let S be a path from $(0, 0)$ to $(2kn - 1, -1)$ that does not go below B and that is not a Dyck path; let q_0 be its first point and q_S its leftmost minimum. The cyclic permutation at q_S is a Dyck path S' . Let q'_0 be the image of the point q_0 in S' . If the point q_S is in

class c and has height h in S , then the point q'_0 in S' is a down point that belongs to class $k - 1 - c$ and has height $-h - 1$; also $h > c - k$. We have to show that q'_0 is one of the points p_1, \dots, p_{k-1} with respect to the Dyck path S' . Since by definition the point p_i is in class $k - i + j$ and has height j , it follows that q'_0 should be the point p_{c-h} with $j = -h - 1$ (note that $c - h$ is a valid index since $c - k < h$). The result will follow if we show that no down point in S' is in class $k - c + h + j$ and has height j for $0 \leq j < -h - 1$, and that any down point in class $k - c - 1$ with height $-h - 1$ comes after q'_0 in S' . It is easy to show that if there were a point satisfying either condition, then its height in the path S would exceed the class minus k , and hence the point would be below the path B , which is a contradiction. \square

9. CONNECTIONS WITH THE TENNIS BALL PROBLEM

The following problem is of current interest in enumerative combinatorics; only a very limited number of cases have been settled (see [9]).

THE $(k + l, l)$ TENNIS BALL PROBLEM. Let $b_1, b_2, \dots, b_{(k+l)n}$ be a sequence of distinct balls. At stage 1, balls b_1, b_2, \dots, b_{k+l} are put in bin A and then l balls are moved from bin A to bin B . At stage i , balls $b_{(i-1)(k+l)+1}, b_{(i-1)(k+l)+2}, \dots, b_{i(k+l)}$ are put in bin A and then some set of l balls from bin A are moved to bin B . (In particular, balls that remain in bin A after stage $i - 1$ can go in bin B at stage i .) How many different sets of ln balls can be in bin B after n iterations?

We show that the answer is the number of bases of the (k, l) -Catalan matroid $M_{n+1}^{k,l}$.

It is well known that free extensions of transversal matroids are also transversal; we use the presentations of free extensions given in the following lemma.

Lemma 9.1. *Assume that M is a transversal matroid of rank r with presentation $(A_j : j \in K)$ where $|K| = r$. Then the free extension $M + e$ is also transversal and the set system $(A_j \cup e : j \in K)$ is a presentation of $M + e$.*

Proof. The partial transversals X of $(A_j \cup e : j \in K)$ with $e \notin X$ are precisely the partial transversals of $(A_j : j \in K)$. Also, for any partial transversal X of $(A_j : j \in K)$ with $|X| < r$, the set $X \cup e$ is a partial transversal of $(A_j \cup e : j \in K)$. \square

There are many ways to add a set $I = \{f_1, f_2, \dots, f_u\}$ of isthmuses to a transversal matroid with presentation $(A_j : j \in K)$; for instance, $(A_j : j \in K)$ together with $\{f_1\}, \{f_2\}, \dots, \{f_u\}$ is such a presentation. The presentation of interest for us is the union of the multiset $(A_j \cup I : j \in K)$ with u copies of I .

By Theorem 3.14, the (k, l) -Catalan matroid $M_{n+1}^{k,l}$ can be constructed from the empty matroid by taking k free extensions, then adding l isthmuses, then taking k free extensions, then adding l isthmuses, etc., for a total of $n + 1$ iterations. With this in mind, as well as the presentations of free extensions and additions of isthmuses just discussed, consider the following bipartite graph $G_{n+1}^{k,l}$. One set of the bipartition of the vertex set is $[(k + l)(n + 1)]$, the ground set of $M_{n+1}^{k,l}$; let v_h^j , with $1 \leq j \leq n + 1$ and $1 \leq h \leq l$, be the remaining vertices. Vertices $(k + l)i + \kappa$, with $1 \leq \kappa \leq k$, are adjacent to all v_h^j with $1 \leq j \leq i$ and $1 \leq h \leq l$; vertices $(k + l)i + \eta$, with $k + 1 \leq \eta \leq k + l$, are adjacent to all v_h^j with $1 \leq j \leq i + 1$ and $1 \leq h \leq l$. The graph $G_3^{2,2}$ is illustrated in Figure 11.

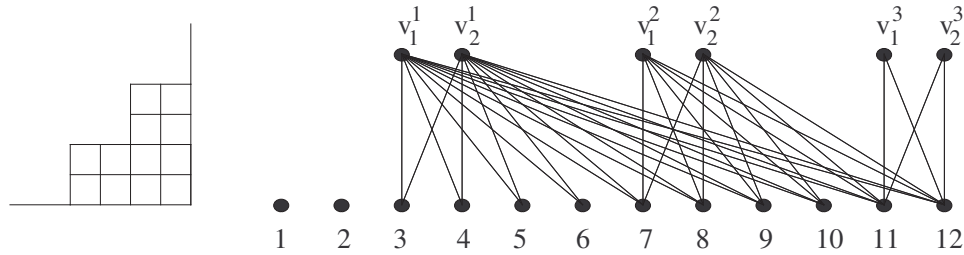


FIGURE 11. The lattice path presentation of $M_3^{2,2}$ and the graph $G_3^{2,2}$.

It follows from the descriptions of presentations of free extensions and extensions by isthmuses that the bases of $M_{n+1}^{k,l}$ are precisely the sets of vertices in $[(k+l)(n+1)]$ of maximal size that can be matched in $G_{n+1}^{k,l}$. Note that $M_{n+1}^{k,l}$ has as many bases as the matroid obtained by deleting the first k elements (which are loops) and the last l elements (which are isthmuses); let $\widehat{G}_{n+1}^{k,l}$ denote graph obtained from $G_{n+1}^{k,l}$ by deleting these vertices. The graph $\widehat{G}_{n+1}^{k,l}$ can be used to model n iterations of the $(k+l, l)$ -tennis ball problem: after relabelling vertices, those adjacent to $v_1^n, v_2^n, \dots, v_l^n$ can be viewed as the balls that could be selected to go in bin B on first iteration; those adjacent to $v_1^{n-1}, v_2^{n-1}, \dots, v_l^{n-1}$ can be viewed as the candidates to go in bin B on the second iteration, and so on. Furthermore, maximal-sized sets of vertices that can be matched in this graph are precisely the sets of balls that can be in bin B at the end of n iterations. Thus, the answer to the $(k+l, l)$ tennis ball problem, with n iterations, is the number of bases of the (k, l) -Catalan matroid $M_{n+1}^{k,l}$.

ACKNOWLEDGEMENTS

The authors thank Lou Shapiro for interesting and useful discussions about the Catalan numbers and the objects they count, and for bringing the tennis ball problem to our attention. The first author also thanks Bill Schmitt for further discussions about Catalan numbers.

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