Boson Normal Ordering Problem and Generalized Bell Numbers

P. Blasiak†, K.A. Penson‡ and A.I. Solomon§

†‡§ *Université Pierre et Marie Curie, Laboratoire de Physique Théorique des Liquides, Tour 16, $5^{ième}$ étage, 4, place Jussieu, F 75252 Paris Cedex 05, France*

† *H. Niewodniczański Institute of Nuclear Physics, ul.Eliasza Radzikowskiego 152, PL 31-342 Kraków, Poland*

For any function $F(x)$ having a Taylor expansion we solve the boson normal ordering problem for $F\left[(a^\dagger)^r a^s\right]$, with r, s positive integers, $\left[a, a^\dagger\right] = 1$, i.e. we provide exact and explicit expressions for its normal form $\mathcal{N}\left\{F\left[(a^\dagger)^r a^s\right]\right\} = F\left[(a^\dagger)^r a^s\right]$, where in $\mathcal{N}(F)$ all a's are to the right. The solution involves integer sequences of numbers which, for $r, s \geq 1$, are generalizations of the conventional Bell and Stirling numbers whose values they assume for $r = s = 1$. A complete theory of such generalized combinatorial numbers is given including closed-form expressions (extended Dobinski - type formulas), recursion relations and generating functions. These last are special expectation values in boson coherent states.

Consider a function $F(x)$ having a Taylor expansion around $x = 0$, i.e. $F(x) = \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} x^k$. In this note we will collect the formulas concerning our solution of the normal ordering problem for $F\left[(a^\dagger)^r a^s\right]$, where $a, a^\dagger$ are the boson annihilation and creation operators, $\left[a, a^\dagger\right] = 1$, and r and s are positive integers. The normally ordered form of the operator $F\left[(a^\dagger)^r a^s\right]$ is denoted by $\mathcal{N}\left\{F\left[(a^\dagger)^r a^s\right]\right\}$, where in $\mathcal{N}(F)$ all the a's are to the right. It satisfies the operator identity:

$$\mathcal{N}\left\{F\left[(a^\dagger)^r a^s\right]\right\} = F\left[(a^\dagger)^r a^s\right]. \tag{1}$$

Furthermore, an auxiliary symbol $:O(a, a^\dagger):$ is used, which means expand O in powers of a and $a^\dagger$ and order normally *without* taking into account the commutation rule [1], [2].

For the moment we restrict ourselves to the case $r \geq s$, the other alternative being treated later. We do not give the proofs of the formulas here; they will be given elsewhere [3]. The case $r = s = 1$ is known [1], [2] and some features of the $r > 1, s = 1$ case have been published [4]. When needed we shall refer to [1], [2] and [4] where particular cases of our more general formulas are discussed.

We define the set of positive integers $S_{r,s}(n, k)$ entering the expansion:

$$\left[(a^\dagger)^r a^s\right]^n = \mathcal{N}\left\{\left[(a^\dagger)^r a^s\right]^n\right\} = (a^\dagger)^{n(r-s)}\left[\sum_{k=s}^{ns} S_{r,s}(n, k)(a^\dagger)^k a^k\right], \tag{2}$$

for $n = 1, 2, \ldots$. Once the $S_{r,s}(n, k)$ are known, the normal ordering of $\left[(a^\dagger)^r a^s\right]^n$ is achieved. The same applies to any operator $F\left[(a^\dagger)^r a^s\right]$ if $F(x)$ has a Taylor expansion around $x = 0$. The row sums of the triangle $S_{r,s}(n, k)$ are given by:

$$B_{r,s}(n) = \sum_{k=s}^{ns} S_{r,s}(n, k), \tag{3}$$

which for any t extend to the polynomials of order ns defined by:

$$B_{r,s}(n, t) = \sum_{k=s}^{ns} S_{r,s}(n, k) t^k. \tag{4}$$

†e-mail: blasiak@lptl.jussieu.fr
‡e-mail: penson@lptl.jussieu.fr
§e-mail: a.i.solomon@open.ac.uk

All the subsequent formulas are consequences of the following relation, linking the polynomial of Eq.(4) to a certain infinite series:

$$\begin{aligned} e^{-t}\sum_{k=s}^{\infty}\frac{1}{k!}\prod_{j=1}^{n}&[(k+(j-1)(r-s))\cdot(k+(j-1)(r-s)-1)\cdot\\ &\ldots\cdot(k+(j-1)(r-s)-s+1)]\,t^k=\\ &=\sum_{k=s}^{ns}S_{r,s}(n,k)t^k, \qquad n=1,2... \end{aligned} \tag{5}$$

which for $r,s\geq 1$ and $t=1$ is an analogue of the celebrated Dobinski relation, which expresses the combinatorial Bell numbers $B_{1,1}(n)$ as a sum of an infinite series [5], [6], [7]:

$$B_{1,1}(n)=\frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}. \tag{6}$$

For a review of various characteristics of Bell numbers $B_{1,1}(n)$ see [8]. It seems that Bell and Stirling numbers are now beginning to appear in textbooks of mathematical physics [9].

By setting $t=1$ in Eq.(5) we obtain equivalent forms of expressions for the *generalized Bell numbers* $B_{r,s}(n), n=1,2...$:

$$\begin{aligned} r>s:\quad B_{r,s}(n)=&\ B_{r,s}(n,1)=\\ &=\frac{1}{e}\sum_{k=s}^{\infty}\frac{1}{k!}\prod_{j=1}^{n}[(k+(j-1)(r-s))\cdot(k+(j-1)(r-s)-1)\cdot\\ &\qquad\qquad\ldots\cdot(k+(j-1)(r-s)-s+1)] && (7)\\ &=\frac{1}{e}\sum_{k=s}^{\infty}\frac{1}{k!}\prod_{j=1}^{n}[k+(j-1)(r-s)]^{\underline{s}} && (8)\\ &=\frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\prod_{j=1}^{n-1}\frac{(k+jr-(j-1)s)!}{(k+j(r-s))!} && (9)\\ &=\frac{(r-s)^{s(n-1)}}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\left[\prod_{j=1}^{s}\frac{\Gamma(n+\frac{k+j}{r-s})}{\Gamma(1+\frac{k+j}{r-s})}\right], && (10)\\ r=s:\quad B_{r,r}(n)=&\ B_{r,r}(n,1)=\frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{(k+r)!}{k!}\right]^{n-1}. && (11) \end{aligned}$$

For all r,s we set $B_{r,s}(0)=1$ by convention. In Eq.(8) we have employed the notation $m^{\underline{s}}:=m\cdot(m-1)\cdot(m-2)\cdot\ldots\cdot(m-s+1)$ for the falling factorial [10], and in Eq.(10) $\Gamma(y)$ is the Euler gamma function.

The numbers $S_{r,s}(n,k)$ are non-zero for $s\leq k\leq ns$, and satisfy by convention $S_{r,s}(n,0)=\delta_{n,0}$. We shall refer to them as *generalized Stirling numbers of the second kind*. Their exact expressions in the form of a finite sum are:

$$\begin{aligned} S_{r,s}(n,k)&=\frac{(-1)^k}{k!}\sum_{p=s}^{k}(-1)^p\binom{k}{p}\prod_{j=1}^{n}[(p+(j-1)(r-s))\cdot(p+(j-1)(r-s)-1)\cdot\\ &\qquad\qquad\ldots\cdot(p+(j-1)(r-s)-s+1)] && (12)\\ &=\frac{(-1)^k}{k!}\sum_{p=s}^{k}(-1)^p\binom{k}{p}\prod_{j=1}^{n}(p+(j-1)(r-s))^{\underline{s}} && (13)\\ &=\frac{(-1)^k}{k!}\left\{\left(x^r\frac{d^s}{dx^s}\right)^n\left[(1-x)^k-\sum_{p=0}^{s-1}\binom{k}{p}(-x)^p\right]\right\}_{x=1}, && (14) \end{aligned}$$

from which for $r=s=1$ one obtains the standard form of the classical Stirling numbers of the second kind [7]:

$$S_{1,1}(n,k) = \frac{(-1)^k}{k!}\sum_{p=1}^{k}(-1)^p \binom{k}{p} p^n, \tag{15}$$

and for $r = 2, s = 1$,

$$S_{2,1}(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}, \tag{16}$$

which are the so-called unsigned Lah numbers [4], [7].

That the numbers $S_{r,s}(n,k)$ are natural extensions of $S_{1,1}(n,k)$ can be neatly seen by observing their action on the space of polynomials generated by falling factorials. Let x be any real number. Then,

$$\prod_{j=1}^{n}(x+(j-1)(r-s))\cdot(x+(j-1)(r-s)-1)\cdot\ldots\cdot(x+(j-1)(r-s)-s+1) =$$

$$= \sum_{k=s}^{ns} S_{r,s}(n,k)x(x-1)\ldots(x-k+1), \tag{17}$$

which, when rewritten with $m^{\underline{s}}$ symbols become:

$$\prod_{j=1}^{n}(x+(j-1)(r-s))^{\underline{s}} = \sum_{k=s}^{ns} S_{r,s}(n,k)x^{\underline{k}}. \tag{18}$$

This last equation when specified to $r = s$ gives a particularly transparent interpretation of $S_{r,r}(n,k)$ as connection coefficients:

$$[x^{\underline{r}}]^n = \sum_{k=r}^{nr} S_{r,r}(n,k)x^{\underline{k}}. \tag{19}$$

By choosing $r = s = 1$ it boils down to the defining equations for $S_{1,1}(n,k)$ in terms of $x^{\underline{k}}$, used by J. Stirling himself [10]:

$$x^n = \sum_{k=1}^{n} S_{1,1}(n,k)x^{\underline{k}}. \tag{20}$$

The relations for $S_{r,s}(n,k)$ become particularly appealing for $r = s$: Eq.(13) can be reformulated as:

$$S_{r,r}(n,k) = \frac{(-1)^k}{k!}\sum_{p=r}^{k}(-1)^p \binom{k}{p} [p(p-1)\ldots(p-r+1)]^n \tag{21}$$

$$= \frac{(-1)^k}{k!}\sum_{p=r}^{k}(-1)^p \binom{k}{p} [p^{\underline{r}}]^n, \tag{22}$$

which differs from Eq.(15) in that the sum in Eq.(22) starts from $p = r$ and p^n becomes $[p^{\underline{r}}]^n$. The recurence relations for $S_{r,r}(n,k)$ are the following:

$$S_{r,r}(1,r) = 1, \quad S_{r,r}(n,k) = 0, \qquad k < r, \quad nr < k \le (n+1)r, \tag{23}$$

$$S_{r,r}(n+1,k) = \sum_{p=0}^{r}\binom{k+p-r}{p} r^{\underline{p}}\, S_{r,r}(n,k+p-r), \qquad r \le k \le nr, \quad n > 1, \tag{24}$$

where the definition $r^{\underline{0}} := 1$ is made. Note that for $r = s = 1$ we get the known relation for conventional Stirling numbers i.e.: $S_{1,1}(n+1,k) = kS_{1,1}(n,k) + S_{1,1}(n,k-1)$ with appropriate initial conditions [10].

The "non-diagonal" generalized Bell numbers $B_{r,s}(n)$ can always be expressed as special values of generalized hypergeometric functions ${}_pF_q$. Algebraic manipulation of Eqs.(7)-(11) yields the following examples:

$r > 1, s = 1$:

$B_{r,1}(n)$ is a combination of $r-1$ different hypergeometric functions of type ${}_1F_{r-1}(\ldots;x)$, each of them evaluated at the same value of argument $x=(r-1)^{1-r}$; here are some lowest order cases:

$$B_{2,1}(n) = \frac{n!}{e}{}_1F_1(n+1;2;1) = (n-1)!L^{(1)}_{n-1}(-1), \tag{25}$$

$$B_{3,1}(n) = \frac{2^{n-1}}{e}\left(\frac{2\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}{}_1F_2\left(n+\frac{1}{2};\frac{1}{2},\frac{3}{2};\frac{1}{4}\right)+n!{}_1F_2\left(n+1;\frac{3}{2},2;\frac{1}{4}\right)\right), \tag{26}$$

$$B_{4,1}(n) = \frac{3^{n-1}}{2e}\left(\frac{3^{3/2}\Gamma(\frac{2}{3})\Gamma(n+\frac{1}{3})}{\pi}{}_1F_3\left(n+\frac{1}{3};\frac{1}{3},\frac{2}{3},\frac{4}{3};\frac{1}{27}\right)+\right.$$
$$\left.\frac{3\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})}{}_1F_3\left(n+\frac{2}{3};\frac{2}{3},\frac{4}{3},\frac{5}{3};\frac{1}{27}\right)+n!{}_1F_3\left(n+1;\frac{4}{3},\frac{5}{3},2;\frac{1}{27}\right)\right), \tag{27}$$

etc. In Eq.(25) $L^{(\alpha)}_m(y)$ is the associated Laguerre polynomial.

Similarly the series $B_{2r,r}(n)$ can be written down in a compact form using the confluent hypergeometric function of Kummer:

$$B_{2r,r}(n) = \frac{(rn)!}{e\cdot r!}{}_1F_1(rn+1,r+1;1). \tag{28}$$

A still more general family of sequences arising from Eqs.(7)-(11) has the form $(p,r=1,2\ldots)$:

$$B_{pr+p,pr}(n) = \frac{1}{e}\left[\prod_{j=1}^{r}\frac{(p(n-1)+j)!}{(pj)!}\right]\cdot$$
$$\cdot{}_rF_r(pn+1,pn+1+p,\ldots,pn+1+p(r-1);1+p,1+2p\ldots,1+rp;1), \tag{29}$$

etc.

In contrast, the "diagonal" numbers $B_{r,r}(n)$ of Eq.(11), which also can be rewritten as:

$$B_{r,r}(n) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{(k+r-1)!}\left[k(k+1)\ldots(k+r-1)\right]^n, \qquad n=1,2,\ldots \tag{30}$$

cannot be expressed through hypergeometric functions. However, the sequences $B_{r,r}(n)$ possess a particularity that they can be always expressed in terms of conventional Bell numbers with r-nomial (binomial, trinomial…) coefficients. For example:

$$B_{2,2}(n) = \sum_{k=0}^{n-1}\binom{n-1}{k}B_{1,1}(n+k). \tag{31}$$

Several low-order triangles of $S_{r,s}(n,k)$ and their associated $B_{r,s}(n)$ are presented in Table 1.

It turns out that various generating functions for $B_{r,s}(n)$ and $S_{r,s}(n,k)$ can be related to special quantum states, called coherent states [1], defined as linear combinations of the eigenstates of the harmonic oscillator, $H=a^\dagger a$, $H|n\rangle = n|n\rangle$, $\langle n|n'\rangle = \delta_{n,n'}$ and defined as:

$$|z\rangle = e^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle, \tag{32}$$

(with $\langle z|z\rangle = 1$), for z complex. The states $|z\rangle$ satisfy:

$$a|z\rangle = z|z\rangle. \tag{33}$$

It has been noticed in [12] that,

$$\langle z|(a^\dagger a)^n|z\rangle \overset{|z|=1}{=} B_{1,1}(n) \tag{34}$$

and the exponential generating function (egf) of the numbers $B_{1,1}(n)$, i.e. $\sum_{n=0}^{\infty}B_{1,1}(n)\frac{\lambda^n}{n!}$ satisfies

$$\langle z|e^{\lambda a^\dagger a}|s\rangle = \langle z| : e^{a^\dagger a(e^\lambda -1)} : |z\rangle \overset{|z|=1}{=} e^{e^\lambda -1} = \sum_{n=0}^{\infty} B_{1,1}(n)\frac{\lambda^n}{n!} \tag{35}$$

which is a restatement of the known fact that [1], [2], [4]:

$$\mathcal{N}\left(e^{\lambda a^\dagger a}\right) =: e^{a^\dagger a(e^\lambda -1)} : \; . \tag{36}$$

How does it generalize to $r, s \geq 1$?

Eqs.(2) and (3) give directly:

$$\langle z| \left[(a^\dagger)^r a^s\right]^n |z\rangle \overset{z=1}{=} B_{r,s}(n), \tag{37}$$

valid for all r, s.

We first treat the case $r = s = 1, 2 \ldots$. Note in this context that a Hermitian Hamiltonian $(a^\dagger)^r a^r$ is of great importance in quantum optics, as for $r = 2, 3, \ldots$ it provides a description of non-linear Kerr-type media [11]. Using Eq.(22) we can find the following egf of $S_{r,r}(n, k)$:

$$\sum_{n=\lceil k/r \rceil} \frac{x^n}{n!} S_{r,r}(n,k) = \frac{(-1)^k}{k!} \sum_{p=r}^{k} (-1)^p \begin{pmatrix} k \\ p \end{pmatrix} \left(e^{xp(p-1)\ldots(p-r+1)} - 1\right), \tag{38}$$

where $\lceil y \rceil$ is the *ceiling* function [10], defined as the nearest integer greater or equal to y. This yields via exchange of order of summation:

$$\mathcal{N}\left(e^{\lambda (a^\dagger)^r a^r}\right) = 1 + : \sum_{k=r}^{\infty} \frac{(-1)^k}{k!} \left[\sum_{p=r}^{k} (-1)^p \begin{pmatrix} k \\ p \end{pmatrix} \left(e^{\lambda p(p-1)\ldots(p-r+1)} - 1\right)\right] (a^\dagger a)^k : \tag{39}$$

and

$$\langle z|e^{\lambda (a^\dagger)^r a^r}|z\rangle \overset{|z|=1}{=} 1 + \sum_{k=r}^{\infty} \frac{(-1)^k}{k!} \left[\sum_{p=r}^{k} (-1)^p \begin{pmatrix} k \\ p \end{pmatrix} \left(e^{\lambda p^{\underline{r}}} - 1\right)\right]. \tag{40}$$

Another case which can be written in a closed form is $r > 1$, $s = 1$, for which the egf for $S_{r,1}(n, k)$ reads:

$$\sum_{n=\lceil k/r \rceil}^{\infty} \frac{x^n}{n!} S_{r,1}(n,k) = \frac{1}{k!} \left\{(1-(r-1)x)^{-\frac{1}{r-1}} - 1\right\}. \tag{41}$$

One then obtains [4], [13]:

$$\mathcal{N}\left(e^{\lambda (a^\dagger)^r a}\right) =: exp\left\{\left[(1-\lambda(a^\dagger)^{r-1}(r-1))^{-\frac{1}{r-1}} - 1\right] a^\dagger a\right\} : \tag{42}$$

For arbitrary $r > s$ the following formula is still valid:

$$\mathcal{N}\left(e^{\lambda (a^\dagger)^r a^s}\right) = 1 + : \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} (a^\dagger)^{n(r-s)} \left(\sum_{k=s}^{ns} S_{r,s}(n,k)(a^\dagger a)^k\right) : \tag{43}$$

where the explicit form of $S_{r,s}(n, k)$ may be used, see Eqs.(12)-(14).

A close look at Eqs.(40) and (37) reveals that

$$\langle z|e^{\lambda (a^\dagger)^r a^r}|z\rangle \overset{|z|=1}{=} \sum_{n=0}^{\infty} B_{r,r}(n)\frac{\lambda^n}{n!}. \tag{44}$$

Comparing Eqs.(44),(42) and (40) we conclude that for $r > 1, s = 1$ and $r = s$ the egf of respective generalized Bell numbers are special matrix elements of $e^{\lambda(a^\dagger)^r a^s}$ in coherent states.

However, for arbitrary $r > s > 1$ examination of Eqs.(7)-(10) confirms that the numbers $B_{r,s}(n)$ increase so rapidly with n that one cannot define their egf's meaningfully. In particular, this is reflected by Eq.(43) which is true in its operator form, but one needs to exercise care when calculating its matrix elements as the convergence of the result may require limitations on λ.

A well-defined and convergent procedure for such sequences is to consider what we call *hypergeometric generating functions* (hgf), i.e. the egf for the ratios $B_{r,s}/(n!)^t$, where t is an appropriately chosen integer. A case in point is the series $B_{3,2}(n)$ which may be written explicitly from Eq.(29) as:

$$B_{3,2}(n) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{(n+k)!(n+k+1)!}{k!(k+1)!(k+2)!}. \tag{45}$$

Its hgf $G_{3,2}(\lambda)$ is then:

$$G_{3,2}(\lambda) = \sum_{n=0}^{\infty}\left[\frac{B_{3,2}(n)}{n!}\right]\frac{\lambda^n}{n!} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{(k+2)!}\,{}_2F_1(k+2,k+1;1;\lambda). \tag{46}$$

Similarly for $G_{4,2}(\lambda)$ one obtains:

$$G_{4,2}(\lambda) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{(k+2)!}\,{}_2F_1\left(\frac{k+2}{2},\frac{k}{2}+1;1;4\lambda\right). \tag{47}$$

Eq.(28) implies more generally:

$$\begin{aligned} G_{2r,r}(\lambda) &= \sum_{n=0}^{\infty}\left[\frac{B_{2r,r}(n)}{(n!)^{r-1}}\right]\frac{\lambda^n}{n!} \\ &= \begin{cases} \frac{1}{1!e}\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\,{}_2F_1\left(\frac{k+1}{2},\frac{k}{2}+1;1;4\lambda\right), & r=2, \\ \frac{1}{2!e}\sum_{k=0}^{\infty}\frac{1}{(k+2)!}\,{}_3F_2\left(\frac{k+1}{3},\frac{k+2}{3},\frac{k+3}{3};1,1;27\lambda\right), & r=3, \\ \frac{1}{3!e}\sum_{k=0}^{\infty}\frac{1}{(k+3)!}\,{}_4F_3\left(\frac{k+1}{4},\ldots,\frac{k+4}{4};1,1,1;256\lambda\right), & r=4\ldots\text{etc.} \end{cases} \end{aligned} \tag{48}$$

See [14] for other instances where this type of hgf appear.

All the previous considerations also apply to the case $r < s$. In this case we define the generalized Stirling numbers by (note the difference with Eq.(2)):

$$r \le s: \qquad \left[(a^\dagger)^r a^s\right]^n = \mathcal{N}\left\{\left[(a^\dagger)^r a^s\right]^n\right\} = \left[\sum_{k=r}^{nr} S_{r,s}(n,k)(a^\dagger)^k a^k\right] a^{n(s-r)}, \tag{49}$$

By taking the Hermitian conjugate of Eq.(2), with the change $r \leftrightarrow s$, we obtain

$$r \le s: \qquad \left[(a^\dagger)^r a^s\right]^n = \left[\sum_{k=r}^{nr} S_{s,r}(n,k)(a^\dagger)^k a^k\right] a^{n(s-r)}, \tag{50}$$

whence the symmetry relation:

$$S_{r,s}(n,k) = S_{s,r}(n,k), \qquad s \le k \le ns, \quad (r \le s). \tag{51}$$

The normal ordering of $F[a^s(a^\dagger)^r]$ involves the *antinormal to normal* order, in the terminology of [15]. With this in mind we define the *anti*-Stirling numbers of the second kind $\tilde{S}_{r,s}(n,k)$ as:

$$r \ge s: \qquad [a^s(a^\dagger)^r]^n = (a^\dagger)^{n(r-s)}\left[\sum_{k=0}^{ns}\tilde{S}_{s,r}(n,k)(a^\dagger)^k a^k\right]. \tag{52}$$

It can be demonstrated [3] that they satisfy the following symmetry properties:

$$\tilde{S}_{s,r}(n,k) = \tilde{S}_{r,s}(n,k), \qquad 0 \le k \le ns, \quad (r \ge s), \tag{53}$$

$$\tilde{S}_{r,s}(n,k) = S_{r,s}(n+1,k+s), \qquad 0 \le k \le ns, \quad (r \ge s). \tag{54}$$

We have obtained recursion relations for $B_{r,s}(n)$ for certain values of r, s. They include the $r \geq 1, s = 1$ case:

$$B_{r,1}(n+1) = \sum_{k=0}^{n} \binom{n}{k} \left[\prod_{j=0}^{n-k} (r + (j-1)(r-1)) \right] B_{r,1}(k), \tag{55}$$

which for $r = 2, s = 1$ may be written explicitly as:

$$B_{2,1}(n+1) = \sum_{k=0}^{n} \binom{n}{k} (n-k+1)! B_{2,1}(k), \tag{56}$$

and for $r = s = 1$ reduces to the known relation [5], [8], which is the binomial transform:

$$B_{1,1}(n+1) = \sum_{k=0}^{n} \binom{n}{k} B_{1,1}(k). \tag{57}$$

If in Eq.(1) $F(x)$ has a Taylor expansion of $F(x)$ around $x_0 \neq 0$, then

$$\mathcal{N}\left\{F\left[(a^\dagger)^r a^s\right]\right\} = \mathcal{N}\left(\sum_{k=0}^{\infty} \frac{F^{(k)}(x_o)}{k!}\left((a^\dagger)^r a^s - x_o\right)^k\right), \tag{58}$$

and our above results should be supplemented by the appropriate binomial expansion coefficients.

All the $B_{r,s}(n)$ described here are the n-th moments of positive functions on the positive half axis; some solutions of the associated Stieltjes moment problem have been recently discussed at some length [16]. In addition we have found that even larger classes of combinatorial sequences are solutions of the moment problem and they can be used for the construction of new families of coherent states [17].

We are considering the possible combinatorial interpretation of the above results.

Acknowledgments

We thank G. Duchamp, L. Haddad, A. Horzela, M. Mendez and J.Y. Thibon for enlightening discussions. N.J.A. Sloane's Encyclopedia of Integer Sequences (http://www.research.att.com/~njas/sequences) was an essential help in sorting out the properties of sequences we have encountered in this work.

---

[1] J.R. Klauder and E.C.G. Sudarshan, *Fundamentals of Quantum Optics*, (Benjamin, New York, 1968)
[2] W.H. Louisell, *Quantum Statistical Properties of Radiation*, (J.Wiley,New York, 1990)
[3] P. Blasiak, K.A. Penson and A.I. Solomon (to be published)
[4] W. Lang, *J. Int. Seqs.* Article 00.2.4 (2000), available electronically at: http://www.research.att.com/~njas/sequences/JIS/
[5] H.S. Wilf, *Generatingfunctionology*, (Academic Press, New York, 1994)
[6] G.M. Constantine and T.H. Savits, *SIAM J. Discrete Math.* **7**, 194 (1994)
[7] L. Comtet, *Advanced Combinatorics*, (Reidel, Dordrecht, 1974)
[8] M. Aigner, *Discr. Math.* **205**, 207 (1999)
[9] R. Aldrovandi, *Special Matrices of Mathematical Physics*, (World Scientific, Singapore, 2001)
[10] R.L. Graham, D.E. Knuth and O.Patashnik, *Concrete Mathematics. A Foundation of Computer Science*, (Addison-Wesley Publishing Co., Reading, 1994)
[11] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, (Cambridge University Press, Cambridge, 1995)
[12] J. Katriel, *Phys. Lett.* A **237**, 159 (2000)
[13] G. Dattoli, P.L. Ottaviani, A. Torre and L. Vàzquez, *Riv. Nuovo Cim.* **20** no.2, 1 (1997)
[14] J.M. Sixdeniers, K.A. Penson and A.I. Solomon, *J. Int. Seqs.* Article 01.1.04 (2001), available electronically at: http://www.research.att.com/~njas/sequences/JIS/

[15] J. Katriel and G. Duchamp, *J. Phys.* A **28**, 7209 (1995)
[16] K.A. Penson and A.I. Solomon, arXiv: quant-ph/0211061
[17] K.A. Penson and A.I. Solomon, *Proceedings of the 2nd International Symposium on Quantum Theory and Symmetries*, E. Kapuścik and A. Horzela, Editors (World Scientific, Singapore, 2002) p.527, arXiv: quant-ph/0111151

Table 1.
Triangles of generalized Stirling and Bell numbers, as defined by Eqs.(2) and(3).

r=1, s=1

	$S_{1,1}(n,k),\ 1 \le k \le n$						$B_{1,1}(n)$
$n=1$	1						1
$n=2$	1	1					2
$n=3$	1	3	1				5
$n=4$	1	7	6	1			15
$n=5$	1	15	25	10	1		52
$n=6$	1	31	90	65	15	1	203

r=2, s=1

	$S_{2,1}(n,k),\ 1 \le k \le n$						$B_{2,1}(n)$
$n=1$	1						1
$n=2$	2	1					3
$n=3$	6	6	1				13
$n=4$	24	36	12	1			73
$n=5$	120	240	120	20	1		501
$n=6$	720	1800	1200	300	30	1	4051

r=3, s=1

	$S_{3,1}(n,k),\ 1 \le k \le n$						$B_{3,1}(n)$
$n=1$	1						1
$n=2$	3	1					4
$n=3$	15	9	1				25
$n=4$	105	87	18	1			211
$n=5$	945	975	285	30	1		2236
$n=6$	10395	12645	4680	705	45	1	28471

r=2, s=2

	$S_{2,2}(n,k),\ 2 \le k \le 2n$											$B_{2,2}(n)$
$n=1$	1											1
$n=2$	2	4	1									7
$n=3$	4	32	38	12	1							87
$n=4$	8	208	625	576	188	24	1					1657
$n=5$	16	1280	9080	16944	12052	3840	580	40	1			43833
$n=6$	32	7744	116656	412800	540080	322848	98292	16000	1390	60	1	1515903

r=3, s=2

	$S_{3,2}(n,k),\ 2 \le k \le 2n$											$B_{3,2}(n)$
$n=1$	1											1
$n=2$	6	6	1									13
$n=3$	72	168	96	18	1							355
$n=4$	1440	5760	6120	2520	456	36	1					16333
$n=5$	43200	259200	424800	285120	92520	15600	1380	60	1			1121881
$n=6$	1814400	15120000	34776000	33566400	16304400	4379760	682200	62400	3270	90	1	106708921

r=3, s=3

	$S_{3,3}(n,k),\ 3 \le k \le 3n$													$B_{3,3}(n)$
$n=1$	1													1
$n=2$	6	18	9	1										34
$n=3$	36	540	1242	882	243	27	1							2971
$n=4$	216	13608	94284	186876	149580	56808	11025	1107	54	1				513559
$n=5$	1296	330480	6148872	28245672	49658508	41392620	18428400	4691412	706833	63375	3285	90	1	149670844