Dobiński-type relations and the Log-normal distribution

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Abstract.
We consider sequences of generalized Bell numbers $B(n),\ n=0,1,\ldots$ which can be represented by Dobiński-type summation formulas, i.e. $B(n)=\frac{1}{C}\sum_{k=0}^{\infty}\frac{[P(k)]^n}{D(k)}$, with $P(k)$ a polynomial, $D(k)$ a function of k and $C=const$. They include the standard Bell numbers ($P(k)=k,\ D(k)=k!,\ C=e$), their generalizations $B_{r,r}(n),\ r=2,3,\ldots$ appearing in the normal ordering of powers of boson monomials ($P(k)=\frac{(k+r)!}{k!},\ D(k)=k!,\ C=e$), variants of "ordered" Bell numbers $B_o^{(p)}(n)$ ($P(k)=k,\ D(k)=(\frac{p+1}{p})^k,\ C=1+p$, p=1,2...), etc. We demonstrate that for α,β,γ,t positive integers ($\alpha,t\neq 0$), $\left[B(\alpha n^2+\beta n+\gamma)\right]^t$ is the n-th moment of a positive function on $(0,\infty)$ which is a weighted infinite sum of log-normal distributions.

In a recent investigation [1] we analysed sequences of integers which appear in the process of normal ordering of powers of monomials of boson creation $a^\dagger$ and annihilation a operators, satisfying the commutation rule $[a, a^\dagger] = 1$. For r, s integers such that $r \geq s$, we define the generalized Stirling numbers of the second kind $S_{r,s}(n, k)$ as

$$\left[(a^\dagger)^r a^s\right]^n = (a^\dagger)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n,k)(a^\dagger)^k a^k \tag{1}$$

and the corresponding Bell numbers $B_{r,s}(n)$ as

$$B_{r,s}(n) = \sum_{k=s}^{ns} S_{r,s}(n,k). \tag{2}$$

In [1] explicit and exact expressions for $S_{r,s}(n, k)$ and $B_{r,s}(n)$ were found. In a parallel study [2] it was demonstrated that $B_{r,s}(n)$ can be considered as the n-th moment of a probability distribution on the positive half-axis. In addition, for every pair (r, s) the corresponding distribution can be explicitly written down. These distributions constitute the solutions of a family of Stieltjes moment problems, with $B_{r,s}(n)$ as moments. Of particular interest to us are the sequences with $r = s$, for which the following representation as an infinite series has been obtained:

$$B_{r,r}(n) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{(k+r)!}{k!}\right]^{n-1} \tag{3}$$

$$= \frac{1}{e}\sum_{k=0}^{\infty} \frac{[k(k+1)\dots(k+r-1)]^n}{(k+r-1)!}, \qquad n > 0. \tag{4}$$

Eqs.(3) and (4) are generalizations of the celebrated Dobiński formula ($r = 1$) [3]:

$$B_{1,1}(n) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^n}{k!}, \qquad n \geq 0, \tag{5}$$

which expresses the conventional Bell numbers $B_{1,1}(n)$ as a rapidly convergent series. Its simplicity has inspired combinatorialists such as G.-C. Rota [4] and H.S. Wilf [5]. Eq.(5) has far-reaching implications in the theory of stochastic processes [6], [7], [8].

The probability distribution whose n-th moment is $B_{r,r}(n)$ is an infinite ensemble of weighted Dirac delta functions located at a specific set of integers (a so-called *Dirac comb*):

$$B_{r,r}(n) = \int_0^\infty x^n \left\{\frac{1}{e}\sum_{k=0}^{\infty} \frac{\delta(x - k(k+1)\dots(k+r-1))}{(k+r-1)!}\right\} dx, \quad n \geq 0. \tag{6}$$

For $r = 1$ the discrete distribution of Eq.(6) is the weight function for the orthogonality relation for Charlier polynomials [9]. In contrast we emphasize that for $r \neq s$ the $B_{r,s}(n)$ are moments of continuous distributions [2].

In this note we wish to point out an intimate relation between the formulas of Eqs. (3), (4), (5) and the log-normal distribution [10], [11]:

$$P_{\sigma,\mu}(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}, \qquad x \geq 0, \quad \sigma, \mu > 0. \tag{7}$$

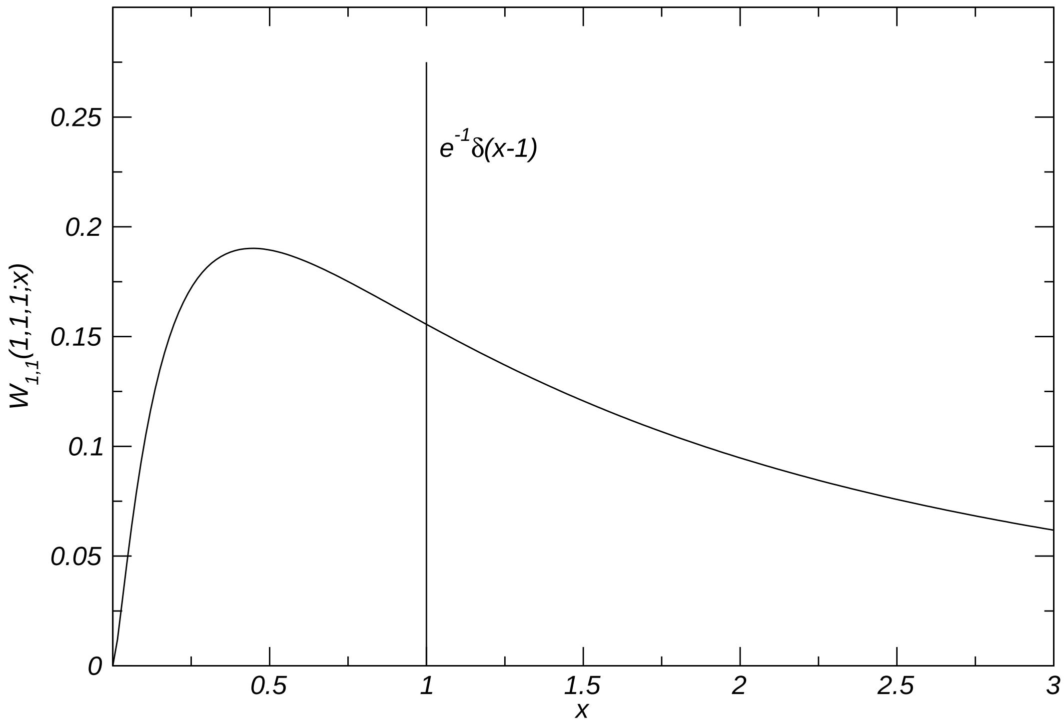

Figure 1. Weight function $W_{1,1}(1,1,1;x)$, see Eq.(13).

First we quote the standard expression for its n-th moment:

$$M_n = \int_0^\infty x^n P_{\sigma,\mu}(x)dx = e^{n\left(\mu+n\frac{\sigma^2}{2}\right)}, \quad n \geq 0, \tag{8}$$

which can be reparametrized for $k > 1$ as

$$M_n = k^{\alpha n^2+\beta n}, \tag{9}$$

with

$$\mu = \beta \ln(k), \tag{10}$$

$$\sigma = \sqrt{2\alpha \ln(k)} > 0. \tag{11}$$

Given three integers α, β, γ (where $\alpha > 0$), we wish to find a weight function $W_{1,1}(\alpha, \beta, \gamma; x) > 0$ such that

$$B_{1,1}(\alpha n^2 + \beta n + \gamma) = \int_0^\infty x^n W_{1,1}(\alpha, \beta, \gamma; x)dx. \tag{12}$$

Eqs.(5), (7) and (9) provide an immediate solution:

$$W_{1,1}(\alpha, \beta, \gamma; x) = \frac{1}{e}\left[\delta(x-1) + \sum_{k=2}^{\infty} \frac{k^\gamma \exp\left(-\frac{(\ln(x)-\beta\ln(k))^2}{4\alpha\ln(k)}\right)}{2xk!\sqrt{\pi\alpha\ln(k)}}\right], \tag{13}$$

which is an *infinite* sum of weighted log-normal distributions supplemented by a single Dirac peak of weight e^{-1} located at $x = 1$. Thus it is a *superposition* of discrete and continuous distributions. Virtually the same approach can be adopted for the sequences $B_{r,r}(n)$, $r > 1$. In this case the $k = 1$ term in the numerator of Eq.(3) is larger than one

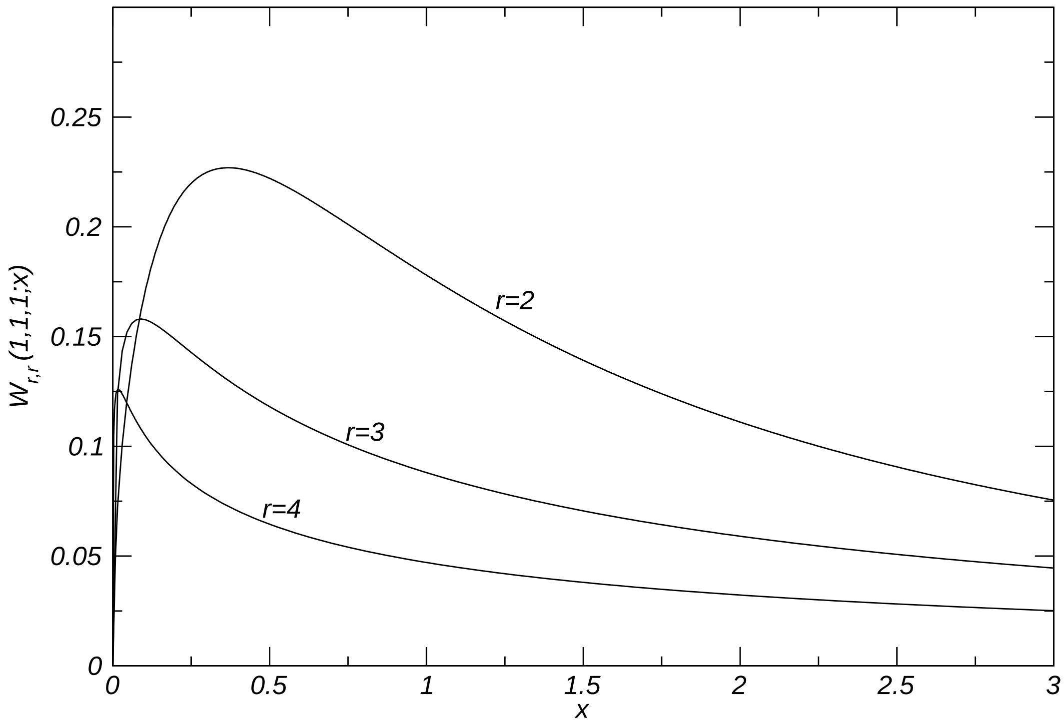

Figure 2. Weight functions $W_{r,r}(1,1,1;x)$ for $r = 2,3,4$.

and so there will be no Dirac peak in the formula. Then the function $W_{r,r}(\alpha,\beta,\gamma;x) > 0$ defined by (α,β,γ integers, $\alpha,\gamma > 0$):

$$B_{r,r}(\alpha n^2 + \beta n + \gamma) = \int_0^\infty x^n W_{r,r}(\alpha,\beta,\gamma;x)dx, \tag{14}$$

is a purely *continuous* probability distribution given again by an infinite sum of weighted log-normal distributions:

$$W_{r,r}(\alpha,\beta,\gamma;x) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{\left[\frac{(k+r)!}{k!}\right]^{\gamma-1} \exp\left(-\frac{\left[\ln(x)-\beta\ln\left(\frac{(k+r)!}{k!}\right)\right]^2}{4\alpha\ln\left[\frac{(k+r)!}{k!}\right]}\right)}{2xk!\sqrt{\pi\alpha\ln\left[\frac{(k+r)!}{k!}\right]}}. \tag{15}$$

The solutions of the moment problems of Eqs.(9), (12) and (14) are not unique. More general solutions may be obtained by the method of the inverse Mellin transform, see [12].

Several other types of combinatorial sequences have properties exemplified by Eqs.(12) and (14). We quote for example the so-called "ordered" Bell numbers $B_o(n)$ defined as [5]:

$$B_o(n) = \sum_{k=1}^{n} S(n,k)k!, \tag{16}$$

where $S(n,k)$ are the Stirling numbers of the second kind, $S_{1,1}(n,k)$ in our notation.

These ordered Bell numbers satisfy the following Dobiński-type relation:

$$B_o(n) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k}, \tag{17}$$

from which a formula analogous to Eq.(13) readily follows. A more general identity of type (17) is [13]:

$$B_o^{(p)}(n) = \frac{1}{p+1}\sum_{k=1}^{\infty}k^n\left(\frac{p}{p+1}\right)^k = \sum_{k=0}^{n}S(n,k)k!\;p^k, \qquad p = 2,3,\ldots. \tag{18}$$

We will not discuss other types of sequences but rather observe that the relations of Eqs. (3), (4), (5), (17),(18) naturally imply that any power of these numbers also satisfies a Dobiński-type relation. As an example we give explicitly the simplest case of Eq.(5). For integer $t > 0$:

$$[B_{1,1}(n)]^t = \frac{1}{e^t}\sum_{k_1,k_2,\ldots k_t=0}^{\infty}\frac{(k_1k_2\ldots k_t)^n}{k_1!k_2!\ldots k_t!}, \tag{19}$$

with correspondingly more complicated formulas of a similar nature for powers of $B_{r,r}(n)$, $B_o(n)$ and $B_o^{(p)}(n)$. For combinatorial applications of Eq.(19) see [14], [15] and [16].

We conclude that for any sequences of the type $B(n)$ specified above $[B(\alpha n^2+\beta n+\gamma)]^t$ is always given as an n-th moment of a positive function on $(0,\infty)$ expressible by sums of weighted log-normal distributions. We illustrate such a function for $B_{1,1}(n)$ in Fig.(1). The application to $B_{r,r}(n)$ for $r = 2,3,4$ is presented in Fig.(2). The area under every curve is equal to 1 on extrapolating to large x (not displayed). In both examples we have chosen $\alpha = \beta = \gamma = 1$. Observe the exceedingly slow decrease of these probabilities for $x \to \infty$. This is confirmed by the fact that the moment sequences $[B(\alpha n^2+\beta n+\gamma)]^t$ are extremely rapidly increasing. In the simplest case $\alpha = \beta = \gamma = t = 1$ we find $B_{1,1}(n^2+n+1) = 1, 5, 877, 27644437, 474869816156751$ for $n = 0\ldots 4$.

The circumstance that we can determine the positive solutions of the Stieltjes moment problem with both $B(n)$ (discrete distribution) and $[B(\alpha n^2+\beta n+\gamma)]^t$ (continuous distribution) is a very specific consequence of the existence of Dobiński-type expansions. To our knowledge it has no equivalent in standard solutions of the moment problem. For instance, if the moments are $n!$ the solution e^{-x} does not give any indication as how one might obtain the solution for the moments equal to $(n^2)!$.

The strict positivity of $W_{r,r}(\alpha,\beta,\gamma;x)$, for $r = 1,2\ldots$, suggests their use in the construction of coherent states, which satisfy the *resolution of identity* property [17], [18], [19], [20]. This can be done by the substitution $n! \to B_{r,r}(\alpha n^2+\beta n+\gamma)$ in the definition of standard coherent states. More precisely for a complete and orthonormal set of wave functions $|n\rangle$ such that $\langle n|n'\rangle = \delta_{n,n'}$ and complex z we define the normalized coherent state as

$$|z;\alpha,\beta,\gamma\rangle = \frac{1}{\mathcal{N}^{1/2}(\alpha,\beta,\gamma;|z|^2)}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{B_{r,r}(\alpha n^2+\beta n+\gamma)}}|n\rangle, \tag{20}$$

with the normalization

$$\mathcal{N}(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{x^n}{B_{r,r}(\alpha n^2+\beta n+\gamma)}, \tag{21}$$

which is a rapidly converging function of x for $0 \le x < \infty$, $x = |z|^2$. Then, using the procedure of [18] we can demonstrate that the states of Eq.(20) along with Eq.(14) automatically satisfy the resolution of unity

$$\int\int_{\mathbb{C}} d^2z|z;\alpha,\beta,\gamma\rangle\tilde{W}_{r,r}(\alpha,\beta,\gamma;|z|^2)\langle z;\alpha,\beta,\gamma| = I = \sum_{n=0}^{\infty}|n\rangle\langle n|, \tag{22}$$

with

$$W_{r,r}(\alpha,\beta,\gamma;|z|^2)=\pi\frac{\tilde{W}_{r,r}(\alpha,\beta,\gamma;|z|^2)}{\mathcal{N}(\alpha,\beta,\gamma;|z|^2)}. \tag{23}$$

We are currently investigating the quantum-optical properties of states defined in Eq.(20).

We close by quoting from [7] that, "the idea of representing the combinatorially defined numbers by an infinite sum or an integral, typically with a probabilistic interpretation, has proved to be a very fruitful one". In our particular case it has allowed us to reveal quite an unexpected relation between the Dobiński-type summation relations, which by themselves are reflections of boson statistics, and the log-normal distribution.

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