Sequences related to the Pell generalized equation

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Abstract

We consider sequences of the type $A_n = 6A_{n-1} - A_{n-2}, \quad A_0 = r,\ A_1 = s$ (r and s integers) and show that all sequences that solve particular cases of the Pell generalized equation are expressible as a constant times one of four particular sequences of the same type.

Let $\alpha = 3 + 2\sqrt{2}$, $\beta = 3 - 2\sqrt{2}$ be the roots of the polynomial $x^2 - 6x + 1$. Note that $\alpha+\beta = 6$, $\alpha\beta = 1$, $\alpha-\beta = 4\sqrt{2}$. Also let $\gamma = 1+\sqrt{2}$, $\delta = 1-\sqrt{2}$. Then $\gamma^2 = \alpha$, $\delta^2 = \beta$, $\gamma\delta = -1$. We take $\gamma = \alpha^{\frac{1}{2}}$, $\delta = -\beta^{\frac{1}{2}}$.
Consider the sequence A_n defined by

$$A_n = 6A_{n-1} - A_{n-2}, \quad A_0 = r,\ A_1 = s, \tag{1}$$

where r and s are integers.
The object of this contribution is to show that all sequences of the type given by Equation 1 that solve particular cases of the Pell generalized equation (see [3]) are expressible as a constant times one of four particular sequences of the same type.
The generating function of A_n is given by

$$g(x) = \frac{r + (s - 7r)x + (6r - s)x^2}{(1 - x)(1 - 6x + x^2)},$$

from which we get the closed form

$$A_n = \frac{2s\gamma\alpha - 2r\gamma}{8\sqrt{2}\gamma^3}\alpha^n - \frac{2r\gamma\beta + 2s\delta\beta}{8\sqrt{2}\delta^3}\beta^n.$$

Two important particular cases of Equation 1 are the sequences

$$T_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}}, \tag{2}$$

$$L_n = \alpha^n + \beta^n. \tag{3}$$

The initial conditions are, respectively, $T_0 = 0$, $T_1 = 1$, and $L_0 = 2$, $L_1 = 6$. They are, respectively, sequences A001109 and A003499 in [4]. T_n is related to triangular numbers: see [2]. The relationships between T_n and L_n are of the same genre as those between Fibonacci and Lucas numbers: so a wealth of known identities relating Fibonacci and Lucas numbers translates to our pair.

Now we establish some identities concerning α and β that allow us to introduce the other sequences necessary for our argument. First note that from $\alpha + \beta + 2\sqrt{\alpha\beta} = 8$ it follows $\left(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}\right)^2 = 8$, that is $\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}} = 2\sqrt{2}$. Also from $\alpha + \beta - 2\sqrt{\alpha\beta} = 4$ it follows $\left(\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}\right)^2 = 4$, that is $\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}} = 2$. Then

1. $$\left(\alpha^{n+\frac{1}{2}} + \beta^{n+\frac{1}{2}}\right)\left(\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}\right) = \alpha^{n+1} - \alpha^n + \beta^n - \beta^{n+1},$$

   from which we get

   $$\frac{\alpha^{n+\frac{1}{2}} + \beta^{n+\frac{1}{2}}}{2\sqrt{2}} = T_{n+1} - T_n = B_n,$$

   which is sequence A001653 in [4]. We have $B_0 = 1$, $B_1 = 5$.

2. $$\left(\alpha^{n+\frac{1}{2}} + \beta^{n+\frac{1}{2}}\right)\left(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}\right) = \alpha^{n+1} + \alpha^n + \beta^n + \beta^{n+1}.$$

   Then

   $$\begin{aligned} \sqrt{2}\left(\alpha^{n+\frac{1}{2}} + \beta^{n+\frac{1}{2}}\right) &= \frac{\alpha^{n+1} + \alpha^n + \beta^n + \beta^{n+1}}{2} \\ &= \frac{L_{n+1} + L_n}{2} \\ &= E_n, \end{aligned}$$

   which is sequence A077445 in [4]. We have $E_0 = 4$, $E_1 = 20$. We see that $E_n = 4B_n$.

3.
$$\left(\alpha^{n+\frac{1}{2}}-\beta^{n+\frac{1}{2}}\right)\left(\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}\right)=\alpha^{n+1}+\alpha^{n}-\beta^{n}-\beta^{n+1}.$$

From this we get

$$\alpha^{n+\frac{1}{2}}-\beta^{n+\frac{1}{2}}=2(T_{n+1}+T_n)=C_n.$$

This is sequence A077444 in [4]. We have $C_0=2$, $C_1=14$. We can write as well

$$C_n=\gamma^{2n+1}+\delta^{2n+1}.$$

Then $\frac{C_n}{2}$ are the NSW numbers (sequence A002315 in [4], also see [1]). Also $\frac{C_n}{2}=R_{2n+1}$, where the R_n sequence gives the numerators of continued fraction convergents to $\sqrt{2}$ (sequence A001333 in [4]).

4.
$$\left(\alpha^{n+\frac{1}{2}}-\beta^{n+\frac{1}{2}}\right)\left(\alpha^{\frac{1}{2}}-\beta^{\frac{1}{2}}\right)=\alpha^{n+1}-\alpha^{n}-\beta^{n}+\beta^{n+1}.$$

From this we get

$$\alpha^{n+\frac{1}{2}}-\beta^{n+\frac{1}{2}}=\frac{L_{n+1}-L_n}{2}=C_n.$$

Now we are going to study in more details the L_n sequence. In the L_n sequence there are hidden some sequences of squares. More precisely, we have that $L_{2n}+2$, $2(L_{2n}-2)$, $L_{2n+1}-2$, $2(L_{2n+1}+2)$ are perfect squares.

1.
$$\begin{aligned} L_n^2 &= (\alpha^n+\beta^n)^2 \\ &= \alpha^{2n}+\beta^{2n}+2 \\ &= L_{2n}+2. \end{aligned}$$

2.
$$\begin{aligned} (8T_n)^2 &= 64\left(\frac{\alpha^n-\beta^n}{4\sqrt{2}}\right)^2 \\ &= 2(\alpha^{2n}+\beta^{2n}-2) \\ &= 2(L_{2n}-2). \end{aligned}$$

3.

$$\begin{aligned} L_{2n+1}-2 &= \alpha^{2n+1}+\beta^{2n+1}-2 \\ &= \left(\alpha^{n+\frac{1}{2}}-\beta^{n+\frac{1}{2}}\right)^2 \\ &= C_n^2. \end{aligned}$$

4.

$$\begin{aligned} 2(L_{2n+1}+2) &= 2(\alpha^{2n+1}+\beta^{2n+1}+2) \\ &= \left(\sqrt{2}\left(\alpha^{n+\frac{1}{2}}+\beta^{n+\frac{1}{2}}\right)\right)^2 \\ &= E_n^2. \end{aligned}$$

Also we obtain easily

$$4(8T_n^2+1) = L_{2n}+2 = L_n^2, \tag{4}$$

$$2C_n^2+8 = 2(L_{2n+1}+2) = E_n^2, \tag{5}$$

and finally

$$4(2B_n^2-1) = L_{2n+1}-2 = C_n^2, \tag{6}$$

which means that $8T_n^2+1$, $2C_n^2+8$ and $2B_n^2-1$ are perfect squares.
After some tedious algebra we can write the following formula involving the square of A_n:

$$32A_n^2+2(r^2+s^2-6rs) = (r^2-s^2)L_{2n-2}+(6s^2-2rs)L_{2n-1}. \tag{7}$$

We would like to find conditions on r and s under which the LHS can be reduced to a perfect square. This happens only if we can find values of r and s for which the RHS can be written as a constant times L_n for some n: this because of the results obtained before for L_n. In this case we would have solutions to particular generalized Pell equations.
The first elementary cases arise when we set equal to zero one of the two coefficients on the RHS. If we set $r^2 = s^2$ we get $s = \pm r$. If $s = r$ then $A_0 = r$, $A_1 = r$, $A_2 = 5r$, $A_3 = 29r$, $\ldots$ so we can conclude $A_n = rB_{n-1}$. If $s = -r$ then $A_0 = r$, $A_1 = -r$, $A_2 = -7r$, $A_3 = -41r$, $\ldots$ so we can conclude $A_n = -r\frac{C_{n-1}}{2}$.
If we set $6s^2-2rs = 0$ we obtain $s = 0$ or $3s = r$. If $s = 0$ then $A_0 = r$, $A_1 = 0$, $A_2 = -r$, $A_3 = -6r$, $\ldots$ so we can conclude $A_n = -rT_{n-1}$. If

$3s = r$, setting $r = 3k$ we have $A_0 = 3k$, $A_1 = k$, $A_2 = 3k$, $A_3 = 17k$, ... so that $A_n = k\frac{L_{n-1}}{2}$.
Now let $a_0 = 2rs - 6s^2$, $a_1 = r^2 - s^2$ and define the recurrence

$$a_n = 6a_{n-1} - a_{n-2}, \qquad n \geq 2.$$

Then

$$\begin{aligned}
32A_n^2 + 2(r^2 + s^2 - 6rs) &= a_1 L_{2n-2} - a_0 L_{2n-1} \\
&= a_1(6L_{2n-1} - L_{2n}) - a_0 L_{2n-1} \\
&= (6a_1 - a_0)L_{2n-1} - a_1 L_{2n} \\
&= a_2 L_{2n-1} - a_1 L_{2n} \\
&= a_2(6L_{2n} - L_{2n+1}) - a_1 L_{2n} \\
&= (6a_2 - a_1)L_{2n} - a_2 L_{2n+1} \\
&= a_3 L_{2n} - a_2 L_{2n+1} \\
&= \cdots \quad \cdots \quad \cdots
\end{aligned}$$

So in general we have

$$32A_n^2 + 2(r^2 + s^2 - 6rs) = a_{m+3} L_{2n+m} - a_{m+2} L_{2n+m+1}, \quad m = -2, -1, 0, 1...$$

Our problem is solved if for some integer h

$$a_{m+2} = -6h = -T_2 h, \qquad a_{m+3} = -h = -T_1 h,$$

since then

$$\begin{aligned}
32A_n^2 + 2(r^2 + s^2 - 6rs) &= -hL_{2n+m} + 6hL_{2n+m+1} \\
&= hL_{2n+m+2}.
\end{aligned}$$

Now

$$\begin{aligned}
a_{m+1} &= 6a_{m+2} - a_{m+3} \\
&= -6T_2 h + T_1 h \\
&= -T_3 h,
\end{aligned}$$

$$\begin{aligned}
a_m &= 6a_{m+1} - a_{m+2} \\
&= -6T_3 h + T_2 h \\
&= -T_4 h,
\end{aligned}$$

and so on. Then the conditions are

$$a_0 = -T_{m+4}h, \qquad a_1 = -T_{m+3}h.$$

For sake of simplicity, write $T_{m+4} = t_1$, $T_{m+3} = t_0$. So we have

$$2rs - 6s^2 + t_1 h = 0, \tag{8}$$

$$r^2 - s^2 + t_0 h = 0. \tag{9}$$

Solving for r Equation 8 and solving for h Equation 9 after insertion of the value for r we get

$$h = \frac{2(-s^2 t_0 + 3s^2 t_1 \pm s^2)}{t_1^2}.$$

In obtaining this result we used the identity

$$T_n^2 - 6T_n T_{n+1} + T_{n+1}^2 = 1.$$

This can be proved in the following way:

$$T_n^2 = \frac{\alpha^{2n} + \beta^{2n} - 2}{32}, \quad T_{n+1}^2 = \frac{\alpha^{2n+2} + \beta^{2n+2} - 2}{32},$$

$$T_n T_{n+1} = \frac{\alpha^{2n+1} + \beta^{2n+1} - 6}{32}.$$

Then

$$T_n^2 - 6T_n T_{n+1} + T_{n+1}^2 = \frac{1}{32}(L_{2n} + L_{2n+2} - 6L_{2n+1} + 32),$$

but

$$L_{2n} + L_{2n+2} - 6L_{2n+1} = 0,$$

from which the desired identity.
If in h we take the plus sign and insert into the expression for r we get

$$r = \frac{s(t_0 - 1)}{t_1}. \tag{10}$$

On the other hand if we take the minus sign we obtain

$$r = \frac{s(t_0 + 1)}{t_1}. \tag{11}$$

Next we will prove the following identities

$$T_{2n+1}B_{n-1} = (1 + T_{2n})B_n, \tag{12}$$

$$T_{2n}L_{n-1} = (1 + T_{2n-1})L_n. \tag{13}$$

Identity 12:

$$\begin{aligned}
T_{2n+1}B_{n-1} &= \frac{\alpha^{2n+1} - \beta^{2n+1}}{4\sqrt{2}} \frac{\alpha^{n-\frac{1}{2}} + \beta^{n-\frac{1}{2}}}{2\sqrt{2}} \\
&= \frac{\alpha^{3n+\frac{1}{2}} + \alpha^{2n+1}\beta^{n-\frac{1}{2}} - \alpha^{n-\frac{1}{2}}\beta^{2n+1} - \beta^{3n+\frac{1}{2}}}{16} \\
&= \frac{\alpha^{3n+\frac{1}{2}} + \alpha^{n+1+\frac{1}{2}} - \beta^{n+1+\frac{1}{2}} - \beta^{3n+\frac{1}{2}}}{16},
\end{aligned}$$

where we used several times the fact that $\alpha\beta = 1$.

$$\begin{aligned}
B_n(1 + T_{2n}) &= \frac{\alpha^{n+\frac{1}{2}} + \beta^{n+\frac{1}{2}}}{2\sqrt{2}} + \frac{\alpha^{3n+\frac{1}{2}} - \alpha^{n+\frac{1}{2}}\beta^{2n} + \alpha^{2n}\beta^{n+\frac{1}{2}} - \beta^{3n+\frac{1}{2}}}{16} \\
&= \frac{4\sqrt{2}\alpha^{n+\frac{1}{2}} + 4\sqrt{2}\beta^{n+\frac{1}{2}}}{16} + \\
&\qquad + \frac{\alpha^{3n+\frac{1}{2}} - \alpha^{n+\frac{1}{2}}\beta^{2n} + \alpha^{2n}\beta^{n+\frac{1}{2}} - \beta^{3n+\frac{1}{2}}}{16} \\
&= \frac{\alpha^{n+1+\frac{1}{2}} - \alpha^{n+\frac{1}{2}}\beta + \alpha\beta^{n+\frac{1}{2}} - \beta^{n+1+\frac{1}{2}}}{16} + \\
&\qquad + \frac{\alpha^{3n+\frac{1}{2}} - \alpha^{n+\frac{1}{2}}\beta^{2n} + \alpha^{2n}\beta^{n+\frac{1}{2}} - \beta^{3n+\frac{1}{2}}}{16} \\
&= \frac{\alpha^{3n+\frac{1}{2}} + \alpha^{n+1+\frac{1}{2}} - \beta^{n+1+\frac{1}{2}} - \beta^{3n+\frac{1}{2}}}{16},
\end{aligned}$$

where we used again $\alpha\beta = 1$ and $\alpha - \beta - 4\sqrt{2}$. Hence

$$T_{2n+1}B_{n-1} = (1 + T_{2n})B_n = \frac{C_{3n} + C_{n+1}}{16}.$$

Identity 13:

$$\begin{aligned}
T_{2n}L_{n-1} &= \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}(\alpha^{n-1} + \beta^{n-1}) \\
&= \frac{\alpha^{3n-1} + \alpha^{2n}\beta^{n-1} - \alpha^{n-1}\beta^{2n} - \beta^{3n-1}}{4\sqrt{2}} \\
&= \frac{\alpha^{3n-1} + \alpha^{n+1} - \beta^{n+1} - \beta^{3n-1}}{4\sqrt{2}}.
\end{aligned}$$

$$\begin{aligned}
(1+T_{2n-1})L_n &= \alpha^n+\beta^n+(\alpha^n+\beta^n)\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}\\
&= \frac{\alpha^{n+1}-\alpha^n\beta+\alpha\beta^n-\beta^{n+1}}{4\sqrt{2}}+\\
&\qquad +\frac{\alpha^{3n-1}-\alpha^n\beta^{2n-1}+\alpha^{2n-1}\beta^n-\beta^{3n-1}}{4\sqrt{2}}\\
&= \frac{\alpha^{3n-1}+\alpha^{n+1}-\beta^{n+1}-\beta^{3n-1}}{4\sqrt{2}}.
\end{aligned}$$

Hence

$$T_{2n}L_{n-1}=(1+T_{2n-1})L_n=T_{3n-1}+T_{n+1}.$$

Along these lines we can prove these two other identities

$$(T_{2n}-1)C_n=T_{2n+1}C_{n-1}=\frac{B_{3n}-B_{n+1}}{2}, \tag{14}$$

$$(T_{2n+1}-1)T_{n+1}=T_nT_{2n+2}=\frac{L_{3n+2}-L_{n+2}}{32}. \tag{15}$$

Going back to Equation 10 if $m=2k$ we get using Identity 15

$$\frac{s}{r}=\frac{T_{k+2}}{T_{k+1}},$$

from which

$$r=\mu T_{k+1}=A_0,\quad s=\mu T_{k+2}=A_1,$$

so that $A_n=\mu T_{n+k+1}$. If $m=2k+1$ we get using Identity 14

$$\frac{s}{r}=\frac{C_{k+2}}{C_{k+1}},$$

from which

$$r=\mu C_{k+1}=A_0,\quad s=\mu C_{k+2}=A_1,$$

so that $A_n=\mu C_{n+k+1}$. On the other hand in the case of Equation 11 if $m=2k$ we get using Identity 9

$$\frac{s}{r}=\frac{L_{k+2}}{L_{k+1}},$$

from which

$$r=\mu L_{k+1}=A_0,\quad s=\mu L_{k+2}=A_1,$$

so that $A_n = \mu L_{n+k+1}$. Finally if $m = 2k+1$ we get using Identity 8

$$\frac{s}{r} = \frac{B_{k+2}}{B_{k+1}},$$

from which

$$r = \mu B_{k+1} = A_0, \quad s = \mu B_{k+2} = A_1,$$

so that $A_n = \mu B_{n+k+1}$.
The conclusion is that any sequence we looked for is expressible as a constant times one of the four sequences L_n, T_n, B_n and C_n.

References

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