Sequences related to convergents to square root of rationals

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1 The Initial Results

This note has its source in [4], in a different setting. Consider the system of recurrences, with $a_0 = b_0 = 1$,

$$a_n = a_{n-1} + kb_{n-1}, \tag{1}$$

$$b_n = a_{n-1} + b_{n-1}. \tag{2}$$

Later on we will generalize and show that the ratio $\frac{a_n}{b_n}$ is related to convergents to the square root of rationals.
We postpone the proof by induction of the following formulas ("summation formulas")

$$a_{2n} = \sum_{i=0}^{n} \binom{2n+1}{2i+1} k^{n-i}, \tag{3}$$

$$a_{2n+1} = \sum_{i=0}^{n+1} \binom{2n+2}{2i} k^{n+1-i}, \tag{4}$$

$$b_{2n} = \sum_{i=0}^{n} \binom{2n+1}{2i} k^{n-i}, \tag{5}$$

$$b_{2n+1} = \sum_{i=0}^{n} \binom{2n+2}{2i+1} k^{n-i}. \tag{6}$$

Let $\alpha = k + 1 + 2\sqrt{k}$, $\beta = k + 1 - 2\sqrt{k}$. Note that $\alpha + \beta = 2(k+1)$, $\alpha\beta = (k-1)^2$, $\alpha - \beta = 4\sqrt{k}$. Using Identities 1.87 and 1.95 in [2] and Pascal's Identity (see [3]) we have the following closed form representations

$$a_{2n} = \frac{\alpha^n + \beta^n}{2} + \sqrt{k}\frac{\alpha^n - \beta^n}{2}, \tag{7}$$

$$a_{2n+1} = \frac{\alpha^{n+1} + \beta^{n+1}}{2}, \tag{8}$$

$$b_{2n} = \frac{\alpha^n + \beta^n}{2} + \frac{\alpha^n - \beta^n}{2\sqrt{k}}, \tag{9}$$

$$b_{2n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{2\sqrt{k}}. \tag{10}$$

Let $w(r,\, s)$ denote the recurrence

$$w_n = 2(k+1)w_{n-1} - (k-1)^2 w_{n-2}, \quad w_0 = r,\, w_1 = s.$$

Furthermore let $d_n = w(1,\, k+1)$, $u_n = w(0,\, 2k)$, $v_n = w(0,\, 2)$. The closed forms are

$$d_n = \frac{\alpha^n + \beta^n}{2},$$

$$u_n = \frac{(\alpha - \beta)(\alpha^n - \beta^n)}{8},$$

$$v_n = 2\frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Note that

$$kv_n = u_n. \tag{11}$$

Then

$$a_{2n} = d_n + u_n, \tag{12}$$

$$a_{2n+1} = d_{n+1}, \tag{13}$$

$$b_{2n} = d_n + v_n, \tag{14}$$

$$b_{2n+1} = v_{n+1}. \tag{15}$$

It follows that $a_{2n} = w(1,\, 3k+1)$, $a_{2n+1} = w(k+1,\, k^2 + 6k + 1)$, $b_{2n} = w(1,\, k+3)$, $b_{2n+1} = w(2,\, 4k+4)$. Finally

$$a_n = 2(k+1)a_{n-2} - (k-1)^2 a_{n-4}$$

$$a_0 = 1,\, a_1 = k+1,\, a_2 = 3k+1,\, a_3 = k^2+6k+1,$$

$$b_n = 2(k+1)b_{n-2} - (k-1)^2 b_{n-4}$$

$$b_0 = 1,\, b_1 = 2,\, b_2 = k+3,\, b_3 = 4k+4.$$

These fourth order recurrences can be transformed in second order recurrences in the following way. First of all note that

$$1 - 2(k+1)x^2 + (k-1)^2x^4 = \left(1 - 2x - (k-1)x^2\right)\left(1 + 2x - (k-1)x^2\right).$$

The generating function of a_n is

$$\frac{1 + (k+1)x + (k-1)x^2 - (k-1)^2x^3}{1 - 2(k+1)x^2 + (k-1)^2x^4}.$$

This can be simplified to

$$\frac{1 + (k-1)x}{1 - 2x - (k-1)x^2},$$

so that we obtain

$$a_n = 2a_{n-1} + (k-1)a_{n-2}, \quad a_0 = 1,\, a_1 = k+1.$$

Analogously, the generating function of b_n is

$$\frac{1 + 2x - (k-1)x^2}{1 - 2(k+1)x^2 + (k-1)^2x^4},$$

and this becomes

$$\frac{1}{1 - 2x - (k-1)x^2},$$

so that we obtain

$$b_n = 2b_{n-1} + (k-1)b_{n-2}, \quad b_0 = 1,\, b_1 = 2.$$

Writing $\epsilon = 1 + \sqrt{k}$, $\eta = 1 - \sqrt{k}$ (so that $\epsilon = \alpha^{\frac{1}{2}}$, $\eta = \beta^{\frac{1}{2}}$) the closed forms are

$$a_n = \frac{\epsilon^{n+1} + \eta^{n+1}}{2},$$

$$b_n = \frac{\epsilon^{n+1} - \eta^{n+1}}{2\sqrt{k}}.$$

Now it is easy to show that $\lim_{n \longrightarrow \infty} \frac{a_n}{b_n} = \sqrt{k}$. Indeed

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{\epsilon^{n+1} + \eta^{n+1}}{2} \cdot \frac{2\sqrt{k}}{\epsilon^{n+1} - \eta^{n+1}} \\ &= \sqrt{k}\frac{\epsilon^{n+1} + \eta^{n+1}}{\epsilon^{n+1} - \eta^{n+1}} \\ &= \sqrt{k}\frac{1 + \left(\frac{\eta}{\epsilon}\right)^{n+1}}{1 - \left(\frac{\eta}{\epsilon}\right)^{n+1}}. \end{aligned}$$

Since $\left(\frac{\eta}{\epsilon}\right)^{n+1}$ converges to zero we have that $\frac{a_n}{b_n}$ converges to $\sqrt{k}$.

Now we are ready to prove by induction the summation formulas. Assume that Equation 3, Equation 4, Equation 5 and Equation 6 hold for some integer n. This is tantamount to say that Equation 12, Equation 13, Equation 14 and Equation 15 hold for some integer n. For $n = 0$ they are satisfied since

$$a_0 = 1 = \binom{1}{1} k^0 = d_0 + u_0,$$

$$a_1 = k + 1 = \sum_{i=0}^{1} \binom{2}{2i} k^{1-i} = d_1,$$

$$b_0 = 1 = \binom{1}{1} k^0 = d_0 + v_0,$$

$$b_1 = 2 = \binom{2}{1} k^0 = v_1.$$

Now

$$\begin{aligned} a_{2(n+1)} &= a_{2n+2} \\ &= a_{2n+1} + kb_{2n+1} \\ &= d_{n+1} + kv_{n+1} \\ &= d_{n+1} + u_{n+1}, \end{aligned}$$

where the third line is due to the induction hypothesis and we used Equation 11. Then Equation 3 is satisfied through Equation 12.

$$\begin{aligned} b_{2(n+1)} &= b_{2n+2} \\ &= a_{2n+1} + b_{2n+1} \\ &= d_{n+1} + v_{n+1}. \end{aligned}$$

Again the third line is due to the induction hypothesis. Then Equation 5 is satisfied through Equation 14.

$$\begin{aligned}
a_{2(n+1)+1} &= a_{2n+3} \\
&= a_{2n+2} + kb_{2n+2} \\
&= d_{n+1} + u_{n+1} + kd_{n+1} + kv_{n+1} \\
&= (1+k)d_{n+1} + 2u_{n+1} \\
&= (1+k)\frac{\alpha^{n+1} + \beta^{n+1}}{2} + \frac{(\alpha^{n+1} - \beta^{n+1}(\alpha - \beta)}{4} \\
&= \frac{(\alpha+\beta)(\alpha^{n+1} + \beta^{n+1})}{2} + \frac{(\alpha^{n+1} - \beta^{n+1}(\alpha - \beta)}{4} \\
&= \frac{2\alpha^{n+2} + 2\beta^{n+2}}{4} \\
&= d_{n+2}.
\end{aligned}$$

Here the third line is due to the fact that we already proved the formulas for a_{2n} and b_{2n}. Then Equation 4 is satisfied through Equation 13. For the last relationship consider first of all that

$$u_n + v_n = \frac{(\alpha^n - \beta^n)(\alpha + \beta)}{\alpha - \beta}.$$

Then

$$\begin{aligned}
b_{2(n+1)+1} &= b_{2n+3} \\
&= a_{2n+2} + b_{2n+2} \\
&= 2d_{n+1} + u_{n+1} + v_{n+1} \\
&= \alpha^{n+1} + \beta^{n+1} + \frac{(\alpha^{n+1} - \beta^{n+1})(\alpha + \beta)}{\alpha - \beta} \\
&= \frac{(\alpha - \beta)(\alpha^{n+1} + \beta^{n+1}) + (\alpha^{n+1} - \beta^{n+1})(\alpha + \beta)}{\alpha - \beta} \\
&= 2\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} \\
&= v_{n+2}.
\end{aligned}$$

Here again the third line is due to the formulas already proven. Then Equation 6 is satisfied through Equation 15.

2 Other Initial Conditions

If we start out with $a_0 = 0$, $b_0 = 1$ and we denote the resulting sequences by $\tilde{a}_n$ and $\tilde{b}_n$ we get

$$\tilde{a}_{2n} = \sum_{i=0}^{n-1} \binom{2n}{2i+1} k^{i+1} = \frac{\sqrt{k}}{2}(\alpha^n - \beta^n),$$

$$\tilde{a}_{2n+1} = \sum_{i=0}^{n} \binom{2n+1}{2i+1} k^{i+1} = \frac{\sqrt{k}}{2}(\alpha^n - \beta^n) + \frac{k}{2}(\alpha^n + \beta^n),$$

$$\tilde{b}_{2n} = \sum_{i=0}^{n} \binom{2n}{2i} k^{i} = \frac{1}{2}(\alpha^n + \beta^n),$$

$$\tilde{b}_{2n+1} = \sum_{i=0}^{n} \binom{2n+1}{2i} k^{i} = \frac{\sqrt{k}}{2}(\alpha^n - \beta^n) + \frac{1}{2}(\alpha^n + \beta^n).$$

Then

$$\tilde{a}_n = 2\tilde{a}_{n-1} + (k-1)\tilde{a}_{n-2}, \quad \tilde{a}_0 = 0,\ \tilde{a}_1 = k,$$
$$\tilde{b}_n = 2\tilde{b}_{n-1} + (k-1)\tilde{b}_{n-2}, \quad \tilde{b}_0 = 1,\ \tilde{b}_1 = 1.$$

It follows easily $(n > 0)$

$$\tilde{a}_n = kb_{n-1},$$
$$\tilde{b}_n = a_{n-1},$$

so that $\frac{\tilde{a}_n}{\tilde{b}_n} \longrightarrow \sqrt{k}$.

3 Generalization

The work done allows to analyze easily the following situation. Let us consider the sequences

$$u_n = u_{n-1} + kv_{n-1},$$
$$v_n = hu_{n-1} + v_{n-1},$$

with k and h positive integers, where we start with $u_0 = 1$, $v_0 = 0$. Then we get the following summation formulas and closed forms (where $\alpha_1 = 1 + hk + 2\sqrt{hk}$, $\beta_1 = 1 + hk - 2\sqrt{hk}$)

$$u_{2n} = \sum_{i=0}^{n} \binom{2n}{2i} (hk)^i = \frac{1}{2}(\alpha_1^n + \beta_1^n),$$

$$u_{2n+1} = \sum_{i=0}^{n} \binom{2n+1}{2i} (hk)^i = \frac{\sqrt{hk}}{2}(\alpha_1^n - \beta_1^n) + \frac{1}{2}(\alpha_1^n + \beta_1^n),$$

$$v_{2n} = h \sum_{i=0}^{n-1} \binom{2n}{2i+1} (hk)^i = \frac{\sqrt{h}}{2\sqrt{k}}(\alpha_1^n - \beta_1^n),$$

$$v_{2n+1} = h \sum_{i=0}^{n} \binom{2n+1}{2i+1} (hk)^i = \frac{\sqrt{h}}{2\sqrt{k}}(\alpha_1^n - \beta_1^n) + \frac{h}{2}(\alpha_1^n + \beta_1^n).$$

From this, using the same approach as before, we obtain

$$u_n = 2u_{n-1} + (hk-1)u_{n-2}, \quad u_0 = 1,\ u_1 = 1,$$

$$v_n = 2v_{n-1} + (hk-1)v_{n-2}, \quad v_0 = 0,\ v_1 = h.$$

Using the closed forms given before we see that

$$\frac{u_{2n}}{v_{2n}} \longrightarrow \sqrt{\frac{k}{h}}, \quad \frac{u_{2n+1}}{v_{2n+1}} \longrightarrow \sqrt{\frac{k}{h}},$$

so that we can conclude

$$\frac{u_n}{v_n} \longrightarrow \sqrt{\frac{k}{h}}.$$

4 Reduction

Returning to the initial case, if k is an odd number the fraction $\frac{a_n}{b_n}$ can be reduced and new sequences can be defined. More precisely, let $k = 2m+1$, $m = 0, 1, 2, \ldots$. Then

$$\begin{aligned} \alpha &= 2m+1+1+2\sqrt{k} \\ &= 2(m+1+\sqrt{k}) \\ &= 2\gamma, \end{aligned}$$

with $\gamma = m+1+\sqrt{k}$. Analogously, with $\delta = m+1-\sqrt{k}$, we have

$$\beta = 2\delta.$$

Note that $\gamma + \delta = 2(m+1)$, $\gamma\delta = m^2$, $\gamma - \delta = 2\sqrt{k} = 2\sqrt{2m+1}$. Now define

$$c_{2n} = \left(\frac{1}{2}\right)^n a_{2n} = \frac{\gamma^n + \delta^n}{2} + \sqrt{k}\frac{\gamma^n - \delta^n}{2}, \tag{16}$$

$$c_{2n+1} = \left(\frac{1}{2}\right)^n a_{2n+1} = \frac{\gamma^{n+1} + \delta^{n+1}}{2}, \tag{17}$$

$$d_{2n} = \left(\frac{1}{2}\right)^n b_{2n} = \frac{\gamma^n + \delta^n}{2} + \frac{\gamma^n - \delta^n}{2\sqrt{k}}, \tag{18}$$

$$d_{2n+1} = \left(\frac{1}{2}\right)^n b_{2n+1} = \frac{\gamma^{n+1} - \delta^{n+1}}{2\sqrt{k}}. \tag{19}$$

Then of course

$$\frac{c_n}{d_n} \longrightarrow \sqrt{k}.$$

Let $u(r,\, s)$ denote the recurrence

$$u_n = 2(m+1)u_{n-1} - m^2 u_{n-2}, \quad u_0 = r,\, u_1 = s.$$

Then $c_{2n} = u(1,\, 3m+2)$, $c_{2n+1} = u(m+1,\, m^2+4m+2)$, $d_{2n} = u(1,\, m+2)$, $d_{2n+1} = u(1,\, 2(m+1))$. And finally

$$c_n = 2(m+1)c_{n-2} - m^2 c_{n-4},$$

$$d_n = 2(m+1)d_{n-2} - m^2 d_{n-4},$$

where the initial conditions are determined by the previous recurrences.
The generating function of c_n is

$$\frac{1 + (1+m)x + mx^2 - m^2x^3}{1 - 2(1+m)x^2 + m^2x^4},$$

that of d_n is

$$\frac{1 + x - mx^2}{1 - 2(1+m)x^2 + m^2x^4}.$$

5 Some Identities

We will work with the sequences

$$a_n = 2a_{n-1} + (k-1)a_{n-2}, \quad a_0 = 1,\, a_1 = k+1,$$

and

$$b_n = 2b_{n-1} + (k-1)b_{n-2}, \quad b_0 = 1,\, b_1 = 2,$$

with closed forms

$$a_n = \frac{\epsilon^{n+1} + \eta^{n+1}}{2},$$

$$b_n = \frac{\epsilon^{n+1} - \eta^{n+1}}{2\sqrt{k}}.$$

where $\epsilon = 1 + \sqrt{k}$, $\eta = 1 - \sqrt{k}$. Using induction and Equations 1 and 2 we get

$$a_n = (k-1)b_{n-1} + b_n,$$

$$\frac{a_{n+1} + (k-1)a_{n-1}}{2k} = b_n,$$

$$kb_n = a_n + (k-1)a_{n-1}.$$

Using the closed forms (and the fact that $\epsilon\eta = 1 - k$) we get

$$a_n^2 - kb_n^2 = (1-k)^{n+1},$$

$$a_{2n} = 2a_{n-1}a_n - (1-k)^n,$$

$$(k-1)b_{m-1}b_n + b_m b_{n+1} = b_{m+n+1},$$

$$(k-1)a_{m-1}a_n + a_m a_{n+1} = kb_{m+n+1}.$$

Writing $n = m - 1$ the last two become

$$(k-1)b_{m-1}^2 + b_m^2 = b_{2m},$$

$$(k-1)a_{m-1}^2 + a_m^2 = kb_{2m}.$$

Using these identities we obtain

$$a_{2n} = \frac{2ka_n}{k-1}\sqrt{\frac{a_n^2 - (1-k)^{n+1}}{k}} - \frac{2a_n^2}{k-1} - (1-k)^n, \tag{20}$$

$$b_{2n} = \frac{(1-k)^{n+1} + 2kb_n^2 - 2b_n\sqrt{kb_n^2 + (1-k)^{n+1}}}{k-1}. \tag{21}$$

Now noting that $a_{2^{n+1}} = a_{2\cdot 2^n}$, writing 2^n instead of n we get recurrences for a_{2^n} and b_{2^n}

$$a_{2^{n+1}} = \frac{2ka_{2^n}}{k-1}\sqrt{\frac{a_{2^n}^2 - (1-k)^{2^n+1}}{k}} - \frac{2a_{2^n}^2}{k-1} - (1-k)^{2^n},$$

$$b_{2^{n+1}} = \frac{(1-k)^{2^n+1} + 2kb_{2^n}^2 - 2b_{2^n}\sqrt{kb_{2^n}^2 + (1-k)^{2^n+1}}}{k-1}.$$

6 Newton's Iteration

The Newton's iteration algorithm (see [8]) to approximate the square root of integers is given by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{k}{x_n}\right),$$

starting with $x_0 = 1$. We assume $k > 1$. The limit of this recursion is the fixed point of the mapping

$$f(x) = \frac{1}{2}\left(x + \frac{k}{x}\right),$$

which is $\sqrt{k}$.
For $n \geq 0$ let us write

$$x_n = \frac{a_n}{b_n}.$$

We extend the sequences a_n and b_n setting $a_0 = b_0 = 1$. Then we have

$$a_n = a_{n-1}^2 + kb_{n-1}^2, \tag{22}$$

$$b_n = 2a_{n-1}b_{n-1}. \tag{23}$$

We can generalize considering the recursion

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{k}{hx_n}\right),$$

where h, k are positive integers with $h \neq k$. This produces an approximation to $\sqrt{\frac{k}{h}}$. In this case we have

$$a_n = ha_{n-1}^2 + kb_{n-1}^2,$$

$$b_n = 2ha_{n-1}b_{n-1}.$$

We are going to prove using induction that, for $n \geq 2$,

$$a_n = 2a_{n-1}^2 - (k-1)^{2^{n-1}}. \tag{24}$$

Using Equation 22 and Equation 23 we have $a_1 = k+1$, $b_1 = 2$, $a_2 = (k+1)^2+4k = 1+k^2+6k$. If we use Equation 24 we have $a_2 = 2a_1^2-(k-1)^2 = 1+k^2+6k$, so that Equation 24 is true for $n = 2$. Now assume that it holds for some n. Coupled with Equation 22 we get

$$2a_{n-1}^2 - w_{n-1} = a_{n-1}^2 + kb_{n-1}^2,$$

that is

$$a_{n-1}^2 = kb_{n-1}^2 + w_{n-1}, \tag{25}$$

where we wrote

$$w_{n-1} = (k-1)^{2^{n-1}}.$$

Now

$$\begin{aligned} a_{n+1} &= a_n^2 + kb_n^2 \\ &= (2a_{n-1}^2 - w_{n-1})^2 + 4ka_{n-1}^2 b_{n-1}^2 \\ &= 4a_{n-1}^4 - 4a_{n-1}^2 w_{n-1} + w_{n-1}^2 + 4ka_{n-1}^2 b_{n-1}^2. \end{aligned}$$

On the other hand

$$\begin{aligned} 2a_n^2 - w_n &= 2(a_{n-1}^2 + kb_{n-1}^2)^2 - w_n \\ &= 2a_{n-1}^4 + 2k^2 b_{n-1}^4 + 4ka_{n-1}^2 b_{n-1}^2 - w_n. \end{aligned}$$

Now using Equation 25 we have

$$\begin{aligned} k^2 b_{n-1}^4 &= (a_{n-1}^2 - w_{n-1})^2 \\ &= a_{n-1}^4 + w_{n-1}^2 - 2a_{n-1}^2 w_{n-1}. \end{aligned}$$

Then

$$\begin{aligned} 2a_n^2 - w_n &= 4a_{n-1}^4 - 4a_{n-1}^2 w_{n-1} + 2w_{n-1}^2 + 4ka_{n-1}^2 b_{n-1}^2 - w_n \\ &= a_{n+1}, \end{aligned}$$

since $w_n = w_{n-1}^2$. This concludes the proof.
Using Equation 25 we get

$$a_{n-1} = \sqrt{kb_{n-1}^2 + w_{n-1}},$$

so that, through Equation 23, we get a recurrence for b_n

$$b_n = 2b_{n-1}\sqrt{kb_{n-1}^2 + w_{n-1}}. \tag{26}$$

Now we are going to prove, again by induction, the following closed forms

$$a_n = \frac{\alpha^{2^n} + \beta^{2^n}}{2}, \tag{27}$$

$$b_n = \frac{\alpha^{2^n} - \beta^{2^n}}{2\sqrt{k}}, \tag{28}$$

where $\alpha = 1 + \sqrt{k}$, $\beta = 1 - \sqrt{k}$. Note that $\alpha + \beta = 2$, $\alpha\beta = 1 - k$. Hence a_n and b_n are doubly exponential sequences (see [1]).
For $n = 0$ the closed form for a_0 gives $\frac{\alpha+\beta}{2} = 1$; for b_0 gives $\frac{\alpha-\beta}{2\sqrt{k}} = 1$ so Equation 27 and Equation 28 are satisfied. Now assume that Equation 27 and Equation 28 hold for some n. Then

$$\begin{aligned} a_{n+1} &= a_n^2 + kb_n^2 \\ &= \left(\frac{\alpha^{2^n} + \beta^{2^n}}{2}\right)^2 + k\left(\frac{\alpha^{2^n} - \beta^{2^n}}{2\sqrt{k}}\right)^2 \\ &= \frac{\alpha^{2^{n+1}} + \beta^{2^{n+1}} + 2(\alpha\beta)^{2^n}}{4} + k\frac{\alpha^{2^{n+1}} + \beta^{2^{n+1}} - 2(\alpha\beta)^{2^n}}{4k} \\ &= \frac{\alpha^{2^{n+1}} + \beta^{2^{n+1}}}{2}. \end{aligned}$$

This concludes the proof for a_n.
For b_n we have

$$\begin{aligned} b_{n+1} &= 2a_nb_n \\ &= 2\frac{\alpha^{2^n} + \beta^{2^n}}{2}\frac{\alpha^{2^n} - \beta^{2^n}}{2\sqrt{k}} \\ &= \frac{\alpha^{2^{n+1}} - \beta^{2^{n+1}}}{2\sqrt{k}}. \end{aligned}$$

This concludes the proof.
Incidentally we have proved, for $n \geq 1$, that $kb_n^2 + w_n$ is a perfect square. With $k = 2$, a_n is sequence A001601 and b_n is sequence A051009 in [6].

Using Identities 1.87 and 1.95 in [2], where n is replaced by 2^n, we obtain the following summation formulas

$$a_n = \sum_{r=0}^{2^{n-1}} \binom{2^n}{2r} k^r, \tag{29}$$

$$b_n = \sum_{r=0}^{2^{n-1}-1} \binom{2^n}{2r+1} k^r, \quad n > 0. \tag{30}$$

From Equation 23 we see that

$$b_n = 2^n \prod_{r=0}^{n-1} a_r,$$

which implies that b_n is divisible by 2^n.

7 Related Sequences

Let us consider the sequence c_n, with $c_0 = 1$, $c_1 = r$, $r > 1$, such that for $n \geq 2$

$$c_n = 2c_{n-1}^2 - 1.$$

From [5] we get the closed form

$$c_{n+1} = \frac{\gamma^{2^n} + \delta^{2^n}}{2},$$

where $\gamma = r + \sqrt{r^2-1}$, $\delta = r - \sqrt{r^2-1}$. Define the sequence d_n by

$$d_{n+1} = \frac{\gamma^{2^n} - \delta^{2^n}}{2\sqrt{r^2-1}}.$$

We set $d_0 = 1$ and we obtain $d_1 = 1$. Note that $d_2 = 2r$. Then, for $n \geq 1$,

$$\begin{aligned} d_{n+1} &= 2\frac{\gamma^{2^{n-1}} + \delta^{2^{n-1}}}{2}\frac{\gamma^{2^{n-1}} - \delta^{2^{n-1}}}{2\sqrt{r^2-1}} \\ &= 2c_n d_n. \end{aligned}$$

Then

$$d_n = 2^{n-1}\prod_{i=1}^{n-1} c_i.$$

From this we can evaluate

$$\lim_{n\longrightarrow\infty}\prod_{i=1}^{n}\left(1+\frac{1}{c_i}\right).$$

Indeed we have

$$\begin{aligned} \prod_{i=1}^{n}\left(1+\frac{1}{c_i}\right) &= \left(1+\frac{1}{c_1}\right)\left(1+\frac{1}{c_2}\right)\cdots\left(1+\frac{1}{c_n}\right) \\ &= \frac{1+c_1}{c_1}\frac{1+c_2}{c_2}\cdots\frac{1+c_n}{c_n} \\ &= \frac{1+c_1}{c_1}\frac{2c_1^2}{c_2}\cdots\frac{2c_{n-1}^2}{c_n} \\ &= \frac{(1+c_1)2^{n-1}c_1c_2\cdots c_{n-1}}{c_n} \\ &= \frac{(r+1)d_n}{c_n}. \end{aligned}$$

Using the closed forms of c_n and d_n it is easy to see that

$$\lim_{n \longrightarrow \infty} \frac{c_n}{d_n} = \sqrt{r^2 - 1}.$$

Hence

$$\lim_{n \longrightarrow \infty} \prod_{i=1}^{n} \left(1 + \frac{1}{c_i}\right) = \frac{r+1}{\sqrt{r^2-1}} = \sqrt{\frac{r+1}{r-1}}.$$

The case $r = 3$ is considered in [7].

References

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