On a sequence related to the Josephus problem

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In this short note, we show that an integer sequence defined on the minimum of differences between divisor complements of its partial products is connected with the Josephus problem (q=3).

We prove the following theorem and, finally, state the relatedness of two constants.

Theorem. *Let a_n and b_n be recursively defined as*

$$a_0 = 4,\ a_n = \min\left(|d_j - p_n/d_j| > 1\right), \quad p_n = \prod_{k=0}^{n-1} a_k,$$
$$d_j \mid p_n, \quad 1 \le j \le \sigma(p_n).$$
$$b_1 = 1,\ b_n = \left\lceil \tfrac{1}{2} \sum_{k=1}^{n-1} b_k \right\rceil.$$

(1) *Then* $a_n = 2^{b_n}$, *for* $n > 2$.

The first terms of a_n and b_n are [S][Z]

$$a_{n\ge 0} = \{4, 3, 4, 2, 4, 8, 16, 64, \ldots\}, \quad b_{n\ge 1} = \{1, 1, 1, 2, 3, 4, 6, 9, 14 \ldots\}.$$

We need two lemmata.

Lemma 1. *For $k > 0$,*

(2) $$\sigma(3 \cdot 2^k) = 2k + 2.$$

Proof. This is true for $k = 1$, and the set of divisors of $3 \cdot 2^{k+1}$ is the set of divisors of $3 \cdot 2^k$ plus 2^{k+1} and $3 \cdot 2^{k+1}$ itself. □

Lemma 2. *Let $\delta(m)$ denote the smallest absolute value of the differences between complementary divisors of $m > 1$:*

$$\delta(m) = \min\left(\left|d_j - \frac{m}{d_j}\right|\right), \quad d_j|m, \quad 1 \le j \le \sigma(m).$$

Then

(3) $$\delta(3 \cdot 2^k) = 2^{\lceil k/2 \rceil}, \quad k > 0.$$

Proof. Let us sort the divisors of $3 \cdot 2^k$ by size and call these D_j:

$$D_{1\le j\le 2k+2} = \{1, (2, 3), \ldots, (2^i, \tfrac{3}{2}2^i), (2^{i+1}, \tfrac{3}{2}2^{i+1}), \ldots, (2^k, \tfrac{3}{2}2^k), 3 \cdot 2^k\}.$$

Any smallest complementary divisor difference must be the one where the divisors are in the exact middle of the sorted list, which, using (2), is $k + 1$. And so, $\delta(3 \cdot 2^k) = D_{k+2} - D_{k+1}$.

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Now, the proposition (3) is true for $k = 1$. For every increase of k by one, $\sigma(m)$ increases by two, and the index of the wanted pair of divisors increases by one, so $D_{k+2} - D_{k+1}$ goes through the values

$$\begin{aligned} \tfrac{3}{2}2^i - 2^i &= 2^{i-1} \\ 2^{i+1} - \tfrac{3}{2}2^i &= 2^{i-1} \\ \tfrac{3}{2}2^{i+1} - 2^{i+1} &= 2^i \\ 2^{i+2} - \tfrac{3}{2}2^{i+1} &= 2^i \end{aligned}$$

so it doubles every second step which is just the meaning of (3). □

Fixing the induction base at $\delta(p_3 = 48) = 2^1 = 2^{b_3}$ to make sure that $D_{k+2} - D_{k+1} > 1$, the main proposition (1) is now obvious, since the powers of two in a_n behave the same way under multiplication as unity does in b_n under addition.

Because the asymptotics of b_n are known[C], with

$$b_n = \left\lceil c \cdot \left(\tfrac{3}{2}\right)^n - \tfrac{1}{2} \right\rceil, \quad c = 0.36050455619661495910154466\ldots,$$

the investigation of $a_{n\geq 3} = 2^{b_n}$ is settled, except for the closed form for c. Reble already proved[R] that b_n is connected to the Josephus problem. Independently, our numerics show that

$$c = \frac{2}{9}K(3), \tag{4}$$

with $K(3)$ the universal constant in the same problem with $q = 3$, a constant already discussed ([OW][HH]), and whose closed form is still unknown.

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