ON THE HEIGHT OF THE SYLVESTER RESULTANT

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Abstract. Let n be a positive integer. We consider the Sylvester resultant of f and g, where f is a generic polynomial of degree 2 or 3 and g is a generic polynomial of degree n. If f is a quadratic polynomial, we find the resultant's height. If f is a cubic polynomial, we find tight asymptotics for the resultant's height.

1. Introduction

Let m and n be positive integers, f and g be generic univariate polynomials of degrees m and n respectively:

$$\begin{array}{rcl} f(x) & := & f_0 + f_1 x + \cdots + f_m x^m, \\ g(x) & := & g_0 + g_1 x + \cdots + g_n x^n. \end{array} \tag{1}$$

Here, f_i, g_j are new variables. The Sylvester resultant of f and g is the determinant of the following square matrix of order $m+n$:

$$\mathrm{Res}(f,g) := \det \begin{bmatrix} f_0 & & & & g_0 & & \\ f_1 & f_0 & & & g_1 & \ddots & \\ \vdots & \vdots & \ddots & & \vdots & \ddots & g_0 \\ f_m & f_{m-1} & & f_0 & g_{n-1} & & g_1 \\ & f_m & \ddots & \vdots & g_n & \ddots & \vdots \\ & & \ddots & f_{m-1} & & \ddots & g_{n-1} \\ & & & f_m & & & g_n \end{bmatrix}, \tag{2}$$

where the first n columns contain coefficients of f and the last m contain coefficients of g.

From the definition, it is very easy to see that $\mathrm{Res}(f,g)$ is a homogeneous polynomial in the variables f_i and g_j. Further $\mathrm{Res}(f,g)$ is homogeneous in each group of variables, having degree n in the f_i's, and m in the g_j's. It is not hard to see that the resultant is ω-homogeneous of "degree" mn, where $\omega = (0,1,\cdots,n,0,1,\cdots,m)$. This means that

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if a monomial $f_0^{\alpha_0}\cdots f_m^{\alpha_m}g_0^{\beta_0}\cdots g_n^{\beta_n}$ appears with nonzero coefficient in the expansion of $\mathrm{Res}(f,g)$ then $\sum_{i=1}^m i\alpha_i+\sum_{j=1}^n j\beta_j=mn$ (see [12, Theorem 6.1]).

Resultants are widely used as a tool for polynomial equation solving, this has sparked a lot interest in their computation (see e.g. [2, 3, 5]). The absolute height of a polynomial $g=\sum_\alpha c_\alpha U^\alpha\in\mathbb{C}[U_1,\cdots,U_p]$ is defined as $H(g):=\max\{|c_\alpha|,\,\alpha\in\mathbb{N}^p\}$. In this paper we will be concerned with the computation of the height of $\mathrm{Res}(f,g)$.

The sharpest upper bound for the height was given in [10, Theorem 1.1], where it is shown that $H\left(\mathrm{Res}(f,g)\right)\le(m+1)^n\,(n+1)^m$. Previous upper bounds were given in [1, 6, 7, 8, 11], for more general resultants which include $R(f,g)$.

However, up to now there have been no known exact expressions for $H(\mathrm{Res}(f,g))$, for any non-trivial cases. We only know the exact value of the coefficients of the resultant for extremal monomials with respect to a generic weight, and they are equal to ±1 (see [12, Corollary 3.1]).

The purpose of this paper is to give non-trivial estimates on the height of the resultant for polynomials f of low degree.

1.1. **Quadratic polynomials.** In the case $m=2$, we get an exact solution for the height of $\mathrm{Res}(f,g)$ in terms of an integer number A_n. To define A_n, first consider $p_n(z):=(n-2z+1)(n-2z+2)-z(n-z)$. It is easy to see that if $n\ge3$, then $p_n(0)>0$ and $p_n\left(\frac{n}{2}\right)<0$. As $p_n(z)$ is a quadratic polynomial in z, we define, for $n\ge3$, r_n as the unique root of $p_n(z)$ lying in $\left[0,\frac{n}{2}\right]$. Set $A_n:=\lfloor r_n\rfloor$, the floor of r_n. In Table 1 page 19, we have listed some values of A_n.

Theorem 1.1. *Let $n\ge3$. The coefficient of highest absolute value in the expansion of $\mathrm{Res}(f_0+f_1x+f_2x^2,g)$ is the coefficient corresponding to $g_0g_nf_0^{A_n}f_1^{n-2A_n}f_2^{A_n}$. Moreover,*

$$\begin{aligned}H\left(\mathrm{Res}(f_0+f_1x+f_2x^2,g)\right)&=H\left(\mathrm{Res}(f_0+f_1x+f_2x^2,g_0+g_nx^n)\right)\\&=n\frac{(n-A_n-1)!}{(n-2A_n)!A_n!}.\end{aligned}$$

Remark 1.2. *As $A_n<\frac{n}{2}$, it turns out that $(n-2A_n)\ge0$.*

Before we give the next result, we must introduce some notation.

Notation 1.3. *Let $\alpha(n)$ be a positive sequence. We say that a sequence $\beta(n)=\mathcal{O}(\alpha(n))$ if there exist positive constants c_1,c_2 and N such that for all $n>N$ we have $c_1\alpha(n)\le\beta(n)\le c_2\alpha(n)$*

Based on Theorem 1.1 we get

Corollary 1.4. *Let $\alpha \approx 1.6180$ be the positive root of $x^2 - x - 1$ and $\beta \approx 2.3644$ be the positive root of $4x^4 - 125$. Then*

$$H\left(\operatorname{Res}(f_0 + f_1 x + f_2 x^2, g)\right) = \frac{\beta}{\sqrt{n\pi}}\alpha^n - \mathcal{O}\left(\frac{\alpha^n}{n\sqrt{n}}\right)$$

1.2. **Cubic polynomials.** In the case $m = 3$, we get a tight bound for the height. In particular, we get the following:

Theorem 1.5. *Let $\beta \approx 8.13488$ be the real root of $x^3 - 18x^2 + 110x - 242$, and $\alpha \approx 1.83928$ be the real root of $x^3 - x^2 - x - 1$. Let $H(n) := H\left(\operatorname{Res}(f_0 + f_1 x + f_2 x^2 + f_3 x^3, g)\right)$ where g is of degree n. Then*

$$H(n) = \frac{\beta}{\pi n}\alpha^n - \mathcal{O}\left(\frac{\alpha^n}{n^2}\right) \tag{3}$$

1.3. **Organization of paper.** Section 2 gives a proof of Theorem 1.1 and Corollary 1.4. A proof of Theorem 1.5 is given in Section 3. Section 4 gives some conclusions, conjectures and list some open questions. Finally, Section 6 contains a number of different tables which are referred to throughout this paper.

2. Quadratic polynomials

Proof of Theorem 1.1. The proof will be made by induction on n. For this section, denote with $H(n)$ the height of the resultant of a degree-two generic polynomial f and a generic polynomial g of degree n.

For $n = 3$, an explicit computation shows that

- $A_3 = 1$,
- $H(3) = 3$, and this is the coefficient of $g_0 g_3 f_0 f_1 f_2$.

Suppose now $n > 3$. As the degree of $\operatorname{Res}(f, g)$ in the g_j's is 2, we will first consider two special cases:

- If we pick a term in the expansion of $\operatorname{Res}(f, g)$ which is not a multiple of g_0, this term will appear in the expansion of
$$\operatorname{Res}(f, g_n x^n + \cdots + g_1 x) = \pm f_0 \operatorname{Res}(f, g_n x^{n-1} + \cdots + g_1),$$
and by the inductive hypothesis, all the coefficients of this expansion are bounded by $H(n-1)$.
- If we pick a term in the expansion of $\operatorname{Res}(f, g)$ which is not a multiple of g_n, this term will appear in the expansion of
$$\operatorname{Res}(f, g) = \pm f_2 \operatorname{Res}(f, g_{n-1} x^{n-1} + \cdots + g_0),$$
and reasoning as in the previous case, all the coefficients in this case will be bounded by $H(n-1)$.

In order to conclude, we have to bound all the coefficients which appear in $\mathrm{Res}(f,g)$ which are coefficients to a monomial of the form $g_0 g_n f_0^a f_1^b f_2^c$, for some a, b and c, and compare this bound with $H(n-1)$.

Without loss of generality we compute $\mathrm{Res}(f_2x^2+f_1x+f_0, g_nx^n+g_0)$. Moreover, we can also set $g_n := f_2 := 1$. Let $f(x) = (x-x_1)(x-x_2)$. Then,

$$\begin{aligned} \mathrm{Res}(f,g) &= \pm({x_1}^n + g_0)({x_2}^n + g_0) \\ &= \pm\left((x_1x_2)^n + ({x_1}^n + {x_2}^n)g_0 + g_0\right). \end{aligned} \tag{4}$$

In order to write the right-hand side of (4) in terms of f_1, f_0, we apply the classical Girard formulas (see for instance [5, Chapter 4 F]):

$$ {x_1}^n + {x_2}^n = (-1)^n n \sum_{i_1+2i_0=n} (-1)^{2i_1+i_0} \frac{(i_1+i_0-1)!}{i_1!i_0!} {f_1}^{i_1} {f_0}^{i_0}. \tag{5}$$

So, we have to maximize $\frac{(i_1+i_0-1)!}{i_1!i_0!}$ subject to the condition $i_1 + 2i_0 = n$. Set $z := i_0$, then $i_1 = n - 2z$, and we have study the behaviour of the function

$$P(z) := \frac{(n-z-1)!}{(n-2z)!z!}, \text{ for } z = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

As

$$P(z) - P(z-1) = \frac{(n-z-1)!}{(n-2z+2)!z!} p_n(z),$$

and due to the fact that $p_n(z)$ is a quadratic equation having r_n as the unique root in the interval $[0, \frac{n}{2}]$, we have

- P is increasing for $z = 0, 1, \ldots, A_n$.
- P decreases for $z = A_n, A_n + 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

Hence, the maximum of P is attained when $z = A_n$, and $H(n) = nP(A_n)$ because of (4) and (5).

In order to conclude, we only have to prove that $H(n) > H(n-1)$. As $H(n-1) = (n-1)\frac{(n-A_{n-1}-2)!}{(n-1-2A_{n-1})!A_{n-1}!}$, and

$$H(n) \geq n \frac{(n-A_{n-1}-1)!}{(n-2A_{n-1})!A_{n-1}!}, \tag{6}$$

it is easy to check that the right-hand-side of (6) is bigger than $H(n-1)$ if and only if $n \geq 3$. □

From here, we can prove Corollary 1.4:

Proof of Corollary 1.4. By noticing that $r_n = \frac{6+5n-\sqrt{5n^2-4}}{10}$, we get

$$\lim_{n\to\infty} \frac{A_n}{n} = \frac{5-\sqrt{5}}{10}.$$

Thus for large n we get

$$\begin{aligned} n\frac{(n-A_n-1)!}{(n-2A_n)!A_n!} &= n\frac{\Gamma(n-A_n)}{\Gamma(n-2A_n+1)\Gamma(A_n+1)} \\ &= \frac{n\Gamma(n-A_n)}{(n-2A_n)A_n\Gamma(n-2A_n)\Gamma(A_n+1)} \\ &= \frac{n^2}{(n-2A_n)A_n} \times \frac{\Gamma(n-A_n)}{n\Gamma(n-2A_n)\Gamma(A_n+1)} \end{aligned}$$

From the comment above, we see that the first fraction will approach to $\frac{5(1+\sqrt{5})}{2}$. This then gives us

$$\begin{aligned} &\approx \frac{5(1+\sqrt{5})}{2}\frac{\Gamma(n/2+n\sqrt{5}/10)}{n\Gamma(n\sqrt{5}/5)\Gamma(n/2-n\sqrt{5}/10)} \\ &= \frac{\beta}{\sqrt{\pi n}}\alpha^n - \mathcal{O}\left(\frac{\alpha^n}{n^{3/2}}\right) \end{aligned}$$

which gives the desired result. The last line of this inequality was derived with the help of Maple.

Here we ignored a number of problems that occur with respect to errors in approximation. These are done in the same way that they are done for the proof of Theorem 3.7. □

3. Cubic polynomials

By an argument similar to Theorem 1.1, if $H(n) > H(n-1)$ then both g_n and g_0 must divide the terms which gives rise to $H(n)$. We will see that this holds for $n \gg 0$. We have then that three g_i's must divide each of the terms of $\mathrm{Res}(f,g)$ and two of them are known if $H(n) > H(n-1)$ (g_n and g_0). This gives rise to the definitions

Definition 3.1. *Define $H_l(m,k,k',m')$ to be the coefficient of $f_0^m f_1^k f_2^{k'} f_3^{m'} g_0 g_l g_n$ in $\mathrm{Res}(f,g)$.*

Definition 3.2. *Define*

$$H_l(n) = \max_{m+k+k'+m'=n} |H_l(m,k,k',m')|.$$

The main results of the paper will be derived by being able to write $H_l(m,k,k',m')$ in terms of some auxiliary functions $F(m,k,k',m')$ which are defined as follows:

Definition 3.3. *Define $F(m,k,k',m')$ to be the number of occurrences of $f_0^m f_1^k f_2^{k'} f_3^{m'}$ in the determinant of the matrix*

$$\begin{bmatrix} f_2 & f_1 & f_0 & & & & \\ f_3 & f_2 & f_1 & f_0 & & & \\ & f_3 & f_2 & f_1 & f_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & f_3 & f_2 & f_1 & f_0 \\ & & & & f_3 & f_2 & f_1 \\ & & & & & f_3 & f_2 \end{bmatrix}$$

of dimension $m+k+k'+m' \geq 1$. For $m+k+k'+m' = 1$ or 2 the determinant would be of the matrices $[f_2]$ and $\begin{bmatrix} f_2 & f_1 \\ f_3 & f_2 \end{bmatrix}$ respectively.

For convenience we define $F(0,0,0,0) = 1$.

For example for $m+k+k'+m' = 3$ we have

$$\det \begin{bmatrix} f_2 & f_1 & f_0 \\ f_3 & f_2 & f_1 \\ 0 & f_3 & f_2 \end{bmatrix} = f_2^3 - 2f_1f_2f_3 + f_0f_3^2$$

Thus we see that $F(1,0,0,2) = 1$, $F(0,1,1,1) = -2$ and $F(0,0,3,0) = 1$.

Lemma 3.4. *$F(m,k,k',m')$ satisfies the recurrence relation*

$$\begin{aligned} F(m,k,k',m') &= F(m,k,k'-1,m') - F(m,k-1,k',m'-1) \\ &\quad +F(m-1,k,k',m'-2) \end{aligned}$$

with $F(0,0,0,0) = 1$ and $F(m,k,k',m') = 0$ if any of m, k, k' or $m' < 0$

Proof. The recurrence follows by considering the three possibilities from the first row.

$$\begin{bmatrix} \boxed{f_2} & \cancel{f_1} & \cancel{f_0} & & & & \\ \cancel{f_3} & f_2 & f_1 & f_0 & & & \\ & f_3 & f_2 & f_1 & f_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & f_3 & f_2 & f_1 & f_0 \\ & & & & f_3 & f_2 & f_1 \\ & & & & & f_3 & f_2 \end{bmatrix}, \begin{bmatrix} \cancel{f_2} & \boxed{f_1} & \cancel{f_0} & & & & \\ \boxed{f_3} & \cancel{f_2} & \cancel{f_1} & \cancel{f_0} & & & \\ & \cancel{f_3} & f_2 & f_1 & f_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & f_3 & f_2 & f_1 & f_0 \\ & & & & f_3 & f_2 & f_1 \\ & & & & & f_3 & f_2 \end{bmatrix}$$

$$\begin{bmatrix} \cancel{f_2} & \cancel{f_1} & \boxed{f_0} & & & & & \\ \boxed{f_3} & \cancel{f_2} & \cancel{f_1} & \cancel{f_0} & & & & \\ & \boxed{f_3} & \cancel{f_2} & \cancel{f_1} & \cancel{f_0} & & & \\ & & \cancel{f_3} & f_2 & f_1 & f_0 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & f_3 & f_2 & f_1 & f_0 \\ & & & & & f_3 & f_2 & f_1 \\ & & & & & & f_3 & f_2 \end{bmatrix}.$$

□

By induction we will prove the following lemma, whose statement was first discovered experimentally via [9].

Lemma 3.5. *If $m' = 2m + k$, then:*

$$F(m, k, k', k + 2m) = (-1)^k \binom{m+k}{k} \binom{k'+k+m}{k+m} \tag{7}$$

If $m' \neq 2m + k$ then $F(m, k, k', m') = 0$.

Proof. By examining the recurrence relation, we see that $F(m, k, k', m') = 0$ if $m' \neq 2m + k$.

Equation (7) is true for $m+k+k'=1$ by some simple calculations. So we have that

$$\begin{aligned}
& F(m,k,k',k+2m) \\
= \; & F(m,k,k'-1,k+2m) - F(m,k-1,k',k+2m-1) \\
& +F(m-1,k,k',k+2m-2) \\
= \; & (-1)^k \binom{m+k}{k}\binom{k'-1+k+m}{k+m} \\
& -(-1)^{k-1}\binom{m+k-1}{k-1}\binom{k'+k-1+m}{k+m-1} \\
& +(-1)^k \binom{m+k-1}{k}\binom{k'+k+m-1}{k+m-1} \\
= \; & (-1)^k \left(\binom{m+k}{k}\binom{k'-1+k+m}{k+m}\right. \\
& \left.+\binom{k'+k-1+m}{k+m-1}\left(\binom{m+k-1}{k-1}+\binom{m+k-1}{k}\right)\right) \\
= \; & (-1)^k \left(\binom{m+k}{k}\left(\binom{k'-1+k+m}{k+m}+\binom{k'+k-1+m}{k+m-1}\right)\right) \\
= \; & (-1)^k \binom{m+k}{k}\binom{k'+k+m}{k+m}
\end{aligned}$$

and the result follows by induction. □

Theorem 3.6. *Let F be as in Definition 3.3. Then*

$$\begin{aligned}
H_0(m,k,k',m') \; &= \; F(m-1,k,k',m'-2) - F(m,k,k'-1,m') \\
& \quad +2F(m,k,k',m') \\
&= \; (-1)^k(3m+2k+k')\frac{(m+k+k'-1)!}{k!m!k'!}.
\end{aligned}$$

The value of $H_l(m,k,k',m')$ is given in Table 3 (page 20) for l from 0 to 5. We will provide only the proof for $H_0(m,k,k',m')$ here. The other cases listed in Table 3 are similar. Code which automates this process is available upon request.

For all l, we can also write $H_l(m,k,k',m')$ as a sum of various F. Instead of three cases, we tend to get six, depending on which column the g_0, the g_l and the g_n are taken from. In each of these cases we get a finite number of ways to account for the terms above the g_l term, and below the g_n term. The terms between the g_l and the g_n can be accounted for with F functions. So each of these finite number of ways will account for some $F(m-?,k-?,k'-?,m'-?)$ which will then be taken into the final sum.

Proof of Theorem 3.6. The second statement of the theorem follows directly from Lemma 3.5, so it suffices to prove the first statement.

We notice that there are three different ways in which we can get $g_0 g_0 g_n$ as a factor. We will do each case separately.

Case 1:

$$\begin{bmatrix}
\cancel{f_0} & & & & & & \boxed{g_0} & & \\
\cancel{f_1} & \cancel{f_0} & & & & & \cancel{g_1} & \boxed{g_0} & \\
f_2 & f_1 & f_0 & & & & g_2 & g_1 & g_0 \\
f_3 & f_2 & f_1 & \ddots & & & \vdots & \vdots & \vdots \\
 & f_3 & f_2 & \ddots & f_0 & & \vdots & \vdots & \vdots \\
 & & f_3 & \ddots & f_1 & f_0 & \vdots & \vdots & \vdots \\
 & & & \ddots & f_2 & f_1 & g_n & g_{n-1} & g_{n-2} \\
 & & & & f_3 & f_2 & & g_n & g_{n-1} \\
 & & & & & \cancel{f_3} & & & \boxed{g_n}
\end{bmatrix}$$

So we get that this case contributes $F(m, k, k', m')$.

Case 2:

$$\begin{bmatrix}
\cancel{f_0} & & & & & & & \boxed{g_0} & & \\
f_1 & f_0 & & & & & & g_1 & g_0 & \\
\cancel{f_2} & \cancel{f_1} & \cancel{f_0} & & & & & \cancel{g_2} & \cancel{g_1} & \boxed{g_0} \\
f_3 & f_2 & f_1 & f_0 & & & & g_3 & g_2 & g_1 \\
 & f_3 & f_2 & f_1 & \ddots & & & \vdots & \vdots & \vdots \\
 & & f_3 & f_2 & \ddots & f_0 & & \vdots & \vdots & \vdots \\
 & & & f_3 & \ddots & f_1 & f_0 & \vdots & \vdots & \vdots \\
 & & & & \ddots & f_2 & f_1 & g_n & g_{n-1} & g_{n-2} \\
 & & & & & \cancel{f_3} & \cancel{f_2} & & \boxed{g_n} & \cancel{g_{n-1}} \\
 & & & & & & f_3 & & & g_n
\end{bmatrix}$$

First notice that this must have a factor of f_3 from the last row. We see that there are two possibilities for the first column. Either it is f_1 or f_3. If it is f_1, then the remainder of the expression is given by $F(m, k-1, k', m'-1)$. If it is f_3, then we see that the second column must contain f_0. After this, the remainder of the expression is given by $-F(m-1, k, k', m'-2)$. Thus we see that this case will contribute

$$-1 \times (F(m, k-1, k', m'-1) - F(m-1, k, k', m'-2)).$$

Here the -1 in front comes from the sign of the matrix of the $g_0^2 g_n$.

Case 3:

$$\left[\begin{array}{cccccccccc}
f_0 & & & & & & & g_0 & & \\
f_1 & f_0 & & & & & & g_1 & \boxed{g_0} & \\
f_2 & f_1 & f_0 & & & & & g_2 & g_1 & \boxed{g_0} \\
f_3 & f_2 & f_1 & f_0 & & & & g_3 & g_2 & g_1 \\
 & f_3 & f_2 & f_1 & \ddots & & & \vdots & \vdots & \vdots \\
 & & f_3 & f_2 & \ddots & f_0 & & \vdots & \vdots & \vdots \\
 & & & f_3 & \ddots & f_1 & f_0 & \vdots & \vdots & \vdots \\
 & & & & \ddots & f_2 & f_1 & \boxed{g_n} & g_{n-1} & g_{n-2} \\
 & & & & & f_3 & f_2 & & g_n & g_{n-1} \\
 & & & & & & f_3 & & & g_n
\end{array}\right]$$

With a little work we see that this will contribute $F(m-1,k,k',m'-2)$.

This combines together to give that

$$\begin{aligned} H_0(m,k,k',m') &= F(m,k,k',m') - F(m,k-1,k',m'-1) \\ &\quad +2F(m-1,k,k',m'-2). \end{aligned}$$

By noticing that $F(m,k,k',m') = F(m-1,k,k',m'-2) - F(m,k-1,k',m'-1) + F(m,k,k'-1,m')$ we get

$$\begin{aligned} H_0(m,k,k',m') &= 2F(m,k,k',m') + F(m-1,k,k',m'-2) \\ &\quad -F(m,k,k'-1,m'). \end{aligned}$$

which is the desired result. □

From here we can prove on of the main results which will help us to prove Theorem 1.5.

Theorem 3.7. *Let* $\beta \approx 8.13488$ *be the real root of* $x^3 - 18x^2 + 110x - 242$*, and* $\alpha \approx 1.83928$ *be the real root of* $x^3 - x^2 - x - 1$*. Then*

$$H_0(n) = \frac{\beta}{n\pi}\alpha^n - \mathcal{O}\left(\frac{\alpha^n}{n^2}\right).$$

In order to prove Theorem 3.7, we will find an asymptotic for $H_0(n)$ by maximizing $H_0(m,k,k',m')$ over the real numbers, and then accounting for the error introduced.

Proof of Theorem 3.7. Let us find where $|H_0(m,k,k',m')|$ is maximized. (Notice that m' is completely determined by k and m, and further that

$n = 3m+2k+k')$. By writing the factorials as Γ functions, and ignoring the $(-1)^k$ we are maximizing

$$\hat{H}(m,k,k') = (3m+2k+k')\frac{\Gamma(m+k+k')}{\Gamma(k+1)\Gamma(m+1)\Gamma(k'+1)}$$

subject to the condition

$$G(m,k,k') = 3m+2k+k' = n.$$

Thus, to solve for the maximums, we use Lagrange multipliers to solve the equations:

$$\nabla\hat{H} = \lambda\nabla G \text{ and } G(m,k,k') = n.$$

Recall that $\Psi(x)$ denotes the digamma function of x, i.e. $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. The latter gives rise to the equations:

$$\begin{aligned}
3\lambda &= (3m+2k+k')\frac{\Gamma(m+k+k')}{\Gamma(k+1)\Gamma(m+1)\Gamma(k'+1)}\Psi(k'+k+m) - \\
&\quad (3m+2k+k')\frac{\Gamma(m+k+k')}{\Gamma(k+1)\Gamma(m+1)\Gamma(k'+1)}\Psi(m+1) + \\
&\quad 3\frac{\Gamma(m+k+k')}{\Gamma(k+1)\Gamma(m+1)\Gamma(k'+1)} \\
2\lambda &= (3m+2k+k')\frac{\Gamma(m+k+k')}{\Gamma(k+1)\Gamma(m+1)\Gamma(k'+1)}\Psi(k'+k+m) - \\
&\quad (3m+2k+k')\frac{\Gamma(m+k+k')}{\Gamma(k+1)\Gamma(m+1)\Gamma(k'+1)}\Psi(k+1) + \\
&\quad 2\frac{\Gamma(m+k+k')}{\Gamma(k+1)\Gamma(m+1)\Gamma(k'+1)} \\
\lambda &= (3m+2k+k')\frac{\Gamma(m+k+k')}{\Gamma(k+1)\Gamma(m+1)\Gamma(k'+1)}\Psi(k'+k+m) - \\
&\quad (3m+2k+k')\frac{\Gamma(m+k+k')}{\Gamma(k+1)\Gamma(m+1)\Gamma(k'+1)}\Psi(k'+1) + \\
&\quad \frac{\Gamma(m+k+k')}{\Gamma(k+1)\Gamma(m+1)\Gamma(k'+1)} \\
n &= 3m+2k+k'
\end{aligned}$$

Upon some simplification this becomes:

$$\begin{aligned} 3\lambda &= F(m,k,k')(\Psi(k'+k+m) - \Psi(m+1) + 3/n) \\ 2\lambda &= F(m,k,k')(\Psi(k'+k+m) - \Psi(k+1) + 2/n) \\ \lambda &= F(m,k,k')(\Psi(k'+k+m) - \Psi(k'+1) + 1/n) \\ n &= 3m + 2k + k'. \end{aligned}$$

By redefining λ, we get

$$\begin{aligned} 3\lambda &= \Psi(k'+k+m) - \Psi(m+1) + 3/n \\ 2\lambda &= \Psi(k'+k+m) - \Psi(k+1) + 2/n \\ \lambda &= \Psi(k'+k+m) - \Psi(k'+1) + 1/n \\ n &= 3m + 2k + k'. \end{aligned}$$

If we solve for $\lambda - 1/n$ in these equations, and equate them, we get the following three equations:

$$\begin{aligned} \Psi(k'+k+m) - \Psi(k'+1) &= \frac{\Psi(k'+k+m) - \Psi(m+1)}{3} \\ \frac{\Psi(k'+k+m) - \Psi(k+1)}{2} &= \Psi(k'+k+m) - \Psi(k'+1) \\ n &= 3m + 2k + k'. \end{aligned}$$

By noticing that $\Psi(n) = \ln(n) + \mathcal{O}(1/n)$, we can rewrite this as

$$\frac{2}{3}\ln(k'+k+m) - \ln(k'+1) + \frac{1}{3}\ln(m+1) = \mathcal{O}\left(\frac{1}{n}\right) \tag{8}$$

$$\frac{1}{2}\ln(k'+k+m) - \ln(k+1) + \frac{1}{2}\ln(k'+1) = \mathcal{O}\left(\frac{1}{n}\right) \tag{9}$$

$$n = 3m + 2k + k'. \tag{10}$$

Here we use the fact that $\mathcal{O}(k) = \mathcal{O}(m) = \mathcal{O}(k') = \mathcal{O}(n)$.

Now, the question is, what sort of error do we get in the solution of the equations. For large k', k and m, the right hand side is approximately 0, so we can find the solution for 0, and then figure out how far off we are. Thus we need to find a bound for how quickly the left hand side can change (i.e. derivative), and then figure out how skewed the solution is.

The gradients of the left hand sides are

$$\left[\frac{2}{3(k'+k+m)}, \frac{2}{3(k'+k+m)} - \frac{1}{k'+1}, \frac{2}{3(k'+k+m)} + \frac{1}{3(m+1)}\right]$$

$$\left[\frac{1}{2(k'+k+m)} - \frac{1}{2(k+1)}, \frac{1}{2(k'+k+m)} + \frac{1}{2(k'+1)}, \frac{1}{2(k'+k+m)}\right].$$

So we notice that the maximal directional derivatives are $\mathcal{O}(1/n)$. This means that the maximal deviation from the actual solution is $\mathcal{O}(1)$.

By solving equations (8), (9) and (10), where the right hand size is 0 (via Maple [4]) and accounting for the $\mathcal{O}(1)$ term, we can write

$$\begin{aligned} m &= \hat{m}n + \Delta m \\ k &= \hat{k}n + \Delta k \\ k' &= \hat{k}'n + \Delta k' \end{aligned}$$

where Δm, Δk and $\Delta k'$ are all $\mathcal{O}(1)$, and such that m, k and k' are integers, and further that

$$\begin{aligned} \hat{m} &= -\frac{1}{66}\sqrt[3]{1331 + 231\sqrt{33}} - 1/3\frac{1}{\sqrt[3]{1331 + 231\sqrt{33}}} + 1/3 \\ \hat{k} &= \frac{1}{66}\sqrt[3]{3267 + 627\sqrt{33}} - 2\frac{1}{\sqrt[3]{3267 + 627\sqrt{33}}} \\ \hat{k}' &= \frac{1}{66}\sqrt[3]{3267 + 561\sqrt{33}} + \frac{1}{\sqrt[3]{3267 + 561\sqrt{33}}}. \end{aligned}$$

We notice that, asymptotically:

$$\begin{aligned} & \hat{H}(\hat{m}n + \Delta m, \hat{k}n + \Delta k, \hat{k}'n + \Delta k') \\ = \;& n\frac{\Gamma((\hat{m} + \hat{k} + \hat{k}')n + \Delta m + \Delta k + \Delta k')}{\Gamma(\hat{m}n + 1 + \Delta m)\Gamma(\hat{k}n + 1 + \Delta k)\Gamma(\hat{k}'n + 1 + \Delta k')} \\ \approx \;& n\frac{((\hat{m} + \hat{k} + \hat{k}')n)^{\Delta m + \Delta k + \Delta k'}\Gamma((\hat{m} + \hat{k} + \hat{k}')n)}{(\hat{m}n + 1)^{\Delta m}\Gamma(\hat{m}n + 1)(\hat{k}n + 1)^{\Delta k}\Gamma(\hat{k}n + 1)(\hat{k}'n + 1)^{\Delta k'}\Gamma(\hat{k}'n + 1)} \\ \approx \;& \frac{(\hat{m} + \hat{k} + \hat{k}')^{\Delta m + \Delta k + \Delta k'}}{\hat{m}^{\Delta m}\hat{k}^{\Delta k}\hat{k}'^{\Delta k'}} \times n\frac{\Gamma((\hat{m} + \hat{k} + \hat{k}')n)}{\Gamma(\hat{m}n + 1)\Gamma(\hat{k}n + 1)\Gamma(\hat{k}'n + 1)} \\ = \;& \mathcal{O}(1)n\frac{\Gamma((\hat{m} + \hat{k} + \hat{k}')n)}{\Gamma(\hat{k}n + 1)\Gamma(\hat{m}n + 1)\Gamma(\hat{k}'n + 1)} \\ = \;& \mathcal{O}(1)\left(\frac{\beta}{\pi n}\alpha^n - \mathcal{O}\left(\frac{\alpha^n}{n^2}\right)\right). \end{aligned}$$

Let us consider this $\mathcal{O}(1)$ term more precisely. Notice that, using the property that $3\Delta m + 2\Delta k + \Delta k' = 0$, we have:

$$\begin{aligned}
&\frac{(\hat{m}+\hat{k}+\hat{k}')^{\Delta m+\Delta k+\Delta k'}}{\hat{m}^{\Delta m}\hat{k}^{\Delta k}\hat{k}'^{\Delta k'}} \\
=\ & \frac{(\hat{m}+\hat{k}+\hat{k}')^{\Delta m+\Delta k-3\Delta m-2\Delta k}}{\hat{m}^{\Delta m}\hat{k}^{\Delta k}\hat{k}'^{-3\Delta m-2\Delta k}} \\
=\ & \frac{(\hat{m}+\hat{k}+\hat{k}')^{-2\Delta m-\Delta k}}{\hat{m}^{\Delta m}\hat{k}^{\Delta k}\hat{k}'^{-3\Delta m-2\Delta k}} \\
=\ & \frac{(\hat{m}+\hat{k}+\hat{k}')^{-2\Delta m}(\hat{m}+\hat{k}+\hat{k}')^{-\Delta k}}{\hat{m}^{\Delta m}\hat{k}^{\Delta k}\hat{k}'^{-3\Delta m}\hat{k}'^{-2\Delta k}} \\
=\ & \frac{\hat{k}'^{3\Delta m}\hat{k}'^{2\Delta k}}{\hat{m}^{\Delta m}(\hat{m}+\hat{k}+\hat{k}')^{2\Delta m}\hat{k}^{\Delta k}(\hat{m}+\hat{k}+\hat{k}')^{\Delta k}} \\
=\ & \frac{\hat{k}'^{3\Delta m}}{\hat{m}^{\Delta m}(\hat{m}+\hat{k}+\hat{k}')^{2\Delta m}}\frac{\hat{k}'^{2\Delta k}}{\hat{k}^{\Delta k}(\hat{m}+\hat{k}+\hat{k}')^{\Delta k}} \\
=\ & \left(\frac{\hat{k}'^{3}}{\hat{m}(\hat{m}+\hat{k}+\hat{k}')^{2}}\right)^{\Delta m}\left(\frac{\hat{k}'^{2}}{\hat{k}(\hat{m}+\hat{k}+\hat{k}')}\right)^{\Delta k} \\
=\ & 1^{\Delta m}1^{\Delta k} \\
=\ & 1
\end{aligned}$$

where this last simplification was done via Maple.

So this becomes

$$H_0(n) = \frac{\beta}{n\pi}\alpha^n - \mathcal{O}\left(\frac{\alpha^n}{n^2}\right)$$

where β is the real root of $x^3 - 18x^2 + 110x - 242$, and α is the real root of $x^3 - x^2 - x - 1$. □

Theorem 1.5 follows directly from Theorem 3.7 and the following Lemma

Lemma 3.8. *For n sufficiently large, $H_l(n) \leq H_0(n)$.*

Proof. From the comments following the statement of Theorem 3.6 we see that

$$\begin{aligned}
H_l(m,k,k',m') &= H_l(m,k,k'-1,m') - H_l(m,k-1,k',m'-1) \\
&\quad + H_l(m-1,k,k',m'-2).
\end{aligned}$$

From this it follows that

$$H_l(n) \leq H_l(n-1) + H_l(n-2) + H_l(n-3)$$

Notice that

$$H_l(n) = H_{n-l}(n) \tag{11}$$

by considering the resultant with the reciprocal polynomial, namely that $\mathrm{Res}(f,g) = \pm\mathrm{Res}(x^3 f(1/x), x^n g(1/x))$. So, we can suppose w.l.o.g. that $l \geq \frac{n}{2}$. We write this as

$$\begin{aligned}
H_l(n) &\leq 1 \times H_l(n-1) + 1 \times H_l(n-2) + 1 \times H_l(n-3) \\
&:= A_1 H_l(n-1) + B_1 H_l(n-2) + C_1 H_l(n-3) \\
&\leq (A_1 + B_1) H_l(n-2) + (A_1 + C_1) H_l(n-3) + A_1 H_l(n-4) \\
&:= A_2 H_l(n-2) + B_2 H_l(n-3) + C_2 H_l(n-4) \\
&\vdots \\
&\leq A_{n-l-2} H_l(l+2) + B_{n-l-2} H_l(l+1) + C_{n-l-2} H_l(l) \\
&= A_{n-l-2} H_2(l+2) + B_{n-l-2} H_1(l+1) + C_{n-l-2} H_0(l),
\end{aligned}$$

where the last equality holds because of (11). The numbers A_m, B_m and C_m satisfy linear recurrence relationships. Namely, we have that $A_m = A_{m-1} + B_{m-1}, B_m = A_{m-1} + C_{m-1}$ and $C_m = A_{m-1}$. This simplifies to $A_1 = 1, A_2 = 2, A_3 = 4, A_m = A_{m-1} + A_{m-2} + A_{m-3}$, and further that $B_m = A_{m-1} + A_{m-2}$ and $C_m = A_{m-1}$.

Solving this gives $A_m = c\alpha^m + c_1\alpha_1^m + c_2\alpha_2^m$, where α is the real root of $x^3 - x^2 - x - 1$, and α_i are its conjugates. Further c is the real root of $44x^3 - 44x^2 + 12x - 1$ and c_1 and c_2 are its conjugates.

Numerically

$$\begin{aligned}
c &\approx .6184199224 \\
c_1 &\approx .1907900391 + .01870058339i \\
c_2 &\approx .1907900391 - .01870058339i
\end{aligned}$$

For $m \geq 3$, this gives us by the triangle inequality, $A_m \leq 0.7\alpha^m$. Similarly, for $m \geq 5$ we get that

$$B_m = A_{m-1} + A_{m-2} \leq \alpha^m(0.7/\alpha + 0.7/\alpha^2) \leq 0.6\alpha^m$$

and for $m \geq 4$ we get that

$$C_m = A_{m-1} \leq \alpha^m(0.7/\alpha) \leq 0.4\alpha^m$$

Now, we have already shown that

$$H_0(n) = \frac{\beta}{\pi n}\alpha^n - \mathcal{O}\left(\frac{\alpha^n}{n^2}\right)$$

where $\beta = 8.13488$ (Theorem 3.7).

Using the same method, we can show that

$$H_l(n) = \frac{\beta_l}{\pi n}\alpha^n - \mathcal{O}\left(\frac{\alpha^n}{n^2}\right)$$

for l from 0 to 6 where

$$\begin{aligned} \beta_0 &= 8.13488 \\ \beta_1 &= 3.71205 \\ \beta_2 &= 0.92093 \\ \beta_3 &= 1.01680 \\ \beta_4 &= 0.31597 \\ \beta_5 &= 0.01923 \\ \beta_6 &= 0.05956 \end{aligned}$$

So, $H_l(n) \leq H_0(n)$ if $n-6 \leq l \leq n$ (this is again due to (11)). Suppose now that $l \leq n-7$. Then $n-l-2 \geq 5$ and all the bounds computed above for A_m, B_m, C_m hold. So, we have, for large n,

$$\begin{aligned} H_l(n) &\leq A_{n-l-2}H_2(l+2) + B_{n-l-2}H_1(l+1) + C_{n-l-2}H_0(l) \\ &\leq 0.7\alpha^{n-l-2}\left(\tfrac{\beta_2}{\pi(l+2)}\alpha^{l+2} - \mathcal{O}(\tfrac{\alpha^{l+2}}{(l+2)^2})\right) \\ &\quad +0.6\alpha^{n-l-2}\left(\tfrac{\beta_1}{\pi(l+1)}\alpha^{l+1} - \mathcal{O}(\tfrac{\alpha^{l+1}}{(l+1)^2})\right) \\ &\quad +0.4\alpha^{n-l-2}\left(\tfrac{\beta_0}{\pi l}\alpha^{l} - \mathcal{O}(\tfrac{\alpha^{l}}{(l)^2})\right) \\ &\leq 0.7\alpha^{n-l-2}\tfrac{\beta_2}{\pi(l+2)}\alpha^{l+2} + 0.6\alpha^{n-l-2}\tfrac{\beta_1}{\pi(l+1)}\alpha^{l+1} \\ &\quad +0.4\alpha^{n-l-2}\tfrac{\beta_0}{\pi l}\alpha^{l} \\ &= 0.7\tfrac{\beta_2}{\pi(l+2)}\alpha^n + 0.6\tfrac{\beta_1}{\pi(l+1)}\alpha^{n-1} + 0.4\tfrac{\beta_0}{\pi l}\alpha^{n-2}. \end{aligned} \tag{12}$$

The last expression of (12) is maximal when l is minimal, i.e. $l = n/2$. So, for large n, we get that $H_l(n)$ is bounded above by

$$\begin{aligned} H_l(n) &\leq 0.7\frac{\beta_2}{\pi(n/2+2)}\alpha^n + 0.6\frac{\beta_1}{\pi(n/2+1)}\alpha^{n-1} + 0.4\frac{\beta_0}{\pi n/2}\alpha^{n-2} \\ &\leq 0.7\frac{\beta_2}{\pi(n/2)}\alpha^n + 0.6\frac{\beta_1}{\pi(n/2)}\alpha^{n-1} + 0.4\frac{\beta_0}{\pi n/2}\alpha^{n-2} \\ &\leq 2\left(0.7 \times \beta_2 + 0.6\frac{\beta_1}{\alpha} + 0.4\frac{\beta_0}{\alpha^2}\right)\frac{\alpha^n}{\pi n} \\ &= \frac{5.6348}{\pi n}\alpha^n \end{aligned}$$

This expression is bounded above by $H_0(n) = \frac{\beta_0}{\pi n}\alpha^n - \mathcal{O}(\frac{\alpha^n}{n^2})$ for large values of n, which gives the desired result. □

Now we are ready for the proof of our main result.

Proof of Theorem 1.5. Due to Theorem 3.7, we will be done if we show that, for $n \gg 0$, $H(n) = H_0(n)$. As it was showed in Lemma 3.8, it turns out that $H_0(n) = \max_{0\leq l\leq n} H_l(n)$ if $n \gg 0$. As explained at the beginning of this section, notice that if $H(n) > H(n-1)$ then $H(n) = \max_l H_l(n)$, so we only have to prove that for infinite values of N, we have $H(N) > H(N-1)$.

Suppose this is not the case, then $H(N)$ is bounded as $N \to \infty$, and this is a contradiction with Theorem 3.7 which says that $H(N) \geq H_0(N)_{N\to\infty} \to +\infty$.

So pick N such that $H(N) > H(N-1)$, and sufficiently large such that $H(N) = H_0(N) \geq \max_l H_l(N)$ (Lemma 3.8) and $H(N+1) \geq H_0(N+1) > H_0(N)$. Hence by induction for all $m \geq N$ we have that $H(m) > H(m-1)$ and $H(m) = H_0(m)$.

□

It should be pointed out that experimentally, $H(n) > H(n-1)$ for all n and $H(n) = H_0(n)$ for all $n \geq 18$.

4. Conclusions and comments

In this paper we give a precise description for $H(\mathrm{Res}(f, g))$ where f is a quadratic polynomial, and tight asymptotics when f is a cubic polynomial. The methods used in this paper should be extendible to the case of f being a polynomial of fixed degree m. In particular, most of Section 3 is done constructively, and can be extended to arbitrary m. So we can most likely find bounds such as $H(n) \leq \mathcal{O}(\alpha^n)$ for arbitrary fixed m, and α dependent on m. It would be interesting and worthwhile to do this.

Let $g(x) = g_0 + \cdots + g_n x^n$ be a degree n polynomial. As a result of Lemma 3.8 we proved that for sufficiently large n that

$$H(\mathrm{Res}(f_0 + \cdots + f_3x^3, g)) = H\left(\mathrm{Res}(f_0 + \cdots + f_3x^3, g_0 + g_nx^n)\right).$$

(Experimentally, this appears to be true for $n \geq 18$.) Notice that if $\deg(f) = 2$, for $n \geq 3$:

$$H(\mathrm{Res}(f_0 + f_1x + f_2x^2, g)) = H(\mathrm{Res}(f_0 + f_1x + f_2x^2, g_0 + g_nx^n)).$$

It is trivial to see that in the linear case:

$$\begin{aligned} H(\mathrm{Res}(f_0 + f_1x, g)) &= H(\mathrm{Res}(f_0 + f_1x, g_0 + g_nx^n)) \\ (&= 1). \end{aligned}$$

It is reasonable to conjecture that

Conjecture 4.1. *For fixed m, and $g(x) = g_0 + \cdots + g_n x^n$, for sufficiently large n (dependent on m),*

$$H(\mathrm{Res}(f_0 + \cdots + f_m x^m, g)) = H(\mathrm{Res}(f_0 + \cdots + f_m x^m, g_0 + g_n x^n)).$$

There is some computational evidence to support this conjecture.

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6. Appendix: Tables

A_n	n	A_n	n	A_n	n
1	3,4	10	34,35,36,37	19	67,68,69,70
2	5,6,7,8	11	38,39,40,41	20	71,72,73
3	9,10,11,12	12	42,43,44	21	74,75,76,77
4	13,14,15	13	45,46,47,48	22	78,79,80,81
5	16,17,18,19	14	49,50,51,52	23	82,83,84
6	20,21,22,23	15	53,54,55	24	85,86,87,88
7	24,25,26	16	56,57,58,59	25	89,90,91
8	27,28,29,30	17	60,61,62	26	92,93,94,95
9	31,32,33	18	63,64,65,66	27	96,97,98,99

Table 1. Values of A_n (Theorem 1.1, page 2)

n	Maximum at H_l	n	Maximum at H_l	n	Maximum at H_l
1	H_0	8	H_0	15	H_3
2	H_1	9	H_3	16	H_3
3	H_0	10	H_3	17	H_3
4	H_1	11	H_0	18	H_0
5	H_1 and H_2	12	H_0	19	H_0
6	H_3	13	H_3	$\vdots$	$\vdots$
7	H_3	14	H_3	72	H_0

Table 2. Maximal H_l value

$H_0(m,k,k',m')$	$=$	$F(m-1,k,k',m'-2) - F(m,k,k'-1,m') + 2F(m,k,k',m')$
$H_1(m,k,k',m')$	$=$	$2F(m-1,k,k'-1,m'-1) - F(m,k-1,k'-1,m') + 2F(m,k-1,k',m') - 3F(m-1,k,k',m'-1)$
$H_2(m,k,k',m')$	$=$	$2F(m-1,k,k'-2,m') - 4F(m-1,k,k'-1,m') - F(m-2,k-1,k',m'-3) - 3F(m-2,k,k',m'-2) + F(m-1,k-2,k',m'-2) - F(m,k-2,k'-1,m') + 2F(m,k-2,k',m')$
$H_3(m,k,k',m')$	$=$	$-2F(m-2,k,k'-2,m'-1) + 3F(m-1,k-1,k'-2,m') - 6F(m-1,k-1,k'-1,m') + F(m-3,k,k',m'-3) + 5F(m-2,k,k',m'-1) - 2F(m-2,k-1,k',m'-2) - F(m-2,k-2,k',m'-3) + F(m-1,k-3,k',m'-2) - F(m,k-3,k'-1,m') + 2F(m,k-3,k',m')$
$H_4(m,k,k',m')$	$=$	$-2F(m-5,k,k',m'-6) - F(m-4,k,k',m'-4) + 3F(m-3,k-1,k'-1,m'-3) - 9F(m-2,k-2,k'-1,m'-2) + F(m-2,k-3,k',m'-3) - 7F(m-2,k-2,k',m'-2) + 13F(m-3,k-1,k',m'-3) + 6F(m-3,k,k'-2,m'-2) + 2F(m-2,k,k'-3,m') + F(m-1,k-4,k',m'-2) - F(m,k-4,k'-1,m') + 2F(m,k-4,k',m') + 4F(m-1,k-2,k'-2,m') - 8F(m-1,k-2,k'-1,m')$
$H_5(m,k,k',m')$	$=$	$2F(m-3,k,k'-3,m'-1) + 18F(m-3,k-1,k'-2,m'-2) - 7F(m-3,k,k'-2,m'-1) + 12F(m-4,k-1,k'-1,m'-4) - 13F(m-4,k,k'-1,m'-3) - F(m-5,k-1,k',m'-6) - 3F(m-5,k,k',m'-5) + 5F(m-2,k-1,k'-2,m') + 2F(m-1,k-5,k',m'-2) + F(m,k-5,k',m') - F(m,k-6,k',m'-1) + 5F(m-1,k-4,k'-1,m'-1) - 5F(m-1,k-3,k'-1,m') - 15F(m-2,k-4,k',m'-3) - 25F(m-2,k-3,k',m'-2) + 10F(m-3,k-2,k'-1,m'-3) + 15F(m-4,k-2,k',m'-5)$

TABLE 3. A table of $H_l(m,k,k',m')$ values, (Theorem 3.6, page 8)

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