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Algebraic Relations Between Harmonic Sums and Associated Quantities

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Abstract

We derive the algebraic relations of alternating and non-alternating finite harmonic sums up to the sums of depth 6. All relations for the sums up to weight 6 are given in explicit form. These relations depend on the structure of the index sets of the harmonic sums only, but not on their value. They are therefore valid for all other mathematical objects which obey the same multiplication relation or can be obtained as a special case thereof, as the harmonic polylogarithms. We verify that the number of independent elements for a given index set can be determined by counting the Lyndon words which are associated to this set. The algebraic relations between the finite harmonic sums can be used to reduce the high complexity of the expressions for the Mellin moments of the Wilson coefficients and splitting functions significantly for massless field theories as QED and QCD up to three loop and higher orders in the coupling constant and are also of importance for processes depending on more scales. The ratio of the number of independent sums thus obtained to the number of all sums for a given index set is found to be $\leq 1/d$ with d the depth of the sum independently of the weight. The corresponding counting relations are given in analytic form for all classes of harmonic sums to arbitrary depth and are tabulated up to depth $d = 10$.

1 Introduction

Single scale problems in massless and massive perturbative calculations in Quantum Field Theory can be expressed in terms of finite harmonic sums [1–3]. These sums occur in the ϵ–expansion of the integrals for higher order corrections to QCD splitting functions and Wilson coefficients for space- and time-like processes [4] and the amplitudes of important high energy scattering processes such as Bhabha scattering [5], $pp \rightarrow 2$ jets [6], $pp \rightarrow \gamma\gamma$ [7], Higgs production in hadron scattering [8], light-by-light scattering [9], and other QED processes [10]. Multiple harmonic sums or mathematical objects being related to them do generally emerge in Taylor expansions of higher transcendental functions occurring in loop integrals, see e.g. [11]. Unlike for representations in x–space multiple harmonic sums which are obtained after a MELLIN transform of the respective expressions account for the genuine MELLIN symmetry in massless field theory, the observation of which can lead to a considerable simplification of the these expressions. The multiple finite harmonic sums are defined by

$$S_{a_1,\ldots,a_n}(N) = \sum_{k_1=1}^{N} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} \frac{\mathrm{sign}(a_1)^{k_1}}{k_1^{|a_1|}} \cdots \frac{\mathrm{sign}(a_n)^{k_n}}{k_n^{|a_n|}} \,. \tag{1.1}$$

Here, a_k are positive or negative integers and N is a positive even or odd integer depending on the observable under consideration. One calls n the depth and $\sum_{k=1}^{n} |a_k|$ the weight of a harmonic sum. Harmonic sums are associated to MELLIN transforms of real functions or SCHWARTZ–distributions $f(x) \in \mathcal{S}'[0,1]$ [12]

$$S_{a_1,\ldots,a_n}(N) = \int_0^1 dx\ x^N\ f_{a_1,\ldots,a_n}(x) \tag{1.2}$$

which emerge in field theoretic calculations. Finite harmonic sums are related to harmonic polylogarithms $H_{b_1,\ldots,b_n}(x)$ [13]. Their $1/(1 \pm x)$–weighted MELLIN transform yields harmonic sums. The inverse MELLIN transform relates the harmonic sums to functions of NIELSEN integrals [14] and the variable x at least for sums of weight $w \leq 4$ as shown in [3], and associated generalizations for higher weight. NIELSEN integrals are a generalization of the usual polylogarithms [15]. In the limit $N \rightarrow \infty$ the convergent multiple harmonic sums, i.e. those where $a_1 \neq 1$, yield (multiple) Zeta–values $\zeta_{a_1,\ldots,a_n}$, which are also called EULER–ZAGIER sums [16]. A generalization of both harmonic polylogarithms and the EULER–ZAGIER sums are the nested Z–sums [17], which form a HOPF algebra [18, 19] and are related to Goncharov's multiple polylogarithms [20] [1]. Likewise one may also consider polylogarithms of a different kind [22]. [2]

For many applications it is very convenient to express the result first in analytic form in terms of finite harmonic sums or rational functions out of them. This is even mandatory studying factorization–scheme invariant evolution [24]. Harmonic sums may be analytically continued from the even or odd integers to complex values of N [25] which, however, requires some effort in general to obtain highly precise representations. The resulting functions $f(x)$ can be easily obtained by the inverse MELLIN transformation through a single numerical contour integral in the complex N-plane. Since the number of harmonic sums up to weight and depth n is $3^n - 1$ the number of necessary analytic continuations of the respective harmonic sums would grow rather rapidly. The finite harmonic sums obey algebraic relations of different kind, which may be used to determine an explicit (polynomial) representation of all sums of a given weight and depth over a basis of independent sums.

[1] The two–dimensional harmonic polylogarithms [21] are a special case of the latter class of functions.

[2] For a recent review see [23].

It is the aim of the present paper to calculate these representations for all harmonic sums up to depth 6. The results will be applied to derive the representations needed up to depth and weight 6 in explicit form leaving, however, the respective indices as general parameters. This level is expected to be sufficient to express the anomalous dimensions and massless coefficient functions, and similarly other quantities, up to $O(\alpha_i^3)$ in QED and QCD.

The paper is organized as follows. In section 2 we derive the algebraic relations of finite harmonic sums up to depth 6 in general form. There are in principle two classes: the product–relations and those implied by integration by parts. We show that the latter relations are fully contained in the former, since always all allowed index permutations can be considered. In section 3–6 the specific cases are considered for depth 3 to depth 6. Here we also complete foregoing investigations [3, 25] concerning the analytic continuation of sums up to depth and weight 4. The relations obtained were extensively tested numerically. The relations for the finite harmonic sums may be directly applied to obtain relations for the associated Euler–Zagier sums in the limit $N \to \infty$. [3] Moreover, as all the relations considered in the present paper derive from the structure of the respective index set and do not depend on the specific value of the harmonic sums these relations hold for all other mathematical objects which obey the same multiplication relation (2.1). The main product–relation considered in the present paper (2.1) reduces to a simpler one (2.30), which is valid for harmonic polylogarithms [13] by deleting all terms which contain the $\wedge$–operator (2.3) in the index set. Therefore all algebraic relations derived cover the respective algebraic relations for harmonic polylogarithms (2.30). Similarly, the algebraic relations for Z–sums with $x_i = 1, \forall\, i$ are easily derived, e.g. using Eq. (6) of Ref. [17]. In section 7 we briefly show that the number of independent harmonic sums for a given index set can be counted determining the number of Lyndon words [33, 34] which belong to this set. Section 8 contains the conclusions. In appendix A we summarize the numbers of sums and basic sums for all individual index pattern from depth $d = 7$ to 10. Appendix B contains an overview on all specific harmonic sums up to depth and weight 6.

2 General Algebraic Relations

2.1 Product–Relations

The product of two finite harmonic sums (1.1) yields

$$\begin{aligned} S_{a_1,\ldots,a_n}(N) \cdot S_{b_1,\ldots,b_m}(N) \;=\; & \sum_{l_1=1}^{N} \frac{\mathrm{sign}(a_1)^{l_1}}{l_1^{|a_1|}} S_{a_2,\ldots,a_n}(l_1)\, S_{b_1,\ldots,b_m}(l_1) \\ & + \sum_{l_2=1}^{N} \frac{\mathrm{sign}(b_1)^{l_2}}{l_2^{|b_1|}} S_{a_1,\ldots,a_n}(l_2)\, S_{b_2,\ldots,b_m}(l_2) \\ & - \sum_{l=1}^{N} \frac{[\mathrm{sign}(a_1)\mathrm{sign}(b_1)]^{l}}{l^{|a_1|+|b_1|}} S_{a_2,\ldots,a_n}(l)\, S_{b_2,\ldots,b_m}(l)\;. \end{aligned} \tag{2.1}$$

[3]For positive a_i's the finite Zeta–values were calculated to weight 9 in [26], to weight 10 in [27] and to weight 12 in [28]. Allowing general values of a_i explicit relations were given up to weight 4 in [3, 29]. More recently also multiple Zeta–values with weights, which are nth roots of unity were considered and the explicit results for the alternating case were given up to weight 7 in [30]. All multiple Zeta–values up to weight 9 were obtained using FORM [31], see [2], and expressed in terms of the more familiar numbers (symbols) $S_1(\infty), \ln(2), \zeta(2), \zeta(3), \mathrm{Li}_4(1/2), \zeta(5), \mathrm{Li}_5(1/2), \mathrm{Li}_6(1/2), S_{-5,-1}(\infty), \zeta(7), \mathrm{Li}_7(1/2), S_{-5,1,1}(\infty)$ and $S_{5,-1,-1}(\infty)$ as also in [3, 29] and complementary to [28, 30]. For a quantum field theoretic representation of Euler-Zagier sums see [32].

One proves by induction that the r.h.s. of Eq. (2.1) consists out of a linear combination of harmonic sums of the argument N and depth $m+n$ or lower. The prerequisite is obtained choosing $n=1$,

$$\begin{aligned} S_{a_1}(N)\cdot S_{b_1,\ldots,b_m}(N) &= S_{a_1,b_1,\ldots,b_m}(N)+S_{b_1,a_1,b_2,\ldots,b_m}(N)+\ldots+S_{b_1,b_2,\ldots,b_m,a_1}(N) \\ &\quad -S_{a_1\wedge b_1,b_2,\ldots,b_m}(N)-\ldots-S_{b_1,b_2,\ldots,a_1\wedge b_m}(N)\ . \end{aligned} \tag{2.2}$$

Here the symbol $\wedge$ is defined as

$$a\wedge b=\mathrm{sign}(a)\mathrm{sign}(b)\left(|a|+|b|\right)\ . \tag{2.3}$$

Relations similar to (2.2) were investigated by Faà di Bruno [35] for the roots of algebraic equations [4] and Hoffman [37] for Euler–Zagier sums. Eq. (2.2) may be used to establish that the linear combination of all permutations over a given index set of finite harmonic sums can be represented in terms of a polynomial of single harmonic sums [3,38][5], cf. section 2.3. This holds likewise also for positive integer sums [41]. If all the indices of a finite harmonic sum are equal, it obeys a determinant–representation, see [3]. Another representation was given in [2].

We introduce the shuffle product $\sqcup\!\sqcup$ of a single and a general finite harmonic sum

$$S_{a_1}(N)\sqcup\!\sqcup S_{b_1,\ldots,b_m}(N) = S_{a_1,b_1,\ldots,b_m}(N)+S_{b_1,a_1,b_2,\ldots,b_m}(N)+\ldots+S_{b_1,b_2,\ldots,b_m,a_1}(N) \tag{2.4}$$

which is the linear combination of the sums of depth $m+1$ which are generated by Eq. (2.2). The shuffle product of two harmonic sums of depth n and m, $S_{a_1,\ldots,a_n}(N)$ and $S_{b_1,\ldots,b_m}(N)$, is the sum of all harmonic sums of depth $m+n$ in the index set of which a_i occurs left of a_j for $i<j$ and likewise for b_k and b_l for $k<l$. [6] Shuffle products are symmetric

$$S_{a_1,\ldots,a_n}(N)\sqcup\!\sqcup S_{b_1,\ldots,b_m}(N)=S_{b_1,\ldots,b_m}(N)\sqcup\!\sqcup S_{a_1,\ldots,a_n}(N)\ , \tag{2.5}$$

i.e. the commutation relation holds. The set of harmonic sums is extended by the constant $\mathsf{1}$, the empty sum, which forms the unit element

$$\mathsf{1}\ \sqcup\!\sqcup\ S_{a_1,\ldots,a_n}(N)=S_{a_1,\ldots,a_n}\ . \tag{2.6}$$

We consider the product of the double sum $S_{a_1,a_2}(N)$ with a general harmonic sum. The first term of the r.h.s. of Eq. (2.1) reduces to a sum over (2.2) which is a linear combination of harmonic sums, likewise the third term. Inverting the relation for the second term of the r.h.s. of (2.2) one may perform the complete recursion for this case into a linear combination of harmonic sums as well. Because products $S_{a_1,\ldots a_k}(N)\cdot S_{b_1,\ldots,b_l}(N)$ with either $k<n$ or $l<m$ can be represented in terms of linear combinations of harmonic sums, Eq. (2.2) is also a linear combination of harmonic sums.

The products of depth n induce algebraic relations between the single harmonic sums of depth n, which allow their calculation in terms of sums of the same and lower depth. Associated to this the corresponding shuffle products of harmonic sums occur, which are

Depth 2 :

$$S_{a_1}(N)\sqcup\!\sqcup S_{a_2}(N) = S_{a_1,a_2}(N)+S_{a_2,a_1}(N) \tag{2.7}$$

[4] For related work on the invariant theory of the algebraic equations with various variables see [36].

[5] In [38] the complete permutations were derived for arbitrary indices up to depth $n=10$. The representation for $n=2$ was first derived in [39], that for $n=3$ in [40] and for $n=4$ in [3].

[6] Shuffle products were introduced in [42], see also [33]. For the application of shuffles for multiple zeta values see e.g. [43].

Depth 3 :

$$S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3}(N) = S_{a_1,a_2,a_3}(N) + S_{a_2,a_1,a_3}(N) + S_{a_2,a_3,a_1}(N) \tag{2.8}$$

Depth 4 :

$$S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4}(N) = S_{a_1,a_2,a_3,a_4}(N) + S_{a_2,a_1,a_3,a_4}(N) + S_{a_2,a_3,a_1,a_4}(N) + S_{a_2,a_3,a_4,a_1}(N) \tag{2.9}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4}(N) &= S_{a_1,a_2,a_3,a_4}(N) + S_{a_1,a_3,a_2,a_4}(N) + S_{a_1,a_3,a_4,a_2}(N) \\ &+ S_{a_3,a_4,a_1,a_2}(N) + S_{a_3,a_1,a_4,a_2}(N) + S_{a_3,a_1,a_2,a_4}(N) \end{aligned} \tag{2.10}$$

Depth 5 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_2,a_1,a_3,a_4,a_5}(N) + S_{a_2,a_3,a_1,a_4,a_5}(N) \\ &+ S_{a_2,a_3,a_4,a_1,a_5}(N) + S_{a_2,a_3,a_4,a_5,a_1}(N) \end{aligned} \tag{2.11}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_1,a_3,a_2,a_4,a_5}(N) + S_{a_1,a_3,a_4,a_2,a_5}(N) \\ &+ S_{a_1,a_3,a_4,a_5,a_2}(N) + S_{a_3,a_1,a_2,a_4,a_5}(N) + S_{a_3,a_1,a_4,a_2,a_5}(N) \\ &+ S_{a_3,a_1,a_4,a_5,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_2}(N) + S_{a_3,a_4,a_1,a_5,a_2}(N) \\ &+ S_{a_3,a_4,a_1,a_2,a_5}(N) \end{aligned} \tag{2.12}$$

Depth 6 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4,a_5,a_6}(N) &= S_{a_1,a_2,a_3,a_4,a_5,a_6}(N) + S_{a_2,a_1,a_3,a_4,a_5,a_6}(N) + S_{a_2,a_3,a_1,a_4,a_5,a_6}(N) \\ &+ S_{a_2,a_3,a_4,a_1,a_5,a_6}(N) + S_{a_2,a_3,a_4,a_5,a_1,a_6}(N) + S_{a_2,a_3,a_4,a_5,a_6,a_1}(N) \end{aligned} \tag{2.13}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4,a_5,a_6}(N) &= S_{a_1,a_2,a_3,a_4,a_5,a_6}(N) + S_{a_1,a_3,a_2,a_4,a_5,a_6}(N) + S_{a_1,a_3,a_4,a_2,a_5,a_6}(N) \\ &+ S_{a_1,a_3,a_4,a_5,a_2,a_6}(N) + S_{a_1,a_3,a_4,a_5,a_6,a_2}(N) + S_{a_3,a_1,a_2,a_4,a_5,a_6}(N) \\ &+ S_{a_3,a_1,a_4,a_2,a_5,a_6}(N) + S_{a_3,a_1,a_4,a_5,a_2,a_6}(N) + S_{a_3,a_1,a_4,a_5,a_6,a_2}(N) \\ &+ S_{a_3,a_4,a_1,a_2,a_5,a_6}(N) + S_{a_3,a_4,a_1,a_5,a_2,a_6}(N) + S_{a_3,a_4,a_1,a_5,a_6,a_2}(N) \\ &+ S_{a_3,a_4,a_5,a_6,a_1,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_6,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_2,a_6}(N) \end{aligned} \tag{2.14}$$

$$\begin{aligned} S_{a_1,a_2,a_3}(N) \sqcup\!\sqcup S_{a_4,a_5,a_6}(N) &= S_{a_1,a_2,a_3,a_4,a_5,a_6}(N) + S_{a_1,a_2,a_4,a_3,a_5,a_6}(N) + S_{a_1,a_2,a_4,a_5,a_3,a_6}(N) \\ &+ S_{a_1,a_2,a_4,a_5,a_6,a_3}(N) + S_{a_1,a_4,a_2,a_3,a_5,a_6}(N) + S_{a_1,a_4,a_2,a_5,a_3,a_6}(N) \\ &+ S_{a_1,a_4,a_2,a_5,a_6,a_3}(N) + S_{a_1,a_4,a_5,a_6,a_2,a_3}(N) + S_{a_1,a_4,a_5,a_2,a_6,a_3}(N) \\ &+ S_{a_1,a_4,a_5,a_2,a_3,a_6}(N) + S_{a_4,a_5,a_6,a_1,a_2,a_3}(N) + S_{a_4,a_5,a_1,a_6,a_2,a_3}(N) \\ &+ S_{a_4,a_5,a_1,a_2,a_6,a_3}(N) + S_{a_4,a_5,a_1,a_2,a_3,a_6}(N) + S_{a_4,a_1,a_5,a_6,a_2,a_3}(N) \\ &+ S_{a_4,a_1,a_5,a_2,a_6,a_3}(N) + S_{a_4,a_1,a_5,a_2,a_3,a_6}(N) + S_{a_4,a_1,a_2,a_3,a_5,a_6}(N) \\ &+ S_{a_4,a_1,a_2,a_5,a_3,a_6}(N) + S_{a_4,a_1,a_2,a_5,a_6,a_3}(N)\ . \end{aligned} \tag{2.15}$$

The number of harmonic sums $n\{m_1 \sqcup\!\sqcup m_2\}$ occurring in the r.h.s. of the binary shuffle products of a sum of depth m_1 and m_2 are

$$n\{m_1 \sqcup\!\sqcup m_2\} = \binom{m_1+m_2}{m_1}\ . \tag{2.16}$$

For a given depth n one may consider also shuffle products of more than two factors. For $n = 3$ the triple shuffle product of single sums induces the sum over the complete permutation of the three indices. This combination is linearly dependent of either two equations of type (2.8). More generally, the shuffle product of m single sums leads to the sum of all harmonic sums of depth m permuting the whole index set. As we will later consider entire sets of all index permutations of the relations (2.8–2.15) this combination is always linearly dependent and does not lead to a new relation. For $n = 4$ the product $S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2}(N) \sqcup\!\sqcup\, S_{a_3,a_4}(N)$ is related to the combination $S_{a_1,a_2}(N) \sqcup\!\sqcup\, S_{a_3,a_4}(N) + S_{a_2,a_1}(N) \sqcup\!\sqcup\, S_{a_3,a_4}(N)$ and forms therefore a linear combination out of the permutation set for (2.10). Likewise the same argument holds for (2.12, 2.14) for $n = 5, 6$, respectively. For $n = 5$ the product $S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2}(N) \sqcup\!\sqcup\, S_{a_3}(N) \sqcup\!\sqcup\, S_{a_4,a_5}(N)$ is linearly dependent of the permutations generated by (2.12) and similarly for the partial products of triple single sums for $n = 6$ being linearly dependent of the permutations of (2.15). Up to depth 6 only the binary shuffles are found to contribute.

In the following we list the genuine relations of the finite harmonic sums up to depth 6. Sums of this type emerge in massless 3–loop calculations.

Depth 2 :

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2}(N) - S_{a_1}(N)S_{a_2}(N) - S_{a_1\wedge a_2}(N) = 0 \qquad [39] \tag{2.17}$$

Depth 3 :

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3}(N) - S_{a_1}(N)S_{a_2,a_3}(N) - S_{a_1\wedge a_2,a_3}(N) - S_{a_2,a_1\wedge a_3}(N) = 0 \tag{2.18}$$

Depth 4 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3,a_4}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4}(N) - S_{a_1\wedge a_2,a_3,a_4}(N) - S_{a_2,a_1\wedge a_3,a_4}(N) \\ &- S_{a_2,a_3,a_1\wedge a_4}(N) = 0 \end{aligned} \tag{2.19}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup\, S_{a_3,a_4}(N) &- S_{a_1,a_2}(N)S_{a_3,a_4}(N) - S_{a_1,a_2\wedge a_3,a_4}(N) - S_{a_1,a_3,a_2\wedge a_4}(N) \\ &- S_{a_3,a_1\wedge a_4,a_2}(N) - S_{a_3,a_1,a_2\wedge a_4}(N) - S_{a_1\wedge a_3,a_2,a_4}(N) \\ &- S_{a_1\wedge a_3,a_4,a_2}(N) + S_{a_1\wedge a_3,a_2\wedge a_4} = 0 \end{aligned} \tag{2.20}$$

Depth 5 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3,a_4,a_5}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4,a_5}(N) - S_{a_1\wedge a_2,a_3,a_4,a_5}(N) - S_{a_2,a_1\wedge a_3,a_4,a_5}(N) \\ &- S_{a_2,a_3,a_1\wedge a_4,a_5}(N) - S_{a_2,a_3,a_4,a_1\wedge a_5}(N) = 0 \end{aligned} \tag{2.21}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup\, S_{a_3,a_4,a_5}(N) &- S_{a_1,a_2\wedge a_3,a_4,a_5}(N) - S_{a_1,a_3,a_2\wedge a_4,a_5}(N) - S_{a_1,a_3,a_4,a_2\wedge a_5}(N) \\ &- S_{a_3,a_1,a_2\wedge a_4,a_5}(N) - S_{a_3,a_1,a_4,a_2\wedge a_5}(N) - S_{a_3,a_4,a_1\wedge a_5,a_2}(N) \\ &- S_{a_3,a_4,a_1,a_2\wedge a_5}(N) - S_{a_3,a_1\wedge a_4,a_2,a_5}(N) - S_{a_3,a_1\wedge a_4,a_5,a_2}(N) \\ &- S_{a_1,a_2}(N)S_{a_3,a_4,a_5}(N) - S_{a_1\wedge a_3,a_2,a_4,a_5}(N) - S_{a_1\wedge a_3,a_4,a_2,a_5}(N) \\ &- S_{a_1\wedge a_3,a_4,a_5,a_2}(N) + S_{a_1\wedge a_3,a_2\wedge a_4,a_5}(N) + S_{a_1\wedge a_3,a_4,a_2\wedge a_5}(N) = 0 \end{aligned} \tag{2.22}$$

Depth 6 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3,a_4,a_5,a_6}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4,a_5,a_6}(N) - S_{a_1\wedge a_2,a_3,a_4,a_5,a_6}(N) \\ &- S_{a_2,a_1\wedge a_3,a_4,a_5,a_6}(N) - S_{a_2,a_3,a_1\wedge a_4,a_5,a_6}(N) \\ &- S_{a_2,a_3,a_4,a_1\wedge a_5,a_6}(N) - S_{a_2,a_3,a_4,a_5,a_1\wedge a_6}(N) = 0 \end{aligned} \tag{2.23}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup\, S_{a_3,a_4,a_5,a_6}(N) &- S_{a_1,a_2\wedge a_3,a_4,a_5,a_6}(N) - S_{a_1,a_3,a_2\wedge a_4,a_5,a_6}(N) - S_{a_1,a_3,a_4,a_2\wedge a_5,a_6}(N) \\ &- S_{a_1,a_3,a_4,a_5,a_2\wedge a_6}(N) - S_{a_3,a_1,a_2\wedge a_4,a_5,a_6}(N) - S_{a_3,a_1,a_4,a_2\wedge a_5,a_6}(N) \\ &- S_{a_3,a_1,a_4,a_5,a_2\wedge a_6}(N) - S_{a_3,a_4,a_1,a_2\wedge a_5,a_6}(N) - S_{a_3,a_4,a_1,a_5,a_2\wedge a_6}(N) \\ &- S_{a_3,a_4,a_5,a_1\wedge a_6,a_2}(N) - S_{a_3,a_4,a_5,a_1,a_2\wedge a_6}(N) - S_{a_3,a_4,a_1\wedge a_5,a_2,a_6}(N) \\ &- S_{a_3,a_4,a_1\wedge a_5,a_6,a_2}(N) - S_{a_3,a_1\wedge a_4,a_2,a_5,a_6}(N) - S_{a_3,a_1\wedge a_4,a_5,a_2,a_6}(N) \\ &- S_{a_3,a_1\wedge a_4,a_5,a_6,a_2}(N) - S_{a_1\wedge a_3,a_2,a_4,a_5,a_6}(N) - S_{a_1\wedge a_3,a_4,a_2,a_5,a_6}(N) \end{aligned}$$

$$
\begin{aligned}
&- && S_{a_1\wedge a_3,a_4,a_5,a_2,a_6}(N) - S_{a_1\wedge a_3,a_4,a_5,a_6,a_2}(N) + S_{a_3,a_4,a_1\wedge a_5,a_2\wedge a_6}(N)\\
&+ && S_{a_3,a_1\wedge a_4,a_2\wedge a_5,a_6}(N) + S_{a_3,a_1\wedge a_4,a_5,a_2\wedge a_6}(N) + S_{a_3,a_4,a_1\wedge a_5,a_2\wedge a_6}(N)\\
&+ && S_{a_1\wedge a_3,a_2\wedge a_4,a_5,a_6}(N) + S_{a_1\wedge a_3,a_4,a_2\wedge a_5,a_6}(N) + S_{a_1\wedge a_3,a_4,a_5,a_2\wedge a_6}(N)\\
&- && S_{a_1,a_2}(N)S_{a_3,a_4,a_5,a_6}(N) = 0 \qquad (2.24)\\
S_{a_1,a_2,a_3}(N) \sqcup\!\sqcup\, S_{a_4,a_5,a_6}(N) &- && S_{a_1,a_2,a_3\wedge a_4,a_5,a_6}(N) - S_{a_1,a_2,a_4,a_3\wedge a_5,a_6}(N) - S_{a_1,a_2,a_4,a_5,a_3\wedge a_6}(N)\\
&- && S_{a_1,a_4,a_2,a_3\wedge a_5,a_6}(N) - S_{a_1,a_4,a_2,a_5,a_3\wedge a_6}(N) - S_{a_1,a_4,a_5,a_2\wedge a_6,a_3}(N)\\
&- && S_{a_1,a_4,a_5,a_2,a_3\wedge a_6}(N) - S_{a_1,a_4,a_2\wedge a_5,a_3,a_6}(N) - S_{a_1,a_4,a_2\wedge a_5,a_6,a_3}(N)\\
&- && S_{a_1,a_2\wedge a_4,a_3,a_5,a_6}(N) - S_{a_1,a_2\wedge a_4,a_5,a_3,a_6}(N) - S_{a_1,a_2\wedge a_4,a_5,a_6,a_3}(N)\\
&- && S_{a_4,a_5,a_1\wedge a_6,a_2,a_3}(N) - S_{a_4,a_5,a_1,a_2\wedge a_6,a_3}(N) - S_{a_4,a_5,a_1,a_2,a_3\wedge a_6}(N)\\
&- && S_{a_4,a_1,a_5,a_2\wedge a_6,a_3}(N) - S_{a_4,a_1,a_5,a_2,a_3\wedge a_6}(N) - S_{a_4,a_1,a_2,a_3\wedge a_5,a_6}(N)\\
&- && S_{a_4,a_1,a_2,a_5,a_3\wedge a_6}(N) - S_{a_4,a_1,a_2\wedge a_5,a_6,a_3}(N) - S_{a_4,a_1,a_2\wedge a_5,a_3,a_6}(N)\\
&- && S_{a_4,a_1\wedge a_5,a_6,a_2,a_3}(N) - S_{a_4,a_1\wedge a_5,a_2,a_6,a_3}(N) - S_{a_4,a_1\wedge a_5,a_2,a_3,a_6}(N)\\
&- && S_{a_1\wedge a_4,a_2,a_3,a_5,a_6}(N) - S_{a_1\wedge a_4,a_2,a_5,a_3,a_6}(N) - S_{a_1\wedge a_4,a_2,a_5,a_6,a_3}(N)\\
&- && S_{a_1\wedge a_4,a_5,a_6,a_2,a_3}(N) - S_{a_1\wedge a_4,a_5,a_2,a_6,a_3}(N) - S_{a_1\wedge a_4,a_5,a_2,a_3,a_6}(N)\\
&+ && S_{a_1,a_4,a_2\wedge a_5,a_3\wedge a_6}(N) + S_{a_1,a_2\wedge a_4,a_3\wedge a_5,a_6}(N) + S_{a_1,a_2\wedge a_4,a_5,a_3\wedge a_6}(N)\\
&+ && S_{a_4,a_1,a_2\wedge a_5,a_3\wedge a_6}(N) + S_{a_4,a_1\wedge a_5,a_2\wedge a_6,a_3}(N) + S_{a_4,a_1\wedge a_5,a_2,a_3\wedge a_6}(N)\\
&+ && S_{a_1\wedge a_4,a_2,a_3\wedge a_5,a_6}(N) + S_{a_1\wedge a_4,a_2,a_5,a_3\wedge a_6}(N) + S_{a_1\wedge a_4,a_5,a_2\wedge a_6,a_3}(N)\\
&+ && S_{a_1\wedge a_4,a_5,a_2,a_3\wedge a_6}(N) + S_{a_1\wedge a_4,a_2\wedge a_5,a_3,a_6}(N) + S_{a_1\wedge a_4,a_2\wedge a_5,a_6,a_3}(N)\\
&- && S_{a_1\wedge a_4,a_2\wedge a_5,a_3\wedge a_6}(N) - S_{a_1,a_2,a_3}(N)S_{a_4,a_5,a_6}(N) = 0\ . \qquad (2.25)
\end{aligned}
$$

We consider algebraic relations of harmonic sums to express the harmonic sums of a given depth in terms of a minimal set of harmonic sums. For larger depth in addition to the relation given above multiple shuffles may contribute in principle.

2.2 Integration-by-Parts Relations

The finite harmonic sums are related to the harmonic polylogarithms [13] by a weighted Mellin transform up to terms of lower weight. The harmonic polylogarithms derive from the following three functions

$$
\begin{aligned}
f(0;x) &= \frac{1}{x}\\
f(1;x) &= \frac{1}{1-x}\\
f(-1;x) &= \frac{1}{1+x} \qquad (2.26)
\end{aligned}
$$

by iterated integrals :

$$
\begin{aligned}
H_a(x) &= \int_0^x dz f(a;z)\\
H_{\pm 1,b_1\ldots b_m}(x) &= \int_0^x dz f(\pm 1;z) H_{b_1,\ldots,b_m}(z)\ ,\\
H_{b_1+1,b_2,\ldots,b_m}(x) &= \int_0^x dz f(0;z) H_{b_1,\ldots,b_m}(z)\ , \qquad (2.27)
\end{aligned}
$$

where in the set $a, b_1, \ldots, b_m$ not all indices are zero and

$$H_{a,b_1\ldots b_m}(x)|_{a=0,b_i=0} = \frac{1}{(m+1)!}\ln^{m+1}(x) \ . \tag{2.28}$$

Since harmonic polylogarithms $H_{a_1,\ldots,a_n}(x)$ obey the integration-by-parts relation

$$\begin{aligned} H_{a_1,\ldots,a_n}(x) &= H_{a_1}(x)H_{a_2,\ldots,a_n}(x) - H_{a_2,a_1}(x)H_{a_3,\ldots,a_n}(x) \\ &\quad + \ldots + (-1)^{n+1}H_{a_n,\ldots,a_1}(x) \end{aligned} \tag{2.29}$$

they can potentially imply new relations between harmonic sums extending the number of relations which were discussed in the foregoing section. Harmonic polylogarithms are somewhat simpler objects than harmonic sums. Their algebraic product relation is

$$H_{a_1,\ldots,a_n}(x)H_{b_1,\ldots,b_m}(x) = H_{a_1,\ldots,a_n}(x) ⧢ H_{b_1,\ldots,b_m}(x) \ . \tag{2.30}$$

Due to this it is evident that all relations derived in the forthcoming sections for harmonic sums turn into those for harmonic polylogarithms simply removing all terms containing the $\wedge$–symbol.

For $n = 2$ (2.29) and (2.30) lead to the same relation. For $n = 3$ one obtains out of a combination of the algebraic relations (2.30) that

$$H_{a_1}(x)H_{a_2,a_3}(x) - H_{a_3}(x)H_{a_2,a_1}(x) = H_{a_1}(x) ⧢ H_{a_2,a_3}(x) - H_{a_3}(x) ⧢ H_{a_2,a_1}(x) \tag{2.31}$$

$$H_{a_1,a_2,a_3}(x) = H_{a_1}(x)H_{a_2,a_3}(x) - H_{a_2,a_1}(x)H_{a_3}(x) + H_{a_3,a_2,a_1}(x) \tag{2.32}$$

and therefore no new relations. The explicit consideration of the cases $n = 4$ to 6 yields no new relations as well. Binary shuffle products of objects of length n_1 and n_2 contain $\binom{n_1+n_2}{n_1}$ summands. Let us rewrite Eq. (2.29) subtracting the l.h.s. written as the first addend in the r.h.s. $H_{a_1,\ldots,a_n}$ is the shuffle product of itself with $\mathsf{1}$. The number of index–shuffled harmonic polylogarithms of depth n with a positive sign are equal to those with a negative sign since

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+1} = 0 \ . \tag{2.33}$$

Combining always two consecutive summands in the rewritten form of (2.29) from the left to the right, one finds that all shuffles starting with a_1 are annihilated in the first pair. The remaining terms are added to the term $H_{a_2,a_1}(x)H_{a_3,\ldots,a_n}(x)$, where all shuffles starting with a_2 are annihilated etc. We conclude that the relations for harmonic sums which correspond to (2.29) are fully contained in the relations derived in the foregoing section.

Finally we note two interesting properties of the (shuffle) product of harmonic polylogarithms. We apply the differential and integral operator, resp., onto the difference of the shuffle product and the product of two harmonic polylogarithms.

$$\begin{aligned} \frac{d}{dx}\left[-H_{a_1,\ldots,a_n}(x)H_{b_1,\ldots,b_m}(x) + H_{a_1,\ldots,a_n}(x) ⧢ H_{b_1,\ldots,b_m}(x)\right] &= \\ -f(a_1;x)\left[H_{a_2,\ldots,a_n}(x)H_{b_1,\ldots,b_m}(x) - H_{a_2,\ldots,a_n}(x) ⧢ H_{b_1,\ldots,b_m}(x)\right] & \\ -f(b_1;x)\left[H_{a_1,\ldots,a_n}(x)H_{b_2,\ldots,b_m}(x) - H_{a_1,\ldots,a_n}(x) ⧢ H_{b_2,\ldots,b_m}(x)\right] &= 0 \ . \end{aligned} \tag{2.34}$$

In (2.34) it is assumed that the harmonic polylogarithm with no indices corresponds to the unit element $\mathsf{1}$. Differentiation for x maps the binary algebraic relations of degree (depth) $n + m$ to those of degree $n+m-1$. Conversely the application of the integral operator maps an algebraic relation of depth $n+m-1$ to a one of degree $n+m$ integrating (2.34) definitely,

$$
\begin{aligned}
\left[-H_{a_1,\ldots,a_n}(x)H_{b_1,\ldots,b_m}(x) + H_{a_1,\ldots,a_n}(x) \sqcup\!\sqcup\, H_{b_1,\ldots,b_m}(x)\right] &= \\
-\int_0^x dz f(a_1;z)\left[H_{a_2,\ldots,a_n}(z)H_{b_1,\ldots,b_m}(z) - H_{a_2,\ldots,a_n}(z) \sqcup\!\sqcup\, H_{b_1,\ldots,b_m}(z)\right] & \\
-\int_0^x dz f(b_1;z)\left[H_{a_1,\ldots,a_n}(z)H_{b_2,\ldots,b_m}(x) - H_{a_1,\ldots,a_n}(z) \sqcup\!\sqcup\, H_{b_2,\ldots,b_m}(z)\right] &= 0\,. \qquad (2.35)
\end{aligned}
$$

In this way differentiation and integration create downwards and upwards moves in the tree of shuffles of harmonic polylogarithms which may be used in the sense outlined above as general representations for shuffles of any type. The connection between harmonic polylogarithms and harmonic sums is easily established [13] by the Mellin transform of the former

$$
\mathbf{M}[H_{a_1,\ldots,a_n}(x)|_{reg}](N) = \int_0^1 dx\; x^{N-1} \;\; H_{a_1,\ldots,a_n}(x)|_{reg} \qquad (2.36)
$$

applying appropriate $+$-distribution regularizations, which are obtained such that the integral (2.36) exists. A unique definition is achieved writing each finite harmonic sum in terms of a linear combination of Mellin transforms of harmonic polylogarithms. The fact that the integration-by-parts relation does not result into new relations for harmonic polylogarithms holds therefore also for the finite harmonic sums.

2.3 Complete Permutations of the Index–Set

We now consider the sums of all finite harmonic sums of a given rank over a complete permutation of the index–set. This combination of harmonic sums can be represented as a polynomial of single harmonic sums and has therefore as well a simple analytic continuation to complex values of N in terms of ψ–functions and their derivatives. Up to rank 4 the corresponding relations were given before [3] and read

$$
\begin{aligned}
S_{a_1,a_2} + S_{a_2,a_1} &= S_{a_1}S_{a_2} + S_{a_1\wedge a_2}, \qquad [39] \qquad (2.37)\\
\sum_{\text{perm}\{a_1,a_2,a_3\}} S_{a_1,a_2,a_3} &= S_{a_1}S_{a_2}S_{a_3} + \sum_{\text{inv perm}} S_{a_1}S_{a_2\wedge a_3} + 2S_{a_1\wedge a_2\wedge a_3}, \qquad [40] \qquad (2.38)\\
\sum_{\text{perm}\{a_1,a_2,a_3,a_4\}} S_{a_1,a_2,a_3,a_4} &= S_{a_1}S_{a_2}S_{a_3}S_{a_4} + \sum_{\text{inv perm}} S_{a_1}S_{a_2}S_{a_3\wedge a_4} \\
&\quad + \sum_{\text{inv perm}} S_{a_1\wedge a_2}S_{a_3\wedge a_4} + 2\sum_{\text{inv perm}} S_{a_1}S_{a_2\wedge a_3\wedge a_4} + 6S_{a_1\wedge a_2\wedge a_3\wedge a_4}, \qquad [3]. \\
&\qquad (2.39)
\end{aligned}
$$

Here 'perm' denotes all permutations and 'inv perm' denotes all permutations in which a single index in a $\wedge$–contraction is only used once.

A general way to derive these relations consists in summing over the general relation Eq. (2.2) accounting for the weight factors. In the r.h.s. still polynomials of harmonic sums of degree larger than one are contained which have to be combined with the help of the complete permutation relations for the sums of lower rank. Finally one obtains a polynomial out of harmonic sums of rank one. The relations for the sums of depth 5 and 6 read [7]

$$
\begin{aligned}
\sum_{\text{perm}} S_{a_1,a_2,a_3,a_4,a_5} &= S_{a_1}S_{a_2}S_{a_3}S_{a_4}S_{a_5} + \sum_{\text{inv perm}} S_{a_1\wedge a_2}S_{a_3}S_{a_4}S_{a_5} \\
&\quad + \sum_{\text{inv perm}} S_{a_1\wedge a_2}S_{a_3\wedge a_4}S_{a_5} + 2\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3}S_{a_4}S_{a_5} \\
&\quad +2\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3}S_{a_4\wedge a_5} + 6\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3\wedge a_4}S_{a_5} \\
&\quad +24\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3\wedge a_4\wedge a_5} \qquad (2.40)\\
\sum_{\text{perm}} S_{a_1,a_2,a_3,a_4,a_5,a_6} &= S_{a_1}S_{a_2}S_{a_3}S_{a_4}S_{a_5}S_{a_6} + \sum_{\text{inv perm}} S_{a_1\wedge a_2}S_{a_3}S_{a_4}S_{a_5}S_{a_6} \\
&\quad + \sum_{\text{inv perm}} S_{a_1\wedge a_2}S_{a_3\wedge a_4}S_{a_5}S_{a_6} + \sum_{\text{inv perm}} S_{a_1\wedge a_2}S_{a_3\wedge a_4}S_{a_5\wedge a_6} \\
&\quad +2\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3}S_{a_4}S_{a_5}S_{a_6} + 2\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3}S_{a_4\wedge a_5}S_{a_6} \\
&\quad +4\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3}S_{a_4\wedge a_5\wedge a_6} + 6\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3\wedge a_4}S_{a_5}S_{a_6} \\
&\quad +6\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3\wedge a_4}S_{a_5\wedge a_6} + 24\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3\wedge a_4\wedge a_5}S_{a_6} \\
&\quad +120\sum_{\text{inv perm}} S_{a_1\wedge a_2\wedge a_3\wedge a_4\wedge a_5\wedge a_6}\ . \qquad (2.41)
\end{aligned}
$$

The construction recipe for the general case is now straightforward. Sum over all possible polynomial structures in $S_{p_i}, S_{p_i\wedge p_j}$ etc. We associate with an l_1-fold $\wedge$-contraction the weight factor $l = (l_1-1)!$. The respective weight factors of the corresponding product of single sums is the product of the weights l. The sum over the complete permutation of a given index set is associated to symmetric polynomials [44, 45].

From the above relations one obtains all k–fold finite harmonic sums with a single index,

$$S_{a,a} = \frac{1}{2}\left[S_a^2 + S_{2a}\right] \qquad (2.42)$$

$$S_{a,a,a} = \frac{1}{6}\left[S_a^3 + 3S_aS_{2a} + 2S_{3a}\right] \qquad (2.43)$$

$$S_{a,a,a,a} = \frac{1}{24}\left[S_a^4 + 6S_a^2 + S_{2a} + 3S_{2a}^2 + 8S_aS_{3a} + 6S_{4a}\right] \qquad (2.44)$$

$$S_{a,a,a,a,a} = \frac{1}{120}\left[S_a^5 + 10S_a^3S_{2a} + 20S_a^2S_{3a} + 30S_aS_{4a} + 15S_aS_{2a}^2 + 20S_{2a}S_{3a} + 24S_{5a}\right] \qquad (2.45)$$

$$
\begin{aligned}
S_{a,a,a,a,a,a} = \frac{1}{720}\Big[&S_a^6 + 15S_{2a}S_a^4 + 40S_{3a}S_a^3 + 90S_{4a}S_a^2 + 144S_aS_{5a} + 45S_a^2S_{2a}^2 \\
&+120S_aS_{2a}S_{3a} + 15S_{2a}^3 + 90S_{2a}S_{4a} + 40S_{3a}^2 + 120S_{6a}\Big]\ . \qquad (2.46)
\end{aligned}
$$

If $a < 0$, i.e. an alternating sum is considered, the symbol na is evaluated as $na = +|na|$ for n even and $na = -|na|$ for n odd. Eqs. (2.42–2.46) are equivalent to determinant relations discussed in Ref. [3] and were studied in different contexts as invariant theory of algebraic equations of

[7] These and the relations up to depth 10 were derived in [38].

various variables and integer sums in Refs. [35, 41] before. Similar type determinant–relations emerge in various aspects for symmetric polynomials, see e.g. [45].

3 The threefold sums revisited

In a previous investigation algebraic relations between the finite non–alternating harmonic sums were obtained by decomposing the sum

$$T = \sum_{k=1}^{N} \frac{1}{k^a} \sum_{l=1}^{k} \frac{1}{l^b} \sum_{m=1}^{k} \frac{1}{m^c} \tag{3.1}$$

by Borwein and Girgensohn Ref. [46]. T can be represented by four different decompositions out of which three relations between the finite harmonic sums S_{a_1,a_2,a_3}. [8] result. These relations were extended to the case of the finite harmonic sums with alternating indices in Ref. [3] and read

$$\begin{aligned}
S_{a,b,c} &= -S_{c,a,b} - S_{a,c,b} + S_c S_{a,b} + S_{c,a\wedge b} - S_c S_{a\wedge b} + S_{a\wedge b,c} + S_{a\wedge c,b} + S_{a,b\wedge c} - S_{a\wedge b\wedge c} && (3.2)\\
S_{b,a,c} &= S_{c,a,b} - S_c S_{a,b} - S_{c,a\wedge b} + S_c S_{a\wedge b} + S_b S_{a,c} + S_{b,a\wedge c} - S_b S_{a\wedge c} && (3.3)\\
S_{b,c,a} &= -S_{c,a,b} - S_{c,b,a} + S_c S_{a,b} + S_{c,a\wedge b} - S_c S_{a\wedge b} + S_{b\wedge c,a} + S_c S_{b,a} - S_b S_{a,c} + S_b S_{a\wedge c} && (3.4)
\end{aligned}$$

The partial permutation of indices induces the relation

$$S_{a,b,c} + S_{b,a,c} + S_{b,c,a} = S_a S_{b,c} + S_{a\wedge b,c} + S_{b,a\wedge c}\ . \tag{3.5}$$

The index structure of this equation can now be permuted leading to six equations. The coefficient matrix of this system of linear equations

$$M_{3a} = \left\| \begin{matrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{matrix} \right\| \tag{3.6}$$

is of rank 4. Due to this one more relation between the threefold finite harmonic sums than found in [46] is obtained in terms of polynomials of harmonic sums of lower rank. All sums with three different indices can be expressed by two chosen sums :

$$\begin{aligned}
S_{a,b,c} &= S_c S_{a,b} + S_{a,b\wedge c} - S_a S_{c,b} - S_{c,a\wedge b} + S_{c,b,a} && (3.7)\\
S_{a,c,b} &= -S_b S_{c,a} - S_{b\wedge c,a} + S_{b,c,a} + S_a S_{c,b} + S_{a\wedge c,b} && (3.8)\\
S_{b,a,c} &= S_{a\wedge b,c} - S_c S_{a,b} - S_{a,b\wedge c} + S_a S_{b,c} - S_{b,c,a} + S_{b,a\wedge c} + S_a S_{c,b} + S_{c,a\wedge b} - S_{c,b,a} && (3.9)\\
S_{c,a,b} &= S_b S_{c,a} + S_{b\wedge c,a} + S_{c,a\wedge b} - S_{b,c,a} - S_{c,b,a}\ . && (3.10)
\end{aligned}$$

The relations Eqs. (3.2–3.4) are contained in Eqs. (3.7–3.10) for three different indices. The corresponding harmonic sums up to weight 4 are expressed by the sums

[8] For the Zeta–values the complete algebraic relations were quoted in [46].

$$S_{-1,1,-2} = S_{-2,1,-1} + S_{-2}S_{-1,1} + S_{-1,-3} - S_{-1}S_{-2,1} - S_{-2,-2} \quad (3.11)$$
$$S_{1,-2,-1} = S_{-1,-2,1} + S_1 S_{-2,-1} + S_{-3,-1} - S_{-1}S_{-2,1} - S_{3,1} \quad (3.12)$$
$$\begin{aligned} S_{-2,-1,1} &= S_{3,1} + S_{1,3} - S_{-3,-1} - S_{-1,-3} + S_{-2,-2} - S_1 S_{-1,-2} + 2S_{-1}S_{-2,1} + S_{-1}S_{1,-2} \\ &\quad -S_1 S_{-2,-1} - S_{-2,1,-1} - S_{-1,-2,1} \end{aligned} \quad (3.13)$$

$$S_{1,-1,-2} = S_{-2}S_{1,-1} + S_{1,3} + S_{3,1} - S_{-2,1,-1} - S_{-1,-2,1} - S_1 S_{-2,-1} + S_{-1}S_{-2,1} \quad (3.14)$$
$$S_{-1,1,2} = S_2 S_{-1,1} + S_{-1,3} - S_{-1}S_{2,1} - S_{2,-2} + S_{2,1,-1} \quad (3.15)$$
$$S_{1,2,-1} = S_{-1,2,1} + S_1 S_{2,-1} + S_{3,-1} - S_{-1}S_{2,1} - S_{-3,1} \quad (3.16)$$
$$S_{2,-1,1} = S_{3,-1} + S_{2,-2} - S_{2,1,-1} - S_{-1,2,1} - S_{3,-1} + S_{-1}S_{2,1} + S_{-3,1} \quad (3.17)$$
$$\begin{aligned} S_{1,-1,2} &= -S_1 S_{2,-1} - S_{3,-1} + 2S_{-1}S_{2,1} + S_{-3,1} - S_{-1,2,1} + S_{-2,2} - S_2 S_{-1,1} - S_{-1,3} \\ &\quad +S_{-1}S_{1,2} + S_{1,-3} + S_{2,-2} - S_{2,1,-1}\ . \end{aligned} \quad (3.18)$$

Comparing with the representations of the harmonic sums in terms of MELLIN transforms Ref. [3], Eq. (39–102) one finds that the MELLIN transform of the functions

$$\frac{\log(1-x)}{1+x}\mathrm{Li}_2(x) \qquad \text{and} \qquad \frac{\log(1+x)}{x-1}\mathrm{Li}_2(-x) \quad (3.19)$$

earlier being counted to the set of basic functions, cf. [3, 25], can be expressed in terms of the algebraic relations (3.2–3.4). The set of basic functions needed up to two–loop order is thus reduced to at most 23 functions.

At given depth n the number of harmonic sums with k_i equal indices such that $\sum_i k_i = n$ is

$$n_{\mathrm{perm}}(\{a_1, \ldots, a_n\}) = \frac{n!}{\prod k_i!}\ . \quad (3.20)$$

For threefold harmonic sums with two different indices three sums emerge and the respective coefficient matrix reads

$$M_{3b} = \left\| \begin{array}{ccc} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{array} \right\|\ . \quad (3.21)$$

The corresponding system of linear equations is of rank 2. One obtains

$$S_{a,b,a} = -2S_{b,a,a} + S_a S_{b,a} + S_{a\wedge b,a} + S_{b,a\wedge a} \quad (3.22)$$
$$S_{a,a,b} = S_{b,a,a} - \frac{1}{2}\left[S_a S_{b,a} + S_{a\wedge b,a} + S_{b,a\wedge a} - S_a S_{a,b} - S_{a\wedge a,b} - S_{a,a\wedge b}\right]\ , \quad (3.23)$$

see Ref. [3].

In summary the threefold harmonic sums are characterized as follows

Index Set	Number	Dep. Sums of Depth 3	min. Weight	Fraction of fund. Sums
$\{a,a,a\}$	1	1	3	0
$\{a,a,b\}$	3	2	3	1/3
$\{a,b,c\}$	6	4	4	1/3

4 The Fourfold Harmonic Sums

Five types of fourfold sums emerge. Their characteristics is summarized in the subsequent table.

Index Set	Number	Dep. Sums of Depth 4	min. Weight	Fraction of fund. Sums
$\{a,a,a,a\}$	1	1	4	0
$\{a,a,a,b\}$	4	3	4	1/4
$\{a,a,b,b\}$	6	5	4	1/6
$\{a,a,b,c\}$	12	9	5	1/4
$\{a,b,c,d\}$	24	18	6	1/4

4.1 Harmonic Sums with 4 Different Indices

For this set of indices 24 different harmonic sums exist. As for the threefold harmonic sums one may write down the associated system of linear equations. The coefficient matrix has a size of 24×48. It is obtained considering all index permutations for Eqs. (2.19,2.20). The rank of the coefficient matrix is 18, i.e. 6 harmonic sums are chosen to express the remaining sums. Since none of the first 18 diagonal elements after bringing matrix into diagonal form vanishes we may use the last 6 sums as basic sums. Here and in the following we will not present the respective coefficient matrices being too large in size. One obtains the following relations :

$$\begin{aligned}
S_{a,b,c,d} &= -S_cS_{a,d,b} - S_cS_{a,b,d} - S_{c\wedge d,a,b} - S_cS_{d,a,b} + S_aS_{d,c,b} - S_{d,c,b,a} + S_{a\wedge d,b,c} \\
&\quad + S_dS_{a,b,c} + S_{a,b}S_{c,d} + S_{a\wedge d,c,b} - S_{a,c\wedge d,b} - S_{a,d,b\wedge c} - S_{d,a,b\wedge c} - S_{a\wedge c,b\wedge d} \\
&\quad + S_{a,c,b\wedge d} + S_{c,a\wedge d,b} + S_{c,a,b\wedge d} + S_{a,b\wedge d,c} + S_{d,c,a\wedge b} && (4.\text{-}1) \\
S_{a,b,d,c} &= S_cS_{a,d,b} + S_cS_{a,b,d} + S_{c\wedge d,a,b} + S_cS_{d,a,b} - S_aS_{d,c,b} + S_{d,c,b,a} - S_{a,b}S_{c,d} \\
&\quad + S_{d,a,b\wedge c} - S_{a\wedge d,b,c} - S_{a\wedge d,c,b} + S_{a,c\wedge d,b} - S_{d,a\wedge b,c} + S_{a,d,b\wedge c} + S_{a\wedge c,b\wedge d} \\
&\quad + S_{a,b,c\wedge d} - S_{a,c,b\wedge d} - S_{c,a\wedge d,b} - S_{c,a,b\wedge d} - S_{d,b,a\wedge c} - S_aS_{d,b,c} - S_{d,c,a\wedge b} \\
&\quad + S_{d,b,a,c} + S_{d,b,c,a} && (4.\text{-}3) \\
S_{a,c,b,d} &= S_dS_{a,c,b} - S_{a,d,b\wedge c} + S_{a\wedge d,b\wedge c} - S_{d,a,b\wedge c} - S_{a,c}S_{d,b} + S_{d,b,a\wedge c} + S_aS_{d,b,c} \\
&\quad + S_{a,c,b\wedge d} - S_{d,b,c,a} && (4.\text{-}3) \\
S_{a,c,d,b} &= -S_aS_{d,c,b} + S_{d,c,b,a} + S_{a,c\wedge d,b} + S_{a,d,b\wedge c} - S_{a\wedge d,b\wedge c} + S_{d,a,b\wedge c} + S_{a,c}S_{d,b} \\
&\quad - S_{d,a\wedge c,b} - S_{d,b,a\wedge c} - S_aS_{d,b,c} - S_{d,c,a\wedge b} + S_{d,c,a,b} + S_{d,b,c,a} && (4.\text{-}3) \\
S_{a,d,b,c} &= -S_{d,b,c,a} - S_{d,b,a,c} - S_{d,a,b,c} + S_aS_{d,b,c} + S_{a\wedge d,b,c} + S_{d,a\wedge b,c} + S_{d,b,a\wedge c} && (4.\text{-}2) \\
S_{a,d,c,b} &= -S_{d,c,b,a} - S_{d,c,a,b} - S_{d,a,c,b} + S_aS_{d,c,b} + S_{a\wedge d,c,b} + S_{d,a\wedge c,b} + S_{d,c,a\wedge b} && (4.\text{-}1) \\
S_{b,a,c,d} &= S_cS_{a,d,b} + S_cS_{a,b,d} + S_{c\wedge d,a,b} + S_cS_{d,a,b} - S_dS_{a,c,b} - S_bS_{c,a,d} - S_bS_{c,d,a} \\
&\quad - S_dS_{a,b,c} - S_{a,b}S_{c,d} + S_{d,a,b\wedge c} - S_{a\wedge d,b,c} - S_{a\wedge d,c,b} + S_aS_{b,c,d} + S_aS_{c,b,d} \\
&\quad + S_{a,d,b\wedge c} + S_{d,a\wedge c,b} + S_{a\wedge c,b\wedge d} - S_{a,b\wedge d,c} + S_aS_{c,d,b} + S_{c,b,a\wedge d} + S_{a\wedge c,d,b} \\
&\quad + S_{a\wedge b,c,d} + S_{b,a\wedge c,d} + S_{b,c,a\wedge d} - 2\,S_{a,c,b\wedge d} - 2\,S_{c,a,b\wedge d} + S_{a\wedge c,b,d} - S_{b\wedge c,a,d} \\
&\quad - S_{b\wedge c,d,a} - S_{c,b\wedge d,a} - S_{d,c,a,b} && (4.\text{-}4) \\
S_{b,a,d,c} &= -S_cS_{a,d,b} - S_cS_{a,b,d} - S_{c\wedge d,a,b} - S_cS_{d,a,b} + S_{a,b}S_{c,d} - S_{a,c\wedge d,b} + S_bS_{a,d,c} \\
&\quad - S_{d,a,b\wedge c} - S_{d,a\wedge c,b} - S_{a\wedge c,b\wedge d} + S_{a,b\wedge d,c} - S_{a,b,c\wedge d} + S_{a\wedge b,d,c} + S_{a,c,b\wedge d}
\end{aligned}$$

$$
\begin{aligned}
& +S_{c,a\wedge d,b}+S_{c,a,b\wedge d}+S_{d,a,b,c}+S_{d,c,a,b}+S_{d,a,c,b} && (4.\text{-}5)\\
S_{b,c,a,d} \;=\;& S_cS_{a,d,b}+S_cS_{a,b,d}+S_{c\wedge d,a,b}+S_cS_{d,a,b}+S_dS_{a,c,b}+S_bS_{c,a,d}+S_bS_{c,d,a} \\
& -S_{a,b}S_{c,d}-S_{a\wedge d,b,c}-S_aS_{b,c,d}-S_{a,b}S_{d,c}-S_aS_{c,b,d}+2\,S_{a,c\wedge d,b}-S_bS_{a,d,c} && (4.\text{-}5)\\
& -S_{a,d,b\wedge c}+S_{a\wedge c,b\wedge d}-S_{a,b\wedge d,c}+S_{a\wedge d,b\wedge c}+S_{b,a,c\wedge d}-S_{a\wedge b,c\wedge d}-S_aS_{c,d,b} \\
& +2\,S_{a,b,c\wedge d}-S_{c,b,a\wedge d}-S_{a\wedge c,d,b}-S_{b,c,a\wedge d}-2\,S_{c,a\wedge d,b}-S_{a\wedge c,b,d}+S_{b\wedge c,a,d} \\
& +S_{b\wedge c,d,a}+S_{c,b\wedge d,a}+S_{a,d}S_{b,c}-S_{d,a,c,b} && (4.\text{-}6)\\
S_{d,b,a,c} \;=\;& -S_cS_{a,d,b}-S_cS_{a,b,d}-S_{c\wedge d,a,b}-S_cS_{d,a,b}-S_aS_{d,c,b}+S_{d,c,b,a}+S_{a,b}S_{c,d} \\
& +S_{a\wedge d,b,c}+S_aS_{b,c,d}+S_{a,b}S_{d,c}-S_{a,c\wedge d,b}+S_bS_{a,d,c}+S_{a,d,b\wedge c}-S_{d,a\wedge c,b} \\
& -S_{a\wedge c,b\wedge d}+S_{a,b\wedge d,c}-S_{a\wedge d,b\wedge c}-S_{b,a,c\wedge d}+S_{a\wedge b,c\wedge d}-2\,S_{a,b,c\wedge d}+S_{b,c,a\wedge d} \\
& +S_{a,c,b\wedge d}+S_{c,a\wedge d,b}+S_{c,a,b\wedge d}-S_{a,d}S_{b,c}-S_{d,c,a\wedge b}+S_{d,c,a,b}+S_{d,a,c,b} && (4.\text{-}8)\\
S_{b,d,a,c} \;=\;& S_{b\wedge d,a,c}+S_bS_{d,a,c}+S_{d,a,b\wedge c}+S_{d,a\wedge b,c}-S_{d,a,b,c}-S_{d,a,c,b}-S_{d,b,a,c} && (4.\text{-}7)\\
S_{b,d,c,a} \;=\;& -S_{d,c,b,a}-S_{d,c,a,b}-S_{d,b,c,a}-S_bS_{d,a,c}-S_{b\wedge d,a,c}-S_{d,a,b\wedge c}-S_bS_{a,d,c} \\
& -S_{a,b\wedge d,c}-S_{a,d,b\wedge c}+S_aS_{b,d,c}+S_{b,a\wedge d,c}+S_{b,d,a\wedge c}+S_aS_{d,b,c}+S_{a\wedge d,b,c} \\
& +S_{d,b,a\wedge c}+S_aS_{d,c,b}+S_{a\wedge d,c,b}+S_{d,a\wedge c,b}+S_{d,c,a\wedge b} && (4.\text{-}8)\\
S_{c,a,b,d} \;=\;& S_cS_{a,b,d}-S_dS_{a,c,b}+S_{a\wedge c,b,d}-S_dS_{a,b,c}+S_{a,d,b\wedge c}-S_{a\wedge d,b\wedge c}+S_{d,a\wedge b,c} \\
& +S_{d,a,b\wedge c}+S_{a,c}S_{d,b}+S_{a,b\wedge c,d}-S_{a,b\wedge d,c}-S_{a,c,b\wedge d}-S_{d,b,a,c} && (4.\text{-}8)\\
S_{c,a,d,b} \;=\;& S_cS_{a,d,b}+S_{a\wedge c,d,b}-S_{a\wedge d,b,c}-S_{a\wedge d,c,b}+S_{a\wedge d,b\wedge c}-S_{d,a\wedge b,c}-S_{d,a,b\wedge c} \\
& -S_{a,c}S_{d,b}+S_{d,a,b,c}+S_{d,a,c,b}+S_{d,b,a,c} && (4.\text{-}8)\\
S_{c,b,a,d} \;=\;& -2\,S_cS_{a,d,b}-2\,S_cS_{a,b,d}-S_{c\wedge d,a,b}-S_cS_{d,a,b}-S_bS_{c,d,a}+S_dS_{a,b,c}+S_{a,b}S_{c,d} \\
& +2\,S_{a\wedge d,b,c}+S_{a\wedge d,c,b}+S_aS_{b,c,d}+S_{a,b}S_{d,c}+S_aS_{c,b,d}-2\,S_{a,c\wedge d,b}+S_bS_{a,d,c} \\
& -S_{a\wedge c,b\wedge d}+2\,S_{a,b\wedge d,c}-S_{a\wedge d,b\wedge c}-S_{b,a,c\wedge d}+S_{a\wedge b,c\wedge d}+S_aS_{c,d,b}-2\,S_{a,b,c\wedge d} \\
& +S_{c,a\wedge b,d}+S_{c,b,a\wedge d}+S_{b,c,a\wedge d}-S_{a,b\wedge c,d}+S_{a,c,b\wedge d}+2\,S_{c,a\wedge d,b}+S_{c,a,b\wedge d} \\
& -S_{b\wedge c,d,a}-S_{c,b\wedge d,a}-S_{a,d}S_{b,c}-S_{d,a,b,c} && (4.\text{-}11)\\
S_{c,b,d,a} \;=\;& 2\,S_cS_{a,d,b}+S_cS_{a,b,d}+S_{c\wedge d,a,b}+S_cS_{d,a,b}+S_bS_{c,d,a}-S_{a,b}S_{c,d}-2\,S_{a\wedge d,b,c} \\
& -S_{a\wedge d,c,b}-S_aS_{b,c,d}-S_{a,b}S_{d,c}+2\,S_{a,c\wedge d,b}-S_bS_{a,d,c}-S_{d,a\wedge b,c}+S_{a\wedge c,b\wedge d} \\
& -S_{a,b\wedge d,c}+S_{a\wedge d,b\wedge c}+S_{b,a,c\wedge d}-S_{a\wedge b,c\wedge d}-S_aS_{c,d,b}+2\,S_{a,b,c\wedge d}-S_{b,c,a\wedge d} \\
& -S_{a,c,b\wedge d}-2\,S_{c,a\wedge d,b}-S_{c,a,b\wedge d}+S_{b\wedge c,d,a}+S_{c,b\wedge d,a}-S_{d,b,a\wedge c}-S_aS_{d,b,c} \\
& +S_{a,d}S_{b,c}+S_{d,a,b,c}+S_{d,b,a,c}+S_{d,b,c,a} && (4.\text{-}14)\\
S_{c,d,a,b} \;=\;& S_{c\wedge d,a,b}+S_cS_{d,a,b}+S_{d,a\wedge c,b}+S_{d,a,b\wedge c}-S_{d,a,b,c}-S_{d,c,a,b}-S_{d,a,c,b} && (4.\text{-}13)\\
S_{c,d,b,a} \;=\;& -S_{d,c,b,a}-S_{d,b,c,a}-S_{d,b,a,c}-S_cS_{d,a,b}-S_{c\wedge d,a,b}-S_{d,a,b\wedge c}-S_cS_{a,d,b} \\
& -S_{a,c\wedge d,b}-S_{a,d,b\wedge c}+S_aS_{c,d,b}+S_{a\wedge d,b,c}+S_{c,a\wedge d,b}+S_{c,d,a\wedge b}+S_aS_{d,b,c}+S_{a\wedge d,c,b} \\
& +S_{d,a\wedge b,c}+S_{d,b,a\wedge c}+S_aS_{d,c,b}+S_{d,c,a\wedge b}\;. && (4.\text{-}14)
\end{aligned}
$$

Harmonic sums of this type occur for the first time at the level of weight 6.

4.2 Harmonic Sums with 3 Different Indices

This class contains 12 different sums. The coefficient matrix is M_{4b} is of rank 9 and again we may choose the last 3 harmonic sums to express the remaining 9. The relations for the sums are

$$
\begin{aligned}
S_{a,a,b,c} \;=\;& \frac{1}{2}\left[S_{a,a\wedge c,b}-S_aS_{a,c,b}-S_aS_{c,a,b}+S_aS_{c,b,a}-S_{c,a,a\wedge b}\right. \\
& \left.+S_{a\wedge c,a,b}+S_{a\wedge c,b,a}-S_{a\wedge a,c,b}-S_{a,c,a\wedge b}+S_{c,b,a\wedge a}-S_{c,a\wedge a,b}\right]+S_cS_{a,a,b}
\end{aligned}
$$

$$
\begin{aligned}
& -S_{c,b,a,a}+S_{a,a,b\wedge c}+\frac{1}{2}S_{c,a\wedge b,a} && (4.\text{-}15)\\
S_{a,a,c,b} &= -\frac{1}{2}S_{a\wedge c,a,b}+S_{c,b,a,a}+S_{c,a,b,a}+S_{c,a,a,b}-\frac{1}{2}\left[S_aS_{c,b,a}+S_{a\wedge c,a,b}+S_{c,a\wedge b,a}\right.\\
&\quad +S_{c,b,a\wedge a}-S_aS_{a,c,b}-S_{a\wedge a,c,b}-S_{a,a\wedge c,b}-S_{a,c,a\wedge b}+S_aS_{c,a,b}+S_{c,a\wedge a,b}\\
&\quad \left.+S_{c,a,a\wedge b}\right] && (4.\text{-}16)\\
S_{a,b,a,c} &= S_{a\wedge a,b,c}+S_{a,b}S_{a,c}-S_{a,a\wedge c,b}-S_{a\wedge a,b\wedge c}-S_{a\wedge c,a,b}+S_{c,a,a\wedge b}+S_{c,a\wedge a,b}\\
&\quad +S_aS_{c,a,b}+S_{a,a\wedge b,c}+S_{a\wedge a,c,b}-2\,S_cS_{a,a,b}-S_{c,a,b,a} && (4.\text{-}16)\\
S_{a,b,c,a} &= S_aS_{a,b,c}+S_{a,b,a\wedge c}+S_aS_{a,c,b}-S_{a,b}S_{a,c}+S_{a,c,a\wedge b}+S_{a\wedge a,b\wedge c}-S_{a\wedge c,b,a}\\
&\quad -S_{c,a\wedge b,a}-2\,S_{a,a,b\wedge c}-S_aS_{c,b,a}-S_{c,b,a\wedge a}+S_{c,a,b,a}+2\,S_{c,b,a,a} && (4.\text{-}16)\\
S_{a,c,a,b} &= -S_{c,a,b,a}-2\,S_{c,a,a,b}+S_aS_{c,a,b}+S_{a\wedge c,a,b}+S_{c,a\wedge a,b}+S_{c,a,a\wedge b} && (4.\text{-}15)\\
S_{a,c,b,a} &= -2\,S_{c,b,a,a}-S_{c,a,b,a}+S_aS_{c,b,a}+S_{a\wedge c,b,a}+S_{c,a\wedge b,a}+S_{c,b,a\wedge a} && (4.\text{-}14)\\
S_{b,a,a,c} &= -S_{c,a,a,b}-S_{a,b}S_{a,c}+\frac{1}{2}\left[S_aS_{a,b,c}+S_aS_{a,c,b}+S_aS_{b,a,c}-S_aS_{b,c,a}-S_aS_{c,a,b}-S_aS_{c,b,a}\right]\\
&\quad +S_bS_{c,a,a}+S_cS_{a,a,b}-S_{a,a,b\wedge c}-\frac{1}{2}\left[S_{c,a\wedge a,b}+S_{a\wedge a,c,b}+S_{c,b,a\wedge a}-S_{a,c,a\wedge b}\right.\\
&\quad \left.-S_{c,a\wedge b,a}-S_{c,a,a\wedge b}-S_{a,a\wedge c,b}-S_{a\wedge c,a,b}+S_{a\wedge c,b,a}+S_{a,a\wedge b,c}+S_{a\wedge a,b,c}\right]\\
&\quad +S_{a\wedge a,b\wedge c}+\frac{1}{2}\left[S_{a,b,a\wedge c}-S_{b,c,a\wedge a}+S_{b,a\wedge a,c}+S_{a\wedge b,a,c}-S_{b,a\wedge c,a}+S_{b,a,a\wedge c}\right.\\
&\quad \left.-S_{a\wedge b,c,a}\right]+S_{b\wedge c,a,a} && (4.\text{-}17)\\
S_{b,a,c,a} &= -S_aS_{a,b,c}-S_{a,b,a\wedge c}-S_aS_{a,c,b}+S_aS_{b,c,a}+2\,S_{c,a,a,b}+S_{a\wedge b,c,a}+S_{b,c,a\wedge a}\\
&\quad +S_{a,b}S_{a,c}+S_{b,a\wedge c,a}-S_{a,c,a\wedge b}-S_{a\wedge a,b\wedge c}+S_{a\wedge c,b,a}-S_{c,a\wedge b,a}-2\,S_{c,a,a\wedge b}\\
&\quad +2\,S_{a,a,b\wedge c}+S_aS_{c,b,a}+S_{c,b,a\wedge a}-2\,S_{b\wedge c,a,a}-2\,S_bS_{c,a,a}+S_{c,a,b,a} && (4.\text{-}18)\\
S_{b,c,a,a} &= -S_{c,b,a,a}-S_{c,a,b,a}-S_{c,a,a,b}+S_bS_{c,a,a}+S_{b\wedge c,a,a}+S_{c,a\wedge b,a}+S_{c,a,a\wedge b}\ . && (4.\text{-}17)
\end{aligned}
$$

$$
M_{4b}=\left\|\begin{array}{cccccccccccc}
2&0&1&1&0&0&0&0&0&0&0&0\\
0&2&0&0&1&1&0&0&0&0&0&0\\
0&0&1&0&0&0&2&1&0&0&0&0\\
0&0&0&1&0&0&0&1&2&0&0&0\\
0&0&0&0&1&0&0&0&0&2&1&0\\
0&0&0&0&0&1&0&0&0&0&1&2\\
1&1&1&0&0&0&1&0&0&0&0&0\\
0&0&0&1&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1\\
1&1&0&0&1&0&0&0&0&1&0&0\\
0&0&1&1&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&1&1&1&0&0&1\\
1&0&1&1&0&0&1&1&1&0&0&0\\
0&1&0&0&1&1&0&0&0&1&1&1\\
2&2&1&0&1&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&2&1&0\\
2&2&1&0&1&0&0&0&0&0&0&0\\
0&0&1&1&0&1&2&1&0&0&0&0\\
0&0&1&1&0&1&2&1&0&0&0&0\\
0&0&0&0&0&0&0&1&2&0&1&2\\
1&0&1&1&0&0&1&1&1&0&0&0\\
0&0&0&1&1&1&0&0&0&2&1&0\\
0&0&0&0&0&0&0&1&2&0&1&2\\
0&1&0&0&1&1&0&0&0&1&1&1
\end{array}\right\| \qquad (4.\text{-}16)
$$

4.3 Harmonic Sums with 2 Different Indices

In this class all harmonic sums can be expressed by a single sum of the respective depth and weight.

4.3.1 Index-Set $\{a, a, b, b\}$

The class contains 6 different sums. The coefficient matrix results from Eqs. (2.19,2.20) and is given by

$$M_{4c} = \left\| \begin{array}{cccccc} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right\| \qquad (4.\text{-}16)$$

It is of rank 5. The following relations are obtained :

$$\begin{aligned} S_{a,a,b,b} &= -S_{b,b,a,a} - \frac{1}{2}\left[S_a S_{b,a,b} - S_a S_{b,b,a} - S_{b,b,a\wedge a} + S_{a,b,a\wedge b} + S_{a\wedge a,b,b} + S_a S_{a,b,b}\right. \\ &\quad \left. + S_{b,a\wedge a,b} - S_{a\wedge b,a,b} - S_{a\wedge b,b,a} - S_{b,a\wedge b,a} + S_{b,a,a\wedge b} - S_{a,a\wedge b,b}\right] + S_{a,a,b\wedge b} \\ &\quad + S_b S_{a,a,b} & (4.\text{-}17) \\ S_{a,b,a,b} &= \frac{1}{2}S_{a,b}^2 - 2\, S_b S_{a,a,b} + 2\, S_{a\wedge a,b,b} - \frac{1}{2}S_{a\wedge a,b\wedge b} - S_{a\wedge b,a,b} - S_{a,a,b\wedge b} + 2\, S_{b,b,a,a} \\ &\quad + S_{a,b,a\wedge b} + S_a S_{a,b,b} - S_{b,b,a\wedge a} - S_a S_{b,b,a} - S_{a\wedge b,b,a} - S_{b,a\wedge b,a} + S_a S_{b,a,b} \\ &\quad + S_{b,a\wedge a,b} + S_{b,a,a\wedge b} & (4.\text{-}18) \\ S_{a,b,b,a} &= -S_{a,a,b\wedge b} - \frac{1}{2}S_{a,b}^2 + S_{a,b,a\wedge b} + \frac{1}{2}S_{a\wedge a,b\wedge b} + S_a S_{a,b,b} & (4.\text{-}17) \\ S_{b,a,a,b} &= -\frac{1}{2}S_{a,b}^2 + S_b S_{a,a,b} - S_{a\wedge a,b,b} + \frac{1}{2}S_{a\wedge a,b\wedge b} + S_{a\wedge b,a,b} & (4.\text{-}16) \\ S_{b,a,b,a} &= S_{a,a,b\wedge b} - 2\, S_{b,b,a,a} + \frac{1}{2}\left[S_{a,b}^2 - S_{a\wedge a,b\wedge b}\right] - S_{a,b,a\wedge b} - S_a S_{a,b,b} + S_{b,b,a\wedge a} \\ &\quad + S_a S_{b,b,a} + S_{a\wedge b,b,a} + S_{b,a\wedge b,a} & (4.\text{-}16) \end{aligned}$$

Depending on the particular structure of the integrals emerging to a given weight it may be useful to choose a different sum than used in (4.-15–4.-15) to express the remaining sums. Up to depth 4 only the weight 4 Mellin transform of

$$\left[\frac{\log^3(1+x)}{x-1}\right]_+ \qquad (4.\text{-}15)$$

contributes. It is related to the harmonic sum $S_{-1,1,1,-1}(N)$.

4.3.2 Index-Set $\{a,a,a,b\}$

This class contains 4 different sums. The coefficient matrix reads

$$M_{4b} = \left\Vert \begin{array}{cccc} 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right\Vert \tag{4.-15}$$

and is of rank 3. All sums can be represented in terms of one. The sums are given by

$$S_{a,b,a,a} = -3S_{b,a,a,a} + S_a S_{b,a,a} + S_{a\wedge b,a,a} + S_{b,a\wedge a,a} + S_{b,a,a\wedge a} \tag{4.-14}$$

$$S_{a,a,b,a} = \frac{1}{2}\left[S_a S_{a,b,a} + S_{a\wedge a,b,a} + S_{a,a\wedge b,a} + S_{a,b,a\wedge a}\right] - S_{a,b,a,a} \tag{4.-13}$$

$$S_{a,a,a,b} = \frac{1}{3}\left[S_a S_{a,a,b} + S_{a\wedge a,a,b} + S_{a,a\wedge a,b} + S_{a,a,a\wedge b} - S_{a,a,b,a}\right] \tag{4.-12}$$

recursively. Up to weight 4 eight sums emerge, which can be all expressed in terms of Mellin polynomials of functions of lower depth and the Mellin transforms of

$$\frac{\log^3(1+x)}{1+x} \tag{4.-11}$$

$$\frac{1}{1+x}\left[\mathrm{Li}_2\left(\frac{1-x}{2}\right)\log\left(\frac{1+x}{4}\right) + 2\mathrm{S}_{1,2}\left(\frac{1-x}{2}\right)\right] - \frac{1}{1+x}\log(2)\log(1-x)\log\left(\frac{1+x}{2}\right) \tag{4.-10}$$

The former function is related to $S_{-1,1,1,1}(N)$ and the latter to $S_{1,-1,-1,-1}(N)$. Note also that the Mellin transforms of the functions $\log^3(1+x)/(1+x)$ and $\log^3(1-x)/(1+x)$ differ only by Mellin polynomials containing functions of lower weight and can be related directly since the associated harmonic sums belong to the same index class.

The functions which represent through Mellin transforms the fourfold harmonic sums of weight 4 depending only on one index (+1 or –1) up to functions of lower rank read :

$$\left\{\frac{1}{x-1}\left[\mathrm{Li}_2\left(\frac{1-x}{2}\right)\log\left(\frac{1+x}{4}\right) + 2\mathrm{S}_{1,2}\left(\frac{1-x}{2}\right)\right] - \frac{1}{x-1}\log(2)\log(1-x)\log\left(\frac{1+x}{2}\right)\right\}_+ \tag{4.-9}$$

$$\left[\frac{\log^3(1-x)}{1-x}\right]_+ \quad . \tag{4.-8}$$

It is interesting to note that the Mellin transforms of these functions can be expressed completely in terms of polynomials of $\psi^k(N)$ and $\beta^{(k)}(N) = (1/2)d^k/dx^k\left[\psi((1+x)/2) - \psi(x/2)\right]$ functions. Finally we mention that instead of (4.-10) one may use

$$\frac{1}{1+x}\left[2\mathrm{Li}_3\left(\frac{1-x}{2}\right) - \ln(1-x)\mathrm{Li}_2\left(\frac{1-x}{2}\right)\right] \tag{4.-7}$$

as basic function which corresponds to $S_{-1,-1,-1,1}(N)$ which yields a shorter expression.

5 The Fivefold Harmonic Sums

In the Table below we specify the different contributing index sets w.r.t. their multiplicity and the level of minimal depth at which they occur.

Index Set	Number	Dep. Sums of Depth 5	min. Weight	Fraction of fund. Sums
$\{a,a,a,a,a\}$	1	1	5	0
$\{a,a,a,a,b\}$	5	4	5	1/5
$\{a,a,a,b,b\}$	10	8	5	1/5
$\{a,a,a,b,c\}$	20	16	6	1/5
$\{a,a,b,b,c\}$	30	24	6	1/5
$\{a,a,b,c,d\}$	60	48	7	1/5
$\{a,b,c,d,e\}$	120	96	9	1/5

The harmonic sums of the type $S_{a,a,b,c,d}(N)$ and $S_{a,b,c,d,e}(N)$ do contribute only at weight 7 or 9 and higher, respectively. Here we list only the number of of independent sums, which is 12 and 24. Since the expressions of the dependent sums in terms of the respective basis become voluminous beginning with the fivefold sums we will give the complete result only for the simpler cases and present one representative relation for the complicated cases. The complete result can be obtained instead from `http://www.desy.de/~blumlein`. Likewise we will present the coefficient matrices only for the simpler cases in explicit form.

5.1 Harmonic Sums with 3 Different Indices

5.1.1 Index-Set $\{a,a,b,b,c\}$

This class contains 30 sums. It turns out that six of the sums are independent. Due to the structure of the coefficient matrix one may choose the latter six sums. As a typical example $S_{a,a,b,b,c}$ reads

$$\begin{aligned}
S_{a,a,b,b,c} = & -\frac{1}{2}S_aS_{a,b,c,b} - \frac{1}{2}S_aS_{b,c,a,b} - \frac{1}{2}S_{b,a,c,a\wedge b} + \frac{1}{2}S_{b,c,a\wedge b,a} - \frac{1}{2}S_{a,b,c,a\wedge b} \\
& + \frac{1}{2}S_{a,a\wedge b,c,b} - \frac{1}{2}S_{b,a\wedge a,c,b} + \frac{1}{2}S_{b,c,b,a\wedge a} - \frac{1}{2}S_{c,a,b,a\wedge b} - \frac{1}{2}S_{b,c,a,a\wedge b} \\
& - \frac{1}{2}S_{a\wedge a,b,c,b} - \frac{1}{2}S_{a\wedge a,c,b,b} + \frac{1}{2}S_{c,b,b,a\wedge a} - \frac{1}{2}S_{b,c,a\wedge a,b} - \frac{1}{2}S_aS_{c,a,b,b} \\
& + \frac{1}{2}S_{c,b,a\wedge a,b} - \frac{1}{2}S_{c,a\wedge a,b,b} + S_{b,a,a,b\wedge c} - S_{b\wedge c,b,a,a} - S_{a\wedge b,a,b\wedge c} \\
& - S_{a,a\wedge b,b\wedge c} + \frac{1}{2}S_{a\wedge c,b,b,a} - S_{a\wedge b,a\wedge c,b} + \frac{1}{2}S_{a,b,a\wedge c,b} + \frac{1}{2}S_{a\wedge c,b,a,b} \\
& + \frac{1}{2}S_{a,a\wedge c,b,b} + \frac{1}{2}S_{b,a\wedge c,a,b} + \frac{1}{2}S_{b,a\wedge c,b,a} + \frac{1}{2}S_{a\wedge b,c,b,a} + \frac{1}{2}S_{c,a\wedge b,a,b} \\
& + \frac{1}{2}S_{a\wedge b,a,c,b} + \frac{1}{2}S_{a\wedge b,c,a,b} + \frac{1}{2}S_{c,a\wedge b,b,a} - \frac{1}{2}S_{a,c,a\wedge b,b} - \frac{1}{2}S_{a,c,b,a\wedge b}
\end{aligned}$$

$$-\frac{1}{2}\,S_{c,b,a\wedge b,a}-\frac{1}{2}\,S_{c,b,a,a\wedge b}+S_{a,b,a,b\wedge c}+S_{a,a,b\wedge c,b}-\frac{1}{2}\,S_aS_{a,c,b,b}$$

$$+\frac{1}{2}\,S_aS_{b,c,b,a}-S_{c,b\wedge b,a,a}+S_{a,a,b,b\wedge c}+S_{b,c}S_{a,a,b}-\frac{1}{2}\,S_aS_{b,a,c,b} \qquad (5.\text{-}7)$$

$$-\frac{1}{2}\,S_{c,a,a\wedge b,b}+\frac{1}{2}\,S_{b,a,a\wedge c,b}+\frac{1}{2}\,S_{a\wedge c,a,b,b}+\frac{1}{2}\,S_aS_{c,b,b,a}+\frac{1}{2}\,S_aS_{c,b,a,b}$$

$$-S_bS_{a,a,b,c}-S_bS_{c,b,a,a}+S_cS_{a,a,b,b}+S_{c,b,b,a,a} \qquad (5.\text{-}7)$$

5.1.2 Index-Set $\{a,a,a,b,c\}$

This class contains 20 sums. Four sums are independent. The expression for $S_{a,a,a,b,c}$ is shown as an example.

$$\begin{aligned}
S_{a,a,a,b,c} \;=\; & S_cS_{a,a,a,b}+S_{c,b,a,a,a}+\frac{1}{6}\,S_aS_{a,c,b,a}+\frac{1}{6}\,S_aS_{c,a,b,a}-\frac{1}{3}\,S_aS_{c,b,a,a}\\
&-\frac{1}{3}\left[S_aS_{c,a,a,b}+S_aS_{a,c,a,b}+S_aS_{a,a,c,b}+S_{c,b,a\wedge a,a}\right]+\frac{1}{6}\,S_{a\wedge a,c,b,a}\\
&-\frac{1}{3}\left[S_{a\wedge a,c,a,b}+S_{c,b,a,a\wedge a}+S_{a,a\wedge a,c,b}+S_{a\wedge a,a,c,b}+S_{c,a\wedge a,a,b}\right]\\
&+\frac{1}{6}\,S_{c,a\wedge a,b,a}-\frac{1}{3}\left[S_{c,a,a\wedge a,b}+S_{a,c,a\wedge a,b}\right]+\frac{1}{6}\left[S_{a,c,b,a\wedge a}+S_{c,a,b,a\wedge a}\right]\\
&-\frac{1}{3}\,S_{c,a\wedge b,a,a}+\frac{1}{6}\,S_{a,c,a\wedge b,a}-\frac{1}{3}\left[S_{a,c,a,a\wedge b}+S_{c,a,a,a\wedge b}+S_{a,a,c,a\wedge b}\right]\\
&+\frac{1}{6}\left[S_{c,a,a\wedge b,a}+S_{a,a\wedge c,b,a}\right]+\frac{1}{3}\left[2S_{a,a\wedge c,a,b}-S_{a\wedge c,b,a,a}+2S_{a\wedge c,a,a,b}\right]\\
&+\frac{1}{6}\,S_{a\wedge c,a,b,a}+\frac{2}{3}\,S_{a,a,a\wedge c,b}+S_{a,a,a,b\wedge c} \qquad (5.\text{-}12)
\end{aligned}$$

5.2 Harmonic Sums with 2 Different Indices

5.2.1 Index-Set $\{a,a,a,b,b\}$

This class contains 10 sums. The coefficient matrix given in Eq. (5.2.1) is of rank 8. All sums can be represented in terms of two. For $S_{a,a,a,b,b}$ one obtains :

$$\begin{aligned}
S_{a,a,a,b,b} \;=\; & S_{b,b,a,a,a}+S_bS_{a,a,a,b}+\frac{1}{6}\,S_aS_{a,b,b,a}-\frac{1}{3}\,S_aS_{b,b,a,a}-\frac{1}{3}\,S_aS_{a,a,b,b}\\
&-\frac{1}{3}\,S_aS_{a,b,a,b}-\frac{1}{3}\,S_aS_{b,a,a,b}+\frac{1}{6}\,S_aS_{b,a,b,a}-\frac{1}{3}\,S_{a\wedge a,a,b,b}-\frac{1}{3}\,S_{a,a\wedge a,b,b}\\
&+\frac{1}{6}\,S_{a\wedge a,b,b,a}-\frac{1}{3}\,S_{b,a,a\wedge a,b}+\frac{1}{6}\,S_{a,b,b,a\wedge a}-\frac{1}{3}\,S_{a\wedge a,b,a,b}-\frac{1}{3}\,S_{b,a\wedge a,a,b}\\
&+\frac{1}{6}\,S_{b,a\wedge a,b,a}-\frac{1}{3}\,S_{a,b,a\wedge a,b}-\frac{1}{3}\,S_{b,b,a,a\wedge a}+\frac{1}{6}\,S_{b,a,b,a\wedge a}-\frac{1}{3}\,S_{b,b,a\wedge a,a}\\
&+\frac{2}{3}\,S_{a,a,a\wedge b,b}-\frac{1}{3}\,S_{a,a,b,a\wedge b}-\frac{1}{3}\,S_{a,b,a,a\wedge b}+\frac{2}{3}\,S_{a,a\wedge b,a,b}+\frac{1}{6}\,S_{a,a\wedge b,b,a}\\
&-\frac{1}{3}\,S_{b,a,a,a\wedge b}+\frac{1}{6}\,S_{a,b,a\wedge b,a}-\frac{1}{3}\,S_{a\wedge b,b,a,a}-\frac{1}{3}\,S_{b,a\wedge b,a,a}+\frac{2}{3}\,S_{a\wedge b,a,a,b}\\
&+\frac{1}{6}\,S_{a\wedge b,a,b,a}+\frac{1}{6}\,S_{b,a,a\wedge b,a}+S_{a,a,a,b\wedge b} \qquad (5.\text{-}17)
\end{aligned}$$

$$M_{5c} = \left\| \begin{array}{cccccccccc} 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 3 \\ 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 3 & 3 \\ 6 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 3 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 5 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 1 & 1 \end{array} \right\| \qquad (5.\text{-}17)$$

5.2.2 Index-Set $\{a, a, a, a, b\}$

Finally this class contains 5 different sums. The coefficient matrix is

$$M_{5d} = \left\| \begin{array}{ccccc} 4 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right\| \qquad (5.\text{-}17)$$

and has rank 4. The sums are given by

$$\begin{aligned} S_{a,a,a,a,b} &= S_{b,a,a,a,a} - \frac{1}{4}\left[S_a S_{b,a,a,a} + S_{a\wedge b,a,a,a} + S_{b,a\wedge a,a,a} + S_{b,a,a\wedge a,a} + S_{b,a,a,a\wedge a}\right] \\ &\quad + \frac{1}{12}\left[S_a S_{a,b,a,a} + S_{a\wedge a,b,a,a} + S_{a,a\wedge b,a,a} + S_{a,b,a\wedge a,a} + S_{a,b,a,a\wedge a} - S_a S_{a,a,b,a} \right. \\ &\quad \left. - S_{a\wedge a,a,b,a} - S_{a,a\wedge a,b,a} - S_{a,a,a\wedge b,a} - S_{a,a,b,a\wedge a}\right] + \frac{1}{4}\left[S_a S_{a,a,a,b} + S_{a\wedge a,a,a,b} \right. \\ &\quad \left. + S_{a,a\wedge a,a,b} + S_{a,a,a\wedge a,b} + S_{a,a,a,a\wedge b}\right] & (5.\text{-}19) \\ S_{a,a,a,b,a} &= -4S_{b,a,a,a,a} + S_a S_{b,a,a,a} + S_{a\wedge b,a,a,a} + S_{b,a\wedge a,a,a} + S_{b,a,a\wedge a,a} + S_{b,a,a,a\wedge a} - \\ &\quad \frac{1}{3}\left[S_a S_{a,b,a,a} + S_{a\wedge a,b,a,a} + S_{a,a\wedge b,a,a} + S_{a,b,a\wedge a,a} + S_{a,b,a,a\wedge a} - S_a S_{a,a,b,a} \right. \\ &\quad \left. - S_{a\wedge a,a,b,a} - S_{a,a\wedge a,b,a} - S_{a,a,a\wedge b,a} - S_{a,a,b,a\wedge a}\right] & (5.\text{-}20) \\ S_{a,a,b,a,a} &= 6S_{b,a,a,a,a} - \frac{3}{2}\left[S_a S_{b,a,a,a} + S_{a\wedge b,a,a,a} + S_{b,a\wedge a,a,a} + S_{b,a,a\wedge a,a} + S_{b,a,a,a\wedge a}\right] \\ &\quad + \frac{1}{2}\left[S_a S_{a,b,a,a} + S_{a\wedge a,b,a,a} + S_{a,a\wedge b,a,a} + S_{a,b,a\wedge a,a} + S_{a,b,a,a\wedge a}\right] & (5.\text{-}20) \\ S_{a,b,a,a,a} &= -4S_{b,a,a,a,a} + S_a S_{b,a,a,a} + S_{a\wedge b,a,a,a} + S_{b,a\wedge a,a,a} + S_{b,a,a\wedge a,a} + S_{b,a,a,a\wedge a} & (5.\text{-}19) \end{aligned}$$

6 The Sixfold Harmonic Sums

The structure of the sixfold harmonic sums is summarized in the following Table. There are six basic classes of sums with up to three cases each.

Index Set	Number	Dep. Sums of Depth 6	min. Weight	Fraction of fund. Sums
$\{a,a,a,a,a,a\}$	1	1	6	0
$\{a,a,a,a,a,b\}$	6	5	6	1/6
$\{a,a,a,a,b,b\}$	15	13	6	2/15
$\{a,a,a,b,b,b\}$	20	17	6	3/20
$\{a,a,a,a,b,c\}$	30	25	7	1/6
$\{a,a,a,b,b,c\}$	60	50	7	1/6
$\{a,a,b,b,c,c\}$	90	76	8	7/45
$\{a,a,a,b,c,d\}$	120	100	8	1/6
$\{a,a,b,b,c,d\}$	180	150	8	1/6
$\{a,a,b,c,d,e\}$	360	300	10	1/6
$\{a,b,c,d,e,f\}$	720	600	12	1/6

6.1 Harmonic Sums with 2 Different Indices

6.1.1 Index-Set $\{a,a,a,b,b,b\}$

This class contains 20 different sums which can be expressed by three basic sums. As an example one obtains for $S_{a,a,a,b,b,b}$

$$\begin{aligned}
S_{a,a,a,b,b,b} = {} & \frac{1}{3}S_aS_{b,a,b,a,b} - \frac{1}{6}S_{b,a\wedge a,b,b,a} - \frac{1}{6}S_{a,b,b,b,a\wedge a} + \frac{1}{3}S_{b,a\wedge a,b,a,b} - \frac{1}{6}S_{a\wedge a,b,b,b,a} \\
& + \frac{1}{3}S_{a\wedge a,b,a,b,b} + \frac{1}{3}S_{a\wedge a,b,b,a,b} + \frac{1}{3}S_{a,b,b,a\wedge a,b} + \frac{1}{3}S_{a\wedge a,a,b,b,b} + \frac{1}{3}S_{a,b,a\wedge a,b,b} \\
& - \frac{1}{6}S_{a,b,a,a\wedge b,b} - \frac{1}{6}S_{b,a\wedge b,a,b,a} - \frac{1}{6}S_{a\wedge b,b,a,a,b} - \frac{1}{6}S_{b,a\wedge b,a,a,b} + \frac{1}{3}S_{a\wedge b,b,b,a,a} \\
& + \frac{1}{3}S_{a,a\wedge b,a,b,b} - \frac{1}{6}S_{a,a,b,a\wedge b,b} + \frac{1}{3}S_{a,a,b,b,a\wedge b} + \frac{1}{3}S_{b,a,a\wedge a,b,b} + \frac{1}{3}S_aS_{a,a,b,b,b} \\
& + \frac{1}{3}S_{b,b,a\wedge b,a,a} + \frac{1}{3}S_{b,a\wedge b,b,a,a} - \frac{1}{6}S_{a,b,b,a\wedge b,a} + \frac{1}{3}S_{b,a,b,a,a\wedge b} - \frac{1}{6}S_{a,a\wedge b,b,b,a} \\
& - \frac{1}{6}S_{a,b,a\wedge b,b,a} - \frac{1}{6}S_{b,a,a\wedge b,a,b} - \frac{1}{6}S_{a,a\wedge b,b,a,b} - \frac{1}{6}S_{a,b,a\wedge b,a,b} - \frac{1}{6}S_{a\wedge b,a,b,a,b} \\
& + \frac{1}{3}S_{a,b,b,a,a\wedge b} - \frac{1}{2}S_{a,b\wedge b,a,a,b} - \frac{1}{2}S_{b\wedge b,a,a,a,b} - \frac{1}{2}S_{a,b,a,a,b\wedge b} - \frac{1}{2}S_{a,a,b\wedge b,a,b} \\
& - \frac{1}{6}S_{b,a,a\wedge b,b,a} + \frac{1}{3}S_{b,a,a,b,a\wedge b} + \frac{1}{3}S_{a,a,a\wedge b,b,b} + \frac{1}{3}S_{a,b,a,b,a\wedge b} - \frac{1}{6}S_{b,b,a,a\wedge b,a} \\
& - \frac{1}{6}S_{a\wedge b,b,a,b,a} + \frac{1}{3}S_{b,b,a,a,a\wedge b} - \frac{1}{6}S_{b,a,a,a\wedge b,b} - \frac{1}{6}S_{b,a,b,a\wedge b,a} - \frac{1}{6}S_{b,b,a,b,a\wedge a} \\
& - \frac{1}{2}S_{b,a,a,a,b\wedge b} - \frac{1}{6}S_{a\wedge b,a,b,b,a} + \frac{1}{3}S_{b,b,a,a\wedge a,b} + \frac{1}{3}S_{b,b,b,a\wedge a,a} + \frac{1}{3}S_{b,b,b,a,a\wedge a}
\end{aligned}$$

$$-\frac{1}{6}S_{b,b,a\wedge a,b,a}+\frac{1}{3}S_{b,a,b,a\wedge a,b}+\frac{1}{3}S_{a,a\wedge a,b,b,b}+\frac{1}{3}S_{b,b,a\wedge a,a,b}+\frac{1}{3}S_{b,a\wedge a,a,b,b}$$
$$-\frac{1}{2}S_{a,a,b,a,b\wedge b}-\frac{1}{6}S_{b,a,b,b,a\wedge a}-S_{b,b,b,a,a,a}+\frac{1}{3}S_{a\wedge b,a,a,b,b}+\frac{1}{3}S_aS_{b,b,a,a,b}$$
$$-\frac{1}{6}S_aS_{a,b,b,b,a}-\frac{1}{6}S_aS_{b,b,a,b,a}+\frac{1}{3}S_aS_{b,a,a,b,b}+\frac{1}{3}S_aS_{a,b,a,b,b}-\frac{1}{6}S_aS_{b,a,b,b,a}$$
$$-\frac{1}{2}S_bS_{a,a,b,a,b}-\frac{1}{2}S_bS_{a,b,a,a,b}+\frac{1}{3}S_aS_{b,b,b,a,a}-\frac{1}{2}S_bS_{b,a,a,a,b}+\frac{1}{3}S_aS_{a,b,b,a,b} \qquad (6.-12)$$

6.1.2 Index-Set $\{a,a,a,a,b,b\}$

This class contains 15 different sums which are represented by two basic sums. For $S_{a,a,a,a,b,b}$ one obtains

$$\begin{aligned}
S_{a,a,a,a,b,b} \;=\;& -\frac{1}{4}S_aS_{b,a,a,a,b}+\frac{3}{4}S_{a\wedge b,a,a,a,b}-\frac{1}{4}S_{b,a,a,a,a\wedge b}+\frac{1}{12}S_aS_{a,a,b,b,a}+S_{a,a,a,a,b\wedge b}\\
&-\frac{1}{12}S_{a\wedge a,b,b,a,a}-\frac{1}{12}S_{a,b,b,a\wedge a,a}-\frac{1}{12}S_{a,b,b,a,a\wedge a}-\frac{1}{4}S_{b,a\wedge a,a,a,b}-\frac{1}{4}S_{b,a,a\wedge a,a,b}\\
&-\frac{1}{4}S_{b,a,a,a\wedge a,b}-\frac{1}{4}S_{a,a\wedge a,a,b,b}-\frac{1}{4}S_{a,a,a\wedge a,b,b}-\frac{1}{4}S_{a,b,a\wedge a,a,b}-\frac{1}{4}S_{a,b,a,a\wedge a,b}\\
&+\frac{1}{12}S_{a\wedge a,b,a,b,a}+\frac{1}{12}S_{a,b,a\wedge a,b,a}-\frac{1}{4}S_{a,a,a,b,a\wedge b}-\frac{1}{4}S_{a,a,b,a,a\wedge b}+\frac{1}{12}S_{a,a,a\wedge b,b,a}\\
&+\frac{3}{4}S_{a,a,a\wedge b,a,b}-S_{b,b,a,a,a,a}+\frac{1}{4}S_{b,b,a,a\wedge a,a}-\frac{1}{4}S_{a\wedge a,a,a,b,b}+\frac{1}{12}S_{a,b,a,b,a\wedge a}\\
&+\frac{1}{12}S_{b,a\wedge a,a,b,a}+\frac{1}{12}S_{b,a,a\wedge a,b,a}+\frac{1}{12}S_{b,a,a,b,a\wedge a}-\frac{1}{12}S_{b,a\wedge a,b,a,a}-\frac{1}{12}S_{b,a,b,a\wedge a,a}\\
&+\frac{1}{4}S_{b,b,a\wedge a,a,a}+\frac{1}{4}S_{b,b,a,a,a\wedge a}-\frac{1}{4}S_{a\wedge a,a,b,a,b}-\frac{1}{4}S_{a,a\wedge a,b,a,b}-\frac{1}{4}S_{a,a,b,a\wedge a,b}\\
&+\frac{1}{12}S_{a\wedge a,a,b,b,a}+\frac{1}{12}S_{a,a\wedge a,b,b,a}+\frac{1}{12}S_{a,a,b,b,a\wedge a}-\frac{1}{4}S_{a\wedge a,b,a,a,b}-\frac{1}{12}S_{b,a,b,a,a\wedge a}\\
&+\frac{1}{12}S_{a\wedge b,a,a,b,a}+\frac{1}{12}S_{b,a,a,a\wedge b,a}-\frac{1}{12}S_{a\wedge b,a,b,a,a}+\frac{1}{4}S_{a\wedge b,b,a,a,a}+\frac{1}{4}S_{b,a\wedge b,a,a,a}\\
&-\frac{1}{12}S_{b,a,a\wedge b,a,a}+\frac{3}{4}S_{a,a,a,a\wedge b,b}+\frac{1}{12}S_{a,a,b,a\wedge b,a}+\frac{3}{4}S_{a,a\wedge b,a,a,b}-\frac{1}{4}S_{a,b,a,a,a\wedge b}\\
&+\frac{1}{12}S_{a,a\wedge b,a,b,a}+\frac{1}{12}S_{a,b,a,a\wedge b,a}-\frac{1}{12}S_{a,a\wedge b,b,a,a}-\frac{1}{12}S_{a,b,a\wedge b,a,a}-\frac{1}{4}S_aS_{a,a,a,b,b}\\
&-\frac{1}{12}S_aS_{a,b,b,a,a}+\frac{1}{12}S_aS_{a,b,a,b,a}-\frac{1}{12}S_aS_{b,a,b,a,a}+\frac{1}{12}S_aS_{b,a,a,b,a}-\frac{1}{4}S_aS_{a,b,a,a,b}\\
&+S_bS_{a,a,a,a,b}+\frac{1}{4}S_aS_{b,b,a,a,a}-\frac{1}{4}S_aS_{a,a,b,a,b}
\end{aligned} \qquad (6.-24)$$

6.1.3 Index-Set $\{a,a,a,a,a,b\}$

This class contains 6 different sums. The coefficient matrix reads

$$M_{6c}=\left\|\begin{array}{cccccc} 5 & 1 & 0 & 0 & 0 & 0\\ 0 & 4 & 2 & 0 & 0 & 0\\ 0 & 0 & 3 & 3 & 0 & 0\\ 0 & 0 & 0 & 2 & 4 & 0\\ 0 & 0 & 0 & 0 & 1 & 5\\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right\| \qquad (6.-24)$$

and is of rank 5. All sums can be represented in terms of one. The sums are given by

$$
\begin{aligned}
S_{a,a,a,a,a,b} &= -S_{b,a,a,a,a,a}+\frac{1}{5}\,S_aS_{a,a,a,a,b}+\frac{1}{30}\,S_aS_{a,a,b,a,a}-\frac{1}{20}\,[S_aS_{a,a,a,b,a}\\
&\quad +S_{a\wedge a,b,a,a,a}]+\frac{1}{5}\,[S_{b,a\wedge a,a,a,a}+S_{b,a,a\wedge a,a,a}+S_{b,a,a,a,a\wedge a}\\
&\quad +S_{a,a,a,a,a\wedge b}]+\frac{1}{30}\,S_{a,a,a\wedge b,a,a}-\frac{1}{20}\,S_{a,a,a,a\wedge b,a}+\frac{1}{30}\,S_{a\wedge a,a,b,a,a}\\
&\quad +\frac{1}{5}\,[S_{a,a,a\wedge a,a,b}+S_{a,a,a,a\wedge a,b}+S_{a\wedge a,a,a,a,b}+S_{a,a\wedge a,a,a,b}]\\
&\quad +\frac{1}{30}\,[S_{a,a,b,a\wedge a,a}+S_{a,a,b,a,a\wedge a}]+\frac{1}{5}\,[S_aS_{b,a,a,a,a}+S_{b,a,a,a\wedge a,a}]\\
&\quad -\frac{1}{20}\,[S_{a,a,a\wedge a,b,a}+S_{a,a,a,b,a\wedge a}+S_{a\wedge a,a,a,b,a}+S_{a,a\wedge a,a,b,a}\\
&\quad +S_{a,b,a\wedge a,a,a}+S_{a,b,a,a\wedge a,a}+S_{a,b,a,a,a\wedge a}+S_aS_{a,b,a,a,a}\\
&\quad +S_{a,a\wedge b,a,a,a}]+\frac{1}{5}\,S_{a\wedge b,a,a,a,a}+\frac{1}{30}\,S_{a,a\wedge a,b,a,a} && (6.\text{-}30)\\
S_{a,a,a,a,b,a} &= 5\,S_{b,a,a,a,a,a}-\frac{1}{6}\,S_aS_{a,a,b,a,a}+\frac{1}{4}\,[S_aS_{a,a,a,b,a}+S_{a\wedge a,b,a,a,a}]-S_{b,a\wedge a,a,a,a}\\
&\quad -S_{b,a,a\wedge a,a,a}-S_{b,a,a,a,a\wedge a}-\frac{1}{6}S_{a,a,a\wedge b,a,a}+\frac{1}{4}\,S_{a,a,a,a\wedge b,a}-\frac{1}{6}\,[S_{a\wedge a,a,b,a,a}\\
&\quad +S_{a,a,b,a\wedge a,a}+S_{a,a,b,a,a\wedge a}]-S_aS_{b,a,a,a,a}-S_{b,a,a,a\wedge a,a}+\frac{1}{4}\,[S_{a,a,a\wedge a,b,a}\\
&\quad +S_{a,a,a,b,a\wedge a}+S_{a\wedge a,a,a,b,a}+S_{a,a\wedge a,a,b,a}+S_{a,b,a\wedge a,a,a}\\
&\quad +S_{a,b,a,a\wedge a,a}+S_{a,b,a,a,a\wedge a}+S_aS_{a,b,a,a,a}+S_{a,a\wedge b,a,a,a}]\\
&\quad -S_{a\wedge b,a,a,a,a}-\frac{1}{6}\,S_{a,a\wedge a,b,a,a} && (6.\text{-}34)\\
S_{a,a,a,b,a,a} &= -10\,S_{b,a,a,a,a,a}+2\,S_aS_{b,a,a,a,a}+2\,[S_{a\wedge b,a,a,a,a}+S_{b,a\wedge a,a,a,a}+S_{b,a,a\wedge a,a,a}\\
&\quad +S_{b,a,a,a\wedge a,a}+S_{b,a,a,a,a\wedge a}]-\frac{1}{2}\,[S_aS_{a,b,a,a,a}+S_{a\wedge a,b,a,a,a}+S_{a,a\wedge b,a,a,a}\\
&\quad +S_{a,b,a\wedge a,a,a}+S_{a,b,a,a\wedge a,a}+S_{a,b,a,a,a\wedge a}]+\frac{1}{3}\,[S_aS_{a,a,b,a,a}\\
&\quad +S_{a\wedge a,a,b,a,a}+S_{a,a\wedge a,b,a,a}+S_{a,a,a\wedge b,a,a}+S_{a,a,b,a\wedge a,a}+S_{a,a,b,a,a\wedge a}] && (6.\text{-}36)\\
S_{a,a,b,a,a,a} &= 10\,S_{b,a,a,a,a,a}-2\,[S_aS_{b,a,a,a,a}+S_{a\wedge b,a,a,a,a}+S_{b,a\wedge a,a,a,a}+S_{b,a,a\wedge a,a,a}\\
&\quad +S_{b,a,a,a\wedge a,a}+S_{b,a,a,a,a\wedge a}]+\frac{1}{2}\,[S_aS_{a,b,a,a,a}+S_{a\wedge a,b,a,a,a}+S_{a,a\wedge b,a,a,a}\\
&\quad +S_{a,b,a\wedge a,a,a}+S_{a,b,a,a\wedge a,a}+S_{a,b,a,a,a\wedge a}] && (6.\text{-}37)
\end{aligned}
$$

$$
\begin{aligned}
S_{a,b,a,a,a,a} &= -5\,S_{b,a,a,a,a,a}+S_aS_{b,a,a,a,a}+S_{a\wedge b,a,a,a,a}+S_{b,a\wedge a,a,a,a}+S_{b,a,a\wedge a,a,a}\\
&\quad +S_{b,a,a,a\wedge a,a}+S_{b,a,a,a,a\wedge a} && (6.\text{-}37)
\end{aligned}
$$

Finally we consider the case of general depth d for the harmonic sums of the index set $a,a,\dots,a,b$ and their respective permutations. They depend on $1/d\cdot n_{\rm perm}$ fundamental sums, i.e. one sum, only. The coefficient matrix reads

$$M_{d;1} = \left\| \begin{array}{ccccccc} d-1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & d-2 & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & d-3 & 3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & d-1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{array} \right\| . \qquad (6.\text{-}37)$$

This matrix is of rank $d-1$, which may be easily seen adding the first line to the $-(d-1)$-fold of the last line, etc. etc., and therefore $d-1$ harmonic sums of this type are dependent, i.e. the fraction of independent sums is $1/d$. In the foregoing tables we calculated the fraction of independent sums out of all possible sums of a given index class up to depth $d = 6$. For these cases

$$\frac{n_{\rm independent}}{n_{\rm perm}} \le \frac{1}{d} . \qquad (6.\text{-}37)$$

7 Number of Independent Harmonic Sums and the Number of Lyndon Words

The algebraic relations between the harmonic sums associated to a given index set studied in the present paper are induced by the multiplication relation between the sums independently of their specific value or structure otherwise. Therefore the foregoing results hold for all mathematical objects which obey these conditions. Since the multiplication relation (2.1) is directly associated to the shuffle product $\sqcup\!\sqcup$ one may determine the number of basic sums by means of mathematical relations derived for shuffle algebras. It turns out that the number of basic harmonic sums of a given index set is given by the number of Lyndon [9] words which can be formed by the letters being contained in this set. The sets of characters (words) being considered in the following are always understood as representatives for which we consider all permutations, as before, to find the maximal set of algebraic relations of harmonic sums. We will make use of results obtained in the theory of words [48] and free algebras [19, 49].

Let $\mathfrak{A} = \{a, b, c, d, \dots\}$ be a finite alphabet. $a, b, c, d, \dots$ are called the letters of this alphabet and any sequence $\{a_1, \dots, a_k\}$ with $a_i \in \mathfrak{A}$ is a word. For brevity one often writes this sequence as a non–commutative product $a_1 a_2 \dots a_k$, the concatenation product. The length of a word is the number of its letters, which corresponds to the depth d of the respective harmonic sums. $\mathfrak{A}^*$ denotes the free monoid on $\mathfrak{A}$ made out of all words w including the empty word 1. The set of non–empty words is denoted as $\mathfrak{A}^+$. The alphabet $\mathfrak{A}$ is ordered by

$$a < b < c < d < \dots . \qquad (7.0)$$

For a word

$$w = pxs \qquad (7.1)$$

any non–empty factor p is called a prefix and any non-empty factor s a suffix of w. In extension of (7) the order relation $<$ applies also to words.

$$\text{i)} \qquad u < v \Leftrightarrow v = ux, \qquad x \in \mathfrak{A}^+ \text{ or} \qquad (7.2)$$

$$\text{ii)} \qquad u = xau', v = xbv', a < b, c, u', v' \in \mathfrak{A}^+ . \qquad (7.3)$$

[9] I thank S. Weinzierl for hinting me to Lyndon words.

DEFINITION.
A word w is called a LYNDON word if it is smaller than any of its suffixes.

The operator $<$ between letters accounts for lexicographic order. This property offers an easy way to construct the set of LYNDON words for a given index set.

One considers a commutative ring with unit, K. A non-commutative polynomial on $\mathfrak{A}$ over K is a linear combination of words on $\mathfrak{A}$ with coefficients in K,

$$P = \sum_{w \epsilon \mathfrak{A}^*} (P, w) w. \tag{7.4}$$

The set of all polynomials is denoted by $K\langle\mathfrak{A}\rangle$. The algebra $K\langle\mathfrak{A}\rangle$ is the free associative $K-$algebra generated by $\mathfrak{A}$.

THEOREM(RADFORD [50]).
The shuffle algebra $K\langle\mathfrak{A}\rangle$ is freely generated by the LYNDON words.

A direct conclusion is that the number of LYNDON words counts the number of basis elements, which are the algebraically independent harmonic sums in our case. In the counting relations for the number of basic sums given below MÖBIUS' function $\mu(n)$ [51] with $n \epsilon \mathbf{N}$ emerges, which is defined by

$$\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & n \text{ is divided by the square of a prime} \\ (-1)^s & n \text{ is the product of } s \text{ different primes.} \end{cases} \tag{7.5}$$

The number of LYNDON words of length n over an alphabet of length q is given by

$$l_n(q) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d} \; . \tag{7.6}$$

We will call this relation the first WITT formula [52]. [10] As mentioned in [52] this relation also counts the number of prime polynomials $\sum_{k=0}^{n} a_k x^{n-k}$ in the GALOIS field of q elements. This relation has been known to GAUSS already, [53]. As we would like to count the number of basic sums for all sums of a given index set individually this relation cannot be used, but we have to count the number of LYNDON words belonging to this set. The respective relation has been given in the same paper as the second WITT formula,

$$l_n(n_1, \ldots, n_q) = \frac{1}{n} \sum_{d|n_i} \mu(d) \frac{\left(\frac{n}{d}\right)!}{\left(\frac{n_1}{d}\right)! \ldots \left(\frac{n_q}{d}\right)!}, \qquad n = \sum_{k=1}^{q} n_k \; . \tag{7.7}$$

One may derive the numbers $l_n(q)$ and $l_n(n_1, \ldots, n_q)$ using the generating functionals [52] :

$$\frac{1}{1 - qx} = \prod_{n=1}^{\infty} \left(\frac{1}{1 - x^n} \right)^{l_n(q)} \tag{7.8}$$

and

$$\frac{1}{1 - x_1 - \ldots - x_{n_q}} = \prod_{n=1}^{\infty} \left(\frac{1}{1 - \sum_{k=1}^{q} x_k^{d_k}} \right)^{l_n(n_1, \ldots, n_q)} \; . \tag{7.9}$$

[10] Usually it is called the WITT formula in the literature, cf. e.g. [49].

Counting relations for multiple Zeta–values were also given in the literature. Multiple Zeta–values are simpler objects than multiple harmonic sums and, consequently, they obey more relations. Besides the shuffle–relations so called stuffle and duality relations exist, see e.g. [22,23]. Hoffman [37] derived the Gauss–Witt relation

$$N(w) = \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 2^d \tag{7.10}$$

for the number of basic multiple Zeta–values of weight w for $\forall n_i > 0$. One verifies that this is actually an upper bound and Broadhurst and Kreimer [54] conjectured that[11] (7.10) can be sharpened to

$$N(w) = \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) P_d \ , \tag{7.11}$$

with P_d the Perrin numbers

$$P_1 = 0, P_2 = 2, P_3 = 3, P_n = P_{n-2} + P_{n-3}, \qquad n \geq 3 \ , \tag{7.12}$$

being checked up to $w = 12$ in [27].

It is clear from (3.20) and (7.7) that the fraction of the number of basic sums to all sums of a given index set $a_1, \ldots, a_n$ is $1/n$ if the greatest common divisor (g.c.d.) of $n_1, \ldots, n_q$ is 1, since the sum (7.7) consists of only one term. If only one prime p aside of 1 divides the numbers $n_1, \ldots, n_q$ (see the cases in the tables above), the fraction is obviously smaller than $1/n$ since $\mu(p) = -1$. For the case of a fully symmetric harmonic sum with the index set $\{a, a, \ldots, a\}$ (7.7) yields

$$l_n(a, a, \ldots, a) = l_n(n) = \sum_{d|n} \mu(d) \frac{\left(\frac{n}{d}\right)!}{\left(\frac{n}{d}\right)!} = \sum_{d|n} \mu(d) = 0, \qquad n > 1 \tag{7.13}$$

due to the well–known property of the Möbius function. In counting basic harmonic sums we will leave out the trivial case of single harmonic sums therefore[12]. Let us now study some explicit examples.

Index-Sets out of one Letter

The letters are Lyndon words. Words which are products out of a single letter $aa \ldots a$ with more than one factor are no Lyndon words. This property corresponds to the fact that harmonic sums of depth $d \geq 1$ with all indices equal can always be decomposed into a (symmetric) polynomial out of single harmonic sums, cf. (2.42–2.46).

Index-Sets out of two Letters

The first letter in a Lyndon word has always to be the smallest letter of the alphabet emerging in the sequence, which can never be the last letter. Words consisting out of (several) factors a and a single factor b are Lyndon words only if they obey the sequence

$$aa \ldots ab \ . \tag{7.13}$$

[11] See also Zagier's conjecture [16b]. Zagier gave numerical checks up to weight $w = 12$.

[12] Their analytic continuation to complex values of N is trivially being obtained by the $\psi(z)$ function and their derivatives.

Obviously ab is a LYNDON word and

$$\underbrace{a\dots a}_{k}b < \underbrace{a\dots a}_{k-1}b\ . \tag{7.13}$$

Any emergence in a or its power in a word right of b would violate the order such that the word is no LYNDON word. (7) confirms the finding that matrix $M_{d;1}$ (6.1.3) is of rank $d-1$, i.e. it exists one independent sum in these cases always. For depth $d=4$ the number of LYNDON words for the sequences of type $\{a,a,b,b\}$ is one since $abba$ and $abab$ are no LYNDON words. The second word is a power of a single LYNDON word, which is no LYNDON word and $aabb$ is the only LYNDON word for this set. Words of length $d=5$ out of two different letters correspond to the set $\{a,a,a,b,b\}$, to which the two LYNDON words $aaabb$ and $aabab$ belong. In general the following inequality holds [49] for LYNDON words u_1 and u_2

$$u_1 < u_1^{\kappa_1}u_2^{\kappa_2} < u_2, \quad \forall\kappa_i \in \mathbf{N}\ , \tag{7.13}$$

where $aab < ab$. For $d=6$ two LYNDON words are associated to the set $\{a,a,a,a,b,b\}$ synonymously. For the set $\{a,a,a,b,b,b\}$ three LYNDON words exist, $aaabbb, aababb$ and $aabbab$. Since $aab < abb$ the first two cases are evident and the latter word is a LYNDON word because $aabb < ab$.

ab	aab	$aaab$	$aaaab$	$aaaaab$
		$aabb$	$aaabb$	$aaaabb$
			$aabab$	$aaabab$
				$aaabbb$
				$aababb$
				$aabbab$

Index-Sets out of only Different Letters

The number of associated LYNDON words can easily be obtained for this case. The LYNDON words have to begin with the letter a. For words of length n the LYNDON words are obtained writing a to the left of the $(n-1)!$ permutations of the letters $b_1,\dots,b_{n-1}$, i.e. the number of LYNDON words is $(n-1)!$ and therefore the $1/n$th of all possible combinations.

Numbers of Basic Sums: Examples

Let us calculate the number of basic sums for a given index set for a few examples. We denote by $n_k(\{a_1,...,a_q\})$

$$n_k(\{a_1,...,a_q\}) = \frac{k!}{n(a_1)!\dots n(a_q)!}\ , \tag{7.14}$$

where $n(a_l)$ is the number of occurrences of a_l in the string $a_1,a_2,\dots,a_q$. For the set $\{a,a,a,a,b,b\}$ one obtains

$$l_6(\{a,a,a,a,b,b\}) = \frac{1}{6}\left[\mu(1)\frac{6!}{4!2!} + \mu(2)\frac{3!}{2!1!}\right] = 2, \tag{7.15}$$

$$\frac{l_6(\{a,a,a,a,b,b\})}{n_6(\{a,a,a,a,b,b\})} = \frac{2}{15} < \frac{1}{6}\ . \tag{7.16}$$

Similarly one obtains for $\{a,a,a,b,b,b\}$

$$l_6(\{a,a,a,b,b,b\}) = \frac{1}{6}\left[\mu(1)\frac{6!}{3!3!} + \mu(3)\frac{2!}{1!1!}\right] = 3, \tag{7.17}$$

$$\frac{l_6(\{a,a,a,b,b,b\})}{n_6(\{a,a,a,b,b,b\})} = \frac{3}{20} < \frac{1}{6}\ . \tag{7.18}$$

and for $\{a, a, b, b, c, c\}$

$$l_6(\{a,a,b,b,c,c\}) = \frac{1}{6}\left[\mu(1)\frac{6!}{2!2!2!} + \mu(2)\frac{3!}{1!1!1!}\right] = 14, \tag{7.19}$$

$$\frac{l_6(\{a,a,b,b,c,c\})}{n_6(\{a,a,b,b,c,c\})} = \frac{7}{45} < \frac{1}{6}\ . \tag{7.20}$$

Finally,

$$\begin{aligned} l_{12}(\{a,a,a,a,a,a,b,b,b,b,b,b\}) &= \frac{1}{12}\left[\mu(1)\frac{12!}{6!6!} + \mu(2)\frac{6!}{3!3!} + \mu(3)\frac{4!}{2!2!} + \mu(6)\frac{2!}{1!1!}\right] \\ &= 75, \qquad (7.20) \\ \frac{l_6(\{a,a,a,a,a,a,b,b,b,b,b,b\})}{n_6(\{a,a,a,a,a,a,b,b,b,b,b,b\})} &= \frac{25}{308} < \frac{1}{12}\ . \qquad (7.21) \end{aligned}$$

We conclude that the fraction of the number of basic sums in all sums of a given index set is primarily determined by the depth of the sum.

In the following Table we compare the number of harmonic sums with the number of basic sums determined applying the algebraic relations.

Weight	# Sums	Cum. # Sums	# Basic Sums	Cum. # Basic Sums	Cum. Fraction
1	2	2	0	0	0.0
2	6	8	1	1	0.1250
3	18	26	6	7	0.2692
4	54	80	16	23	0.2875
5	162	242	46	69	0.2851
6	486	728	114	183	0.2513

The number of sums for a given weight is $2 \cdot 3^{w-1}$, with the cumulative value $3^w - 1$. The numbers of basic sums as a function of the weight w are given as sums over the 2nd Witt formula. The corresponding sequence seems not yet to be contained in Sloane's on-line encyclopedia of integer sequences [55].

8 Conclusions

The product of finite alternating or non–alternating harmonic sums is given by the shuffle product of harmonic sums and polynomials of harmonic sums of lower depth. These representations imply algebraic relations between the harmonic sums. If one considers all harmonic sums associated to an index set $\{a_1, \ldots, a_k\}$ one may express these sums by a number of basic sums. It turns out that this number is given by the 2nd Witt formula which counts the number of Lyndon words corresponding to the respective index set. The set of these Lyndon words generates in this sense all harmonic sums of this class freely. By solving the corresponding linear equations we derived the explicit representation of all harmonic sums up to depth $d = 6$ without specifying the indices numerically and gave all expression which are structurally needed to express the sums up to weight $w = 6$. The counting relations for the basis of the finite harmonic sums were given up to depth $d = 10$. The relations derived hold likewise for other mathematical objects obeying the same multiplication relation or a simpler one, which is being contained, as that for harmonic polylogarithms. This is due to the fact that the relations derived depend on the index set and the multiplication relation but on no further properties of the objects considered.

The ratio of the number of basic sums for a given index set and the number of all sums is mainly determined by the depth d rather than the weight of the respective sums, due to the pre-factor $1/d$ in the WITT formula. Modifications occur due to common non-trivial divisors of the numbers of individual indices in the set being considered. Up to $d = 10$ we showed that the fraction of basic sums is always $\leq 1/d$ compared to all sums. The use of these algebraic relations leads to a considerable reduction in the set of functions needed to express the results of higher order calculations in massless QED and QCD and related subjects. For practical applications such as the description of the QCD scaling violation of the structure functions in deeply inelastic scattering the harmonic sums occurring in the MELLIN N space calculation have to be translated to x–space by the inverse MELLIN transform. For this reason the respective harmonic sums have to be analytically continued in the argument N to complex values, which requires a high effort using numerical procedures. It is therefore recommended to use as many as possible relations between the N space objects before to perform the last step only for a reduced set.

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9 Appendix A: Number of Harmonic Sums and Basic Sums for Individual Index-Pattern of Depth 7 to 10

Index Set	Number of Sums	Basic Sums	Frac. of Sums
$\{a,a,a,a,a,a,a\}$	1	0	0
$\{a,a,a,a,a,a,b\}$	7	1	1/7
$\{a,a,a,a,a,b,b\}$	21	3	1/7
$\{a,a,a,a,b,b,b\}$	35	5	1/7
$\{a,a,a,a,a,b,c\}$	42	6	1/7
$\{a,a,a,a,b,b,c\}$	105	15	1/7
$\{a,a,a,b,b,b,c\}$	140	20	1/7
$\{a,a,a,b,b,c,c\}$	210	30	1/7
$\{a,a,a,a,b,c,d\}$	210	30	1/7
$\{a,a,a,b,b,c,d\}$	420	60	1/7
$\{a,a,b,b,c,c,d\}$	630	90	1/7
$\{a,a,a,b,c,d,e\}$	840	120	1/7
$\{a,a,b,b,c,d,e\}$	1260	180	1/7
$\{a,a,b,c,d,e,f\}$	2550	360	1/7
$\{a,b,c,d,e,f,g\}$	5040	720	1/7

Index Set	Number of Sums	Basic Sums	Frac. of Sums
$\{a,a,a,a,a,a,a,a\}$	1	0	0
$\{a,a,a,a,a,a,a,b\}$	8	1	1/8
$\{a,a,a,a,a,a,b,b\}$	28	3	3/28
$\{a,a,a,a,a,b,b,b\}$	56	7	1/7
$\{a,a,a,a,b,b,b,b\}$	70	8	4/35
$\{a,a,a,a,a,a,b,c\}$	56	7	1/8
$\{a,a,a,a,a,b,b,c\}$	168	21	1/8
$\{a,a,a,a,b,b,b,c\}$	280	35	1/8
$\{a,a,a,a,b,b,c,c\}$	420	51	17/140
$\{a,a,a,b,b,b,c,c\}$	560	70	1/8
$\{a,a,a,a,a,b,c,d\}$	336	42	1/8
$\{a,a,a,a,b,b,c,d\}$	840	105	1/8
$\{a,a,a,b,b,b,c,d\}$	1120	140	1/8
$\{a,a,a,b,b,c,c,d\}$	1680	210	1/8
$\{a,a,b,b,c,c,d,d\}$	2520	312	13/105
$\{a,a,a,a,b,c,d,e\}$	1680	210	1/8
$\{a,a,a,b,b,c,d,e\}$	3360	420	1/8
$\{a,a,b,b,c,c,d,e\}$	5040	630	1/8
$\{a,a,a,b,c,d,e,f\}$	6720	840	1/8
$\{a,a,b,b,c,d,e,f\}$	10080	1260	1/8
$\{a,a,b,c,d,e,f,g\}$	20160	2520	1/8
$\{a,b,c,d,e,f,g,h\}$	40320	5040	1/8

Index Set	Number of Sums	Basic Sums	Frac. of Sums
$\{a,a,a,a,a,a,a,a,a\}$	1	0	0
$\{a,a,a,a,a,a,a,a,b\}$	9	1	1/9
$\{a,a,a,a,a,a,a,b,b\}$	36	4	1/9
$\{a,a,a,a,a,a,a,b,b\}$	72	8	1/9
$\{a,a,a,a,a,a,b,b,b\}$	84	9	3/28
$\{a,a,a,a,a,b,b,b,b\}$	126	14	1/9
$\{a,a,a,a,a,a,b,b,c\}$	252	28	1/9
$\{a,a,a,a,a,b,b,b,c\}$	504	56	1/9
$\{a,a,a,a,a,b,b,c,c\}$	756	84	1/9
$\{a,a,a,a,b,b,b,b,c\}$	630	70	1/9
$\{a,a,a,a,b,b,b,c,c\}$	1260	140	1/9
$\{a,a,a,b,b,b,c,c,c\}$	1680	186	31/280
$\{a,a,a,a,a,a,b,c,d\}$	504	56	1/9
$\{a,a,a,a,a,b,b,c,d\}$	1512	168	1/9
$\{a,a,a,a,b,b,b,c,d\}$	2550	280	1/9
$\{a,a,a,a,b,b,c,c,d\}$	3780	420	1/9
$\{a,a,a,b,b,b,c,c,d\}$	5040	560	1/9
$\{a,a,a,b,b,c,c,d,d\}$	7560	840	1/9
$\{a,a,a,a,a,b,c,d,e\}$	3024	336	1/9
$\{a,a,a,a,b,b,c,d,e\}$	7560	840	1/9
$\{a,a,a,b,b,b,c,d,e\}$	10080	1120	1/9
$\{a,a,a,b,b,c,c,d,e\}$	15120	1680	1/9
$\{a,a,b,b,c,c,d,d,e\}$	22680	2520	1/9
$\{a,a,a,a,b,c,d,e,f\}$	15120	1680	1/9
$\{a,a,a,b,b,c,d,e,f\}$	30240	3360	1/9
$\{a,a,b,b,c,c,d,e,f\}$	45360	5040	1/9
$\{a,a,a,b,c,d,e,f,g\}$	60480	6720	1/9
$\{a,a,b,b,c,d,e,f,g\}$	90720	10080	1/9
$\{a,a,b,c,d,e,f,g,h\}$	90720	10080	1/9
$\{a,b,c,d,e,f,g,h,i\}$	362880	40320	1/9

Index Set	Number of Sums	Basic Sums	Frac. of Sums
$\{a,a,a,a,a,a,a,a,a,a\}$	1	0	0
$\{a,a,a,a,a,a,a,a,a,b\}$	10	1	$1/10$
$\{a,a,a,a,a,a,a,a,b,b\}$	45	4	$4/45$
$\{a,a,a,a,a,a,a,b,b,b\}$	120	12	$1/10$
$\{a,a,a,a,a,a,b,b,b,b\}$	210	20	$2/21$
$\{a,a,a,a,a,b,b,b,b,b\}$	252	25	$25/252$
$\{a,a,a,a,a,a,a,a,b,c\}$	90	9	$1/10$
$\{a,a,a,a,a,a,a,b,b,c\}$	360	36	$1/10$
$\{a,a,a,a,a,a,b,b,b,c\}$	840	84	$1/10$
$\{a,a,a,a,a,a,b,b,c,c\}$	1260	124	$31/315$
$\{a,a,a,a,a,b,b,b,b,c\}$	1260	126	$1/10$
$\{a,a,a,a,a,b,b,b,c,c\}$	2520	252	$1/10$
$\{a,a,a,a,b,b,b,b,c,c\}$	3150	312	$52/525$
$\{a,a,a,a,b,b,b,c,c,c\}$	4200	420	$1/10$
$\{a,a,a,a,a,a,a,b,c,d\}$	720	72	$1/10$
$\{a,a,a,a,a,a,b,b,c,d\}$	2520	252	$1/10$
$\{a,a,a,a,a,b,b,b,c,d\}$	5040	504	$1/10$
$\{a,a,a,a,a,b,b,c,c,d\}$	7560	756	$1/10$
$\{a,a,a,a,b,b,b,b,c,d\}$	6300	630	$1/10$
$\{a,a,a,a,b,b,b,c,c,d\}$	12600	1260	$1/10$
$\{a,a,a,a,b,b,c,c,d,d\}$	18900	1884	$157/1575$
$\{a,a,a,b,b,b,c,c,c,d\}$	16800	1680	$1/10$
$\{a,a,a,b,b,b,c,c,d,d\}$	25200	2520	$1/10$
$\{a,a,a,a,a,a,b,c,d,e\}$	5040	504	$1/10$
$\{a,a,a,a,a,b,b,c,d,e\}$	15120	1520	$1/10$
$\{a,a,a,a,b,b,b,c,d,e\}$	25200	2520	$1/10$
$\{a,a,a,a,b,b,c,c,d,e\}$	37800	3780	$1/10$
$\{a,a,a,b,b,b,c,c,d,e\}$	50400	5040	$1/10$
$\{a,a,a,b,b,c,c,d,d,e\}$	75600	7560	$1/10$
$\{a,a,b,b,c,c,d,d,e,e\}$	113400	11328	$472/4725$
$\{a,a,a,a,a,b,c,d,e,f\}$	30240	3024	$1/10$
$\{a,a,a,b,b,b,c,d,e,f\}$	75600	7560	$1/10$
$\{a,a,a,a,b,b,c,d,e,f\}$	100800	10080	$1/10$
$\{a,a,a,b,b,c,c,d,e,f\}$	151200	15120	$1/10$
$\{a,a,b,b,c,c,d,d,e,f\}$	226900	22680	$1/10$
$\{a,a,a,a,b,c,d,e,f,g\}$	151200	15120	$1/10$
$\{a,a,a,b,b,c,d,e,f,g\}$	302400	30240	$1/10$
$\{a,a,b,b,c,c,d,e,f,g\}$	453600	45360	$1/10$
$\{a,a,a,b,c,d,e,f,g,h\}$	302400	30240	$1/10$
$\{a,a,b,b,c,d,e,f,g,h\}$	907200	90720	$1/10$
$\{a,a,b,c,d,e,f,g,h,i\}$	1814400	181440	$1/10$
$\{a,b,c,d,e,f,g,h,i,j\}$	3628800	362880	$1/10$

10 Appendix B: Overview on all Harmonic Sums up to Depth and Weight 6

Index Set	Number	Weight	Relations
$\{-1\}$	1	1	1
$\{1\}$	1	1	1
$\{-2\}$	1	2	1
$\{2\}$	1	2	1
$\{-1,-1\}$	1	2	1
$\{-1,1\}$	2	2	1
$\{1,1\}$	1	2	1
$\{-3\}$	1	3	1
$\{3\}$	1	3	1
$\{-2,-1\}$	2	3	1
$\{-2,1\}$	2	3	1
$\{2,-1\}$	2	3	1
$\{2,1\}$	2	3	1
$\{-1,-1,-1\}$	1	3	1
$\{-1,-1,1\}$	3	3	2
$\{-1,1,1\}$	3	3	2
$\{1,1,1\}$	1	3	1
$\{-4\}$	1	4	1
$\{4\}$	1	4	1
$\{-2,-2\}$	1	4	1
$\{-2,2\}$	2	4	1
$\{2,2\}$	1	4	1
$\{-3,-1\}$	2	4	1
$\{-3,1\}$	2	4	1
$\{3,-1\}$	2	4	1
$\{3,1\}$	2	4	1
$\{-2,-1,-1\}$	3	4	2
$\{-2,-1,1\}$	6	4	4
$\{-2,1,1\}$	3	4	2
$\{2,-1,-1\}$	3	4	2
$\{2,-1,1\}$	6	4	4
$\{2,1,1\}$	3	4	2
$\{-1,-1,-1,-1\}$	1	4	1
$\{-1,-1,-1,1\}$	4	4	3
$\{-1,-1,1,1\}$	6	4	5
$\{-1,1,1,1\}$	4	4	3
$\{1,1,1,1\}$	1	4	1

Index Set	Number	Weight	Relations
$\{-5\}$	1	5	1
$\{5\}$	1	5	1
$\{-4,-1\}$	2	5	1
$\{-4,1\}$	2	5	1
$\{4,-1\}$	2	5	1
$\{4,1\}$	2	5	1
$\{-3,-2\}$	2	5	1
$\{-3,2\}$	2	5	1
$\{3,-2\}$	2	5	1
$\{3,2\}$	2	5	1
$\{-3,-1,-1\}$	3	5	2
$\{-3,-1,1\}$	6	5	4
$\{-3,1,1\}$	3	5	2
$\{3,-1,-1\}$	3	5	2
$\{3,-1,1\}$	6	5	4
$\{3,1,1\}$	3	5	2
$\{-2,-2,-1\}$	3	5	2
$\{-2,-2,1\}$	3	5	2
$\{-2,2,-1\}$	6	5	4
$\{-2,2,1\}$	6	5	4
$\{2,2,-1\}$	3	5	2
$\{2,2,1\}$	3	5	2
$\{-2,-1,-1,-1\}$	4	5	3
$\{-2,-1,-1,1\}$	12	5	9
$\{-2,-1,1,1\}$	12	5	9
$\{-2,1,1,1\}$	4	5	3
$\{2,-1,-1,-1\}$	4	5	3
$\{2,-1,-1,1\}$	12	5	9
$\{2,-1,1,1\}$	12	5	9
$\{2,1,1,1\}$	4	5	3
$\{-1,-1,-1,-1,-1\}$	1	5	1
$\{-1,-1,-1,-1,1\}$	5	5	4
$\{-1,-1,-1,1,1\}$	10	5	8
$\{-1,-1,1,1,1\}$	10	5	8
$\{-1,1,1,1,1\}$	5	5	4
$\{1,1,1,1,1\}$	1	5	1

Index Set	Number	Weight	Relations
$\{-6\}$	1	6	1
$\{6\}$	1	6	1
$\{-5,-1\}$	2	6	1
$\{-5,1\}$	2	6	1
$\{5,-1\}$	2	6	1
$\{5,1\}$	2	6	1
$\{-4,-2\}$	2	6	1
$\{-4,2\}$	2	6	1
$\{4,-2\}$	2	6	1
$\{4,2\}$	2	6	1
$\{-3,-3\}$	1	6	1
$\{-3,3\}$	2	6	1
$\{3,3\}$	1	6	1
$\{-2,-2,-2\}$	1	6	1
$\{-2,-2,2\}$	3	6	2
$\{-2,2,2\}$	3	6	2
$\{2,2,2\}$	1	6	1
$\{-4,-1,-1\}$	3	6	2
$\{-4,-1,1\}$	6	6	4
$\{-4,1,1\}$	3	6	2
$\{4,-1,-1\}$	3	6	2
$\{4,-1,1\}$	6	6	4
$\{4,1,1\}$	3	6	2
$\{-3,-2,-1\}$	6	6	4
$\{-3,-2,1\}$	6	6	4
$\{-3,2,-1\}$	6	6	4
$\{-3,2,1\}$	6	6	4
$\{3,-2,-1\}$	6	6	4
$\{3,-2,1\}$	6	6	4
$\{3,2,-1\}$	6	6	4
$\{3,2,1\}$	6	6	4
$\{-3,-1,-1,-1\}$	4	6	3
$\{-3,-1,-1,1\}$	12	6	9
$\{-3,-1,1,1\}$	12	6	9
$\{-3,1,1,1\}$	4	6	3
$\{3,-1,-1,-1\}$	4	6	3
$\{3,-1,-1,1\}$	12	6	9
$\{3,-1,1,1\}$	12	6	9
$\{3,1,1,1\}$	4	6	3

Index Set	Number	Weight	Relations
$\{-2,-2,-1,-1\}$	6	6	5
$\{-2,-2,-1,1\}$	12	6	9
$\{-2,-2,1,1\}$	6	6	5
$\{-2,2,-1,-1\}$	12	6	9
$\{-2,2,-1,1\}$	24	6	18
$\{-2,2,1,1\}$	12	6	9
$\{2,2,-1,-1\}$	6	6	5
$\{2,2,-1,1\}$	12	6	9
$\{2,2,1,1\}$	6	6	5
$\{-2,-1,-1,-1,-1\}$	5	6	4
$\{-2,-1,-1,-1,1\}$	20	6	16
$\{-2,-1,-1,1,1\}$	30	6	24
$\{-2,-1,1,1,1\}$	20	6	16
$\{-2,1,1,1,1\}$	5	6	4
$\{2,-1,-1,-1,-1\}$	5	6	4
$\{2,-1,-1,-1,1\}$	20	6	16
$\{2,-1,-1,1,1\}$	30	6	24
$\{2,-1,1,1,1\}$	20	6	16
$\{2,1,1,1,1\}$	5	6	4
$\{-1,-1,-1,-1,-1,-1\}$	1	6	1
$\{-1,-1,-1,-1,-1,1\}$	6	6	5
$\{-1,-1,-1,-1,1,1\}$	15	6	13
$\{-1,-1,-1,1,1,1\}$	20	6	17
$\{-1,-1,1,1,1,1\}$	15	6	13
$\{-1,1,1,1,1,1\}$	6	6	5
$\{1,1,1,1,1,1\}$	1	6	1

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