M-PARTITIONS: OPTIMAL PARTITIONS OF WEIGHT FOR ONE SCALE PAN

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Abstract. An M-partition of a positive integer m is a partition with as few parts as possible such that any positive integer less than m has a partition made up of parts taken from that partition of m. This is equivalent to partitioning a weight m so as to be able to weigh any integer weight $l < m$ with as few weights as possible and only one scale pan.

We show that the number of parts of an M-partition is a log-linear function of m and the M-partitions of m correspond to lattice points in a polytope. We exhibit a recurrence relation for counting the number of M-partitions of m and, for "half" of the positive integers, this recurrence relation will have a generating function. The generating function will be, in some sense, the same as the generating function for counting the number of distinct binary partitions for a given integer.

1. Introduction

Let m be a positive integer and let $\{\lambda_i : i = 0, 1, \ldots, n\}$ be a finite collection of, not necessarily distinct, positive integers with $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ and $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$. In this case, we say $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a *partition* of m with $n+1$ *parts*. We will also refer to the expression $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ as a partition. We call $\lambda_{i_0} + \lambda_{i_1} + \cdots + \lambda_{i_k}$ a *subpartition* of the partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ if $\{\lambda_{i_0}, \lambda_{i_1}, \ldots, \lambda_{i_k}\}$ is a subcollection of $\{\lambda_i : i = 0, 1, \ldots, n\}$.

In [3], MacMahon called a partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ *perfect* if every positive integer less than m can be expressed uniquely as a subpartition of $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$. In this paper, we introduce partitions that are close in spirit to MacMahon's. We maintain the subpartition property of perfect partitions but drop the uniqueness constraint and we demand that the number of parts in the partition be minimal.

Definition 1.1. An **M-partition** of m is a partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ with n being minimal such that $\{\sum_{i \in I} \lambda_i : I \subseteq \{0, 1, \ldots, n\}\} = \{0, 1, 2, \ldots, m\}$.

We denote the set of all M-partitions for m by $Mp(m)$. In Section 2 we will show that the number of parts in an M-partition is a log-linear function of m and that M-partitions correspond to the lattice points in a certain polytope. In particular, one can decide in polynomial time whether a given partition is an M-partition or not.

Theorem 2.10. *An M-partition of m has precisely $\lfloor \log_2 m \rfloor + 1$ parts.*

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Theorem 2.13. *The partition* $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ *is an* M*-partition if and only if* $\lambda_i \leq 1 + \lambda_0 + \cdots + \lambda_{i-1}$ *for each* $i \leq n$ *and* $2^n \leq \lambda_0 + \lambda_1 + \cdots + \lambda_n$.

In Section 3 we develop algorithms for generating M-partitions. These algorithms will be of great benefit when proving the main result of Section 4 which is a recurrence relation for counting the number of elements in $Mp(m)$, for each m. The following is a special case of that recurrence relation.

Theorem 4.7. *Let* m *be a positive integer with* $2^n + 2^{n-1} - 1 \leq m < 2^{n+1}$ *for some positive integer* n*. Then* $|\,Mp(m)\,| = \sum_{i=\lfloor \frac{m}{2} \rfloor}^{2^n-1} |\,Mp(i)\,|$.

In Section 5 we show that the recurrence relation of Theorem 4.7 is, in some sense, simultaneously counting the number of M-partitions for an integer m and counting the number of distinct binary partitions for a given integer.

Corollary 5.5. *If* $2^n + 2^{n-1} - 1 \leq m \leq 2^{n+1} - 1$ *and* $m = 2^{n+1} - 1 - k$ *then* $|\,Mp(m)\,|$ *equals the coefficient of* $x^{\lfloor \frac{k}{2} \rfloor}$ *in the generating function*

$$(1-x)^{-1} \prod_{j=0}^{\infty} (1 - x^{2^j})^{-1}.$$

In this paper $\mathbb{Z}^+$ will denote the positive integers and $m \in \mathbb{Z}^+$. For every $r \in \mathbb{R}$ we denote by $\lceil r \rceil$ the smallest integer greater than or equal to r; $\lfloor r \rfloor$ denotes largest integer less than or equal to r. By $\log_2 m$ we mean the logarithm of m base 2.

2. The parts of an M-partition

We begin by investigating the subpartition property of M-partitions. We define a weaker form of an M-partition by dropping the minimality of parts constraint.

Definition 2.1. A **weakM-partition** of m is a partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ with $\{\sum_{i \in I} \lambda_i \,:\, I \subseteq \{0, 1, \ldots, n\}\} = \{0, 1, 2, \ldots, m\}$.

If $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a $weakM$-partition of m then we must have $\lambda_0 = 1$. If $\lambda_1 \geq 3$ then it would not be possible to express 2 as a subpartition of $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ and so we must have $1 \leq \lambda_1 \leq 2$. In general, we have the following bounds on the parts of a $weakM$-partition.

Lemma 2.2. *If* $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ *is a* $weakM$*-partition then* $\lambda_i \leq 1 + \lambda_0 + \cdots + \lambda_{i-1}$ *for all* $i \leq n$.

Proof: Since $\lambda_i - 1 < \lambda_i$ then $\lambda_i - 1$ can be expressed as $\lambda_i - 1 = \sum_{j \in J} \lambda_j$ for some subset $J \subseteq \{0, 1, \ldots, i-1\}$. Consequently, $\lambda_i - 1 \leq \lambda_0 + \lambda_1 + \cdots + \lambda_{i-1}$. □

Lemma 2.3. *If* $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ *is any partition with* $\lambda_0 = 1$ *and* $\lambda_i \leq 1 + \lambda_0 \cdots + \lambda_{i-1}$ *for all* $i \leq n$ *then* $\lambda_i \leq 2^i$ *for all* $i \leq n$.

Proof: By assumption, $\lambda_0 = 1$. Proving by induction on i, assume $\lambda_k \leq 2^k$ for all $k \leq i - 1$. We are given that $\lambda_i \leq 1 + \lambda_0 \cdots + \lambda_{i-1}$ and so by the induction hypothesis, we have $\lambda_i \leq 1 + \sum_{k=0}^{i-1} 2^k = 2^i$. □

The upshot of Lemma 2.2 and Lemma 2.3 is a lower bound on the number of necessary parts in a $weakM$-partition.

Corollary 2.4. *If* $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ *is a* $weakM$*-partition then* $n \geq \lfloor \log_2 m \rfloor$.

Proof: If $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a *weak*M-partition of m then $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n \leq 2^{n+1} - 1 < 2^n$. This implies that $\lfloor \log_2 m \rfloor < n + 1$. Since $\lfloor \log_2 m \rfloor$ is an integer then it is no more than n. □

Remark 2.5. Lemma 2.2 and Lemma 2.3 apply equally to M-partitions since every M-partition is a *weak*M-partition. Corollary 2.4 provides a lower bound for the minimality of parts criterion of M-partitions.

It is well known that every postive integer has a unique binary representation and this has the following implication for *weak*M-partitions.

Lemma 2.6. *The partition* $1 + 2 + 4 + \cdots + 2^n$ *is a weak*M*-partition of* $2^{n+1} - 1$.

Remark 2.7. In order to show that a partition of m is a *weak*M-partition it is sufficient to show that for all $l \leq \lceil \frac{m}{2} \rceil$ there is some $J \subseteq \{0, 1, \ldots, n\}$ with $\sum_{j \in J} \lambda_j = l$, since $m - l = \sum_{j \in J^c} \lambda_j$ where J^c is the complement of J.

The following algorithm shows that the lower bound presented for the number of parts in Corollary 2.4 is sufficient.

Algorithm 2.8. There exists a *weak*M-partition of m with $\lfloor \log_2 m \rfloor + 1$ parts.

Proof: Let $n = \lfloor \log_2 m \rfloor$ and list the $n+1$ integers $2^0, 2^1, 2^2, \ldots, 2^{n-1}, m - (2^n - 1)$ in increasing order and set a one-to-one correspondence with $\lambda_0, \lambda_1, \ldots, \lambda_n$. Then $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a partition and we claim that every $l < m$ can be expressed a subpartition of this partition.

If $m = 2^{n+1} - 1$ then by Lemma 2.6 we are done. Otherwise, by Corollary 2.4, $m \leq 2^{n+1} - 2$ and so $\lceil \frac{m}{2} \rceil \leq 2^n - 1$. Since the parts of $2^n - 1 = 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}$ are all parts of the partition given then, combining Lemma 2.6 with Remark 2.7, we see that 2^0, 2^1, $2^2, \ldots, 2^{n-1}$, $m - (2^n - 1)$ are the parts of a *weak*M-partition of m. □

Example 2.9. Let $m = 53$. Using Algorithm 2.8 we have the *weak*M-partition $53 = 1 + 2 + 4 + 8 + 16 + 22$.

The first main result of this section is that the above algorithm describes a way to find an M-partition for any m.

Theorem 2.10. *An* M*-partition of* m *has precisely* $\lfloor \log_2 m \rfloor + 1$ *parts.*

Proof: Corollary 2.4 asserts that at least $\lfloor \log_2 m \rfloor + 1$ parts are needed for an M-partition of m. Algorithm 2.8 tells us that this is sufficient. □

Example 2.11.

$$\begin{aligned}
Mp(7) &= \{1 + 2 + 4\},\\
Mp(8) &= \{1 + 1 + 2 + 4,\ 1 + 1 + 3 + 3,\ 1 + 2 + 2 + 3\},\\
Mp(9) &= \{1 + 1 + 2 + 5,\ 1 + 1 + 3 + 4,\ 1 + 2 + 2 + 4,\ 1 + 2 + 3 + 3\},\\
Mp(10) &= \{1 + 1 + 3 + 5,\ 1 + 2 + 2 + 5,\ 1 + 2 + 3 + 4\},\\
Mp(11) &= \{1 + 1 + 3 + 6,\ 1 + 2 + 2 + 6,\ 1 + 2 + 3 + 5,\ 1 + 2 + 4 + 4\},\\
Mp(12) &= \{1 + 2 + 3 + 6,\ 1 + 2 + 4 + 5\},\ Mp(13) = \{1 + 2 + 3 + 7,\ 1 + 2 + 4 + 6\},\\
Mp(14) &= \{1 + 2 + 4 + 7\},\ Mp(15) = \{1 + 2 + 4 + 8\}.
\end{aligned}$$

You will need 5 parts for each M-partition of 16 and there 12 such M-partitions.

At first sight, it appears that deciding whether a partition is a $weakM$-partition or not could be an arduous endeavor. However, we have a relatively painless way of deciding so which avoids checking that the subpartition property holds for every $l < m$.

Lemma 2.12. *The partition $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a $weakM$-partition if and only if $\lambda_i \leq 1 + \lambda_0 + \cdots + \lambda_{i-1}$ for each $i \leq n$.*

Proof: The "only if" follows from Lemma 2.2. Conversely let $\mathcal{S}_n$ be the set of all partitions with $n+1$ parts that satisfy $\lambda_i \leq 1 + \lambda_0 + \cdots + \lambda_{i-1}$ for each $i \leq n$. We will argue the "if" by showing that $\mathcal{S}_n$ is contained in the set of $weakM$-partitions with $n+1$ parts. We will do so by induction on n.

It is clear that $\mathcal{S}_0 = \{1\}$. Assume the induction hypothesis on $\mathcal{S}_i$ for all $i \leq n-1$. Let $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ be a partition in $\mathcal{S}_n$ and let $l < \lambda_0 + \lambda_1 + \cdots + \lambda_n$. We need to show that l can be expressed as a subpartition of $\lambda_0 + \lambda_1 + \cdots + \lambda_n$. Note that $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1}$ is in $\mathcal{S}_{n-1}$ and so by our induction hypothesis if $l \leq \lambda_0 + \lambda_1 + \cdots + \lambda_{n-1}$ then there is nothing to show. Hence, we only need concern ourselves with $\lambda_{n-1} < l < \lambda_n$ and $l > \lambda_n$.

If $\lambda_{n-1} < l < \lambda_n$ then $l - \lambda_{n-1} < \lambda_n - \lambda_{n-1}$. By virtue of $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ being in $\mathcal{S}_n$ we have $\lambda_n - \lambda_{n-1} \leq (1 + \lambda_0 + \cdots + \lambda_{n-1}) - (\lambda_{n-1})$ and so $l - \lambda_{n-1} < \lambda_0 + \lambda_1 + \cdots + \lambda_{n-2}$. But the partition $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-2}$ is in $\mathcal{S}_{n-2}$ and so l can be expressed in terms of a subpartition of $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1}$ which is a subpartition of $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1} + \lambda_n$. Similarly, since $l < \lambda_0 + \lambda_1 + \cdots + \lambda_n$, $l > \lambda_n$ implies that $0 < l - \lambda_n < \lambda_0 + \lambda_1 \cdots + \lambda_{n-1}$. By our inductive hypothesis, $l - \lambda_n$ can be expressed a subpartition of $\lambda_0 + \lambda_1 \cdots + \lambda_{n-1}$ and so l can be expressed as a subpartition of $\lambda_0 + \lambda_1 \cdots + \lambda_n$. □

The second main result of this section is that there is an efficient way of deciding whether a given partition is an M-partition or not. This is achieved by a polyhedral characterization of M-partitions.

Theorem 2.13. *The partition $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ is an M-partition if and only if $\lambda_i \leq 1 + \lambda_0 + \cdots + \lambda_{i-1}$ for each $i \leq n$ and $2^n \leq \lambda_0 + \lambda_1 + \cdots + \lambda_n$.*

Proof: The "only if" follows from Lemma 2.2 and Theorem 2.10. As for the converse we need to show that $n = \lfloor \log_2(\lambda_0 + \lambda_1 + \cdots + \lambda_n) \rfloor$ and that $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a $weakM$-partition.

From Lemma 2.3 we have that $\lambda_0 + \lambda_1 + \cdots + \lambda_n < 2^{n+1}$ and, by assumption, we have $2^n \leq \lambda_0 + \lambda_1 + \cdots + \lambda_n$. Therefore, the partition $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ has the desired number of parts. From Lemma 2.12 we have $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a $weakM$-partition. □

An important consequence of Theorem 2.13 is that M-partitions are both built upon, and can be extended to, other M-partitions.

Corollary 2.14. *Let $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ be an M-partition. Then $\lambda_0 + \lambda_1 + \cdots + \lambda_j$ is an M-partition for all $j \leq n$. Also, if $r \in \mathbb{Z}^+$ then the partition $m + r = \lambda_0 + \lambda_1 + \cdots + \lambda_n + r$ is an M-partition of $m + r$ if and only if $\lambda_n \leq r$, $r \leq m + 1$ and $2^{n+1} \leq m + r$.*

Proof: Since $\lambda_i \leq 1 + \lambda_0 + \cdots + \lambda_{i-1}$ for each $i \leq n$ then $\lambda_i \leq (1 + \lambda_0 + \cdots + \lambda_{i-2}) + \lambda_{i-1} \leq (1 + \lambda_0 + \cdots + \lambda_{i-2}) + (1 + \lambda_0 + \cdots + \lambda_{i-2}) = 2(1 + \lambda_0 + \cdots + \lambda_{i-2})$.

Continuing in this fashion we can see that $\lambda_i \le 2^{i-j-1}(1+\lambda_0+\cdots+\lambda_j)$ for all $i > j$. Since $2^n \le \lambda_0+\lambda_1+\cdots+\lambda_n$ then $2^n \le 2^{n-j}(\lambda_0+\cdots+\lambda_j)+(2^{n-j}-1)$. Therefore, $2^j \le \lambda_0+\cdots+\lambda_j$ since $\frac{(2^{n-j}-1)}{2^{n-j}} < 1$. Since $\lambda_i \le 1+\lambda_0+\cdots+\lambda_{i-1}$ for each $i \le j$, then $\lambda_0+\lambda_1+\cdots+\lambda_j$ is an M-partition for all $j \le n$.

Next, $m+r = \lambda_0+\lambda_1+\cdots+\lambda_n+r$ is a partition which, by definition, means $\lambda_n \le r$. We assumed $m = \lambda_0+\lambda_1+\cdots+\lambda_n$ to be an M-partition so, by Theorem 2.13, both $r \le m+1$ and $2^{n+1} \le m+r$ are necessary and sufficient for our claim. □

Remark 2.15. An important reformulation of the extension statement in Corollary 2.14 is the following: Let $m \in \mathbb{Z}^+$ with $n = \lfloor \log_2 m \rfloor$ and let $m^{(1)} < m$. Then $m = \lambda_0+\lambda_1+\cdots+\lambda_{n-1}+(m-m^{(1)})$ is an M-partition if and only if $\lambda_{n-1} \le m-m^{(1)}$, $m^{(1)} = \lambda_0+\lambda_1+\cdots+\lambda_{n-1}$ is an M-partition and $m-m^{(1)} \le m^{(1)}+1$.

For the rest of this exposition, in light of Theorem 2.10 and Theorem 2.13, all partitions will be M-partitions unless otherwise stated, and n will always refer implicitly to some m via $n = n(m) := \lfloor \log_2 m \rfloor$.

3. Algorithms for generating M-partitions

In this brief section we give two more algorithms for generating M-partitions. These algorithms, in addition to Algorithm 2.8, will assist us in attaining an exact count for the number of M-partitions of m for all $m \in \mathbb{Z}^+$.

Algorithm 3.1. Letting $m \in \mathbb{Z}^+$, assign $\lambda_n = \lceil \frac{m}{2} \rceil$ and recursively define

$$\lambda_i = \lceil \frac{m-(\lambda_n+\lambda_{n-1}+\cdots+\lambda_{i+1})}{2} \rceil$$

for all non-negative $i < n$. Then $m = \lambda_0+\lambda_1+\cdots+\lambda_n$ is a partition of m.

Proof: By construction, $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_n$. Let T_n be the statement "if $m \in \mathbb{Z}^+$ with $n = \lfloor \log_2 m \rfloor$ then $m = \sum_{i=0}^{n} \lambda_i$." We will show by induction that T_n is true for all n.

The statement T_0 is true since $1 = \lceil \frac{1}{2} \rceil$. Assume that T_{n-1} is true. Let $m \in \mathbb{Z}^+$ with $n = \lfloor \log_2 m \rfloor$. Then $\lambda_n = \lceil \frac{m}{2} \rceil$ and so $m - \lambda_n = \lfloor \frac{m}{2} \rfloor$. But $\log_2 \lfloor \frac{m}{2} \rfloor = n-1$ and so $\lambda_0+\lambda_1+\cdots+\lambda_{n-1}$ is a partition of $\lfloor \frac{m}{2} \rfloor$ since T_{n-1} is assumed to be true. Hence $\lambda_0+\lambda_1+\cdots+\lambda_{n-1}+\lambda_n = \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil = m$. □

Corollary 3.2. *The partition $m = \lambda_0+\lambda_1+\cdots+\lambda_n$ given by Algorithm 3.1 is an M-partition.*

Proof: Since $n = \lfloor \log_2 m \rfloor$ then $2^n \le \lambda_0+\lambda_1+\cdots+\lambda_n$. Since $m = \lambda_0+\lambda_1+\cdots+\lambda_n$ then $\lambda_i = \lceil \frac{\lambda_0+\lambda_1+\cdots+\lambda_i}{2} \rceil < \frac{\lambda_0+\lambda_1+\cdots+\lambda_i}{2}+1$. Therefore, $\lambda_i \le 1+\lambda_0+\cdots+\lambda_{i-1}$ for all $i \le n$. By Theorem 2.13 the partition described is an M-partition. □

Algorithm 2.8 and Algorithm 3.1 provide M-partitions with $n+1$ parts for all m such that $2^n \le m < 2^{n+1}$. The next algorithm offers an M-partition for m if there is the further restriction that $2^n \le m \le 2^n+2^{n-1}-2$. The need for such a special case will become apparent in Section 4.

Algorithm 3.3. Let $m \in \mathbb{Z}^+$ with $2^n \le m \le 2^n+2^{n-1}-2$. Define $\lambda_i = 2^i$ for all $i \le n-2$, $\lambda_{n-1} = \lfloor \frac{m-(2^{n-1}-1)}{2} \rfloor$ and $\lambda_n = \lceil \frac{m-(2^{n-1}-1)}{2} \rceil$. Then $m = \lambda_0+\lambda_1+\cdots+\lambda_n$ is an M-partition.

Proof: It is clear that this algorithm provides a partition of m. By Theorem 2.10 the partition has the desired number of parts. All we need show is that $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a *weak*M-partition.

By Remark 2.7 all we need show that every $l \leq \lceil \frac{m}{2} \rceil \leq 2^{n-1} + 2^{n-2} - 1$ can be expressed as a subpartition of $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$. If $l \leq 2^{n-1} - 1$ then Lemma 2.6 applies and l can be expressed as a subpartition of $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-2}$. Alternatively, suppose $2^{n-1} \leq l \leq \lceil \frac{m}{2} \rceil \leq 2^{n-1} + 2^{n-2} - 1$. By our restrictions on m and our choice of λ_n we have $\lambda_n \geq 2^{n-2}$ and hence, $l - \lambda_n \leq 2^{n-1} - 1$. By Lemma 2.6, $l - \lambda_n$ can be expressed as a subpartition of $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-2}$ and thus l can be expressed as a subpartition of $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$. □

Example 2.9 continued. Algorithm 3.1 yields the M-partition $53 = 1 + 2 + 3 + 7 + 13 + 27$. Algorithm 3.3 produces the partition $53 = 1 + 2 + 4 + 8 + 9 + 9$ but this is not an M-partition as we have no way of expressing 16 as a subpartition.

4. Counting the number of elements in the set $Mp(m)$

For each $m \in \mathbb{Z}^+$ define $Mp(m)$ to be the set of all M-partitions of m. By Corollary 2.14 and Remark 2.15 we know that every M-partition must be constructed upon another of one less part. Letting $a_m := |Mp(m)|$ we construct a recurrence relation for a_m by way of finding sharp bounds on the largest part of an M-partition of m.

Fix $m \in \mathbb{Z}^+$. Let $m^{(1)} \in \mathbb{Z}^+$ be any integer whose M-partitions can be extended to an M-partition of m in the sense of Remark 2.15. Similarly, for each such $m^{(1)}$, let $m^{(12)} \in \mathbb{Z}^+$ be any integer whose M-partitions can be extended to an M-partition of $m^{(1)}$.

Remark 2.15 continued. The number of M-partitions of m, a_m equals the cardinality of the set of partitions given by

$$\left\{ \lambda_0 + \lambda_1 + \cdots + \lambda_{n-1} \ : \ \begin{array}{l} m^{(1)} < m,\ \lambda_{n-1} \leq m - m^{(1)},\ m - m^{(1)} \leq m^{(1)} + 1 \\ \text{and } \lambda_0 + \lambda_1 + \cdots + \lambda_{n-1} \text{ is an } M\text{-partition of } m^{(1)} \end{array} \right\}.$$

We now turn our attention to determining what values these $m^{(1)}$'s can take on for a given m. We do so by determining sharp bounds on the largest part of an M-partition of m.

Lemma 4.1. *Let $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ be an M-partition. Then*

$$\lceil \frac{m - 2^{n-i+1} + 1}{i} \rceil \leq \lambda_n.$$

Proof: By Lemma 2.3 we have $\lambda_i \leq 2^i$ for all $i \leq n$. Since $\lambda_{i-1} \leq \lambda_i$ for all $i \leq n-1$ then $i\lambda_n \geq \lambda_{n-i+1} + \cdots + \lambda_n = m - (\lambda_0 + \lambda_1 + \cdots + \lambda_{n-i}) \geq m - 2^{n-i+1} + 1$. Hence, $\lambda_n \geq \lceil \frac{m-2^{n-i+1}+1}{i} \rceil$ □

Remark 4.2. It is unnecessary to consider all of the bounds in Lemma 4.1 – we only need consider the bounds given by $i = 1$ and $i = 2$. When $2^n + 2^{n-1} - 1 \leq m \leq 2^{n+1} - 1$ then $m - 2^n + 1 \geq \lceil \frac{m-2^{n-i+1}+1}{i} \rceil$ for all $i \leq n$. If $2^n \leq m \leq 2^n + 2^{n-1} - 2$ then $\lceil \frac{m-2^{n-1}+1}{2} \rceil \geq \lceil \frac{m-2^{n-i+1}+1}{i} \rceil$ for all $i \leq n$.

Lemma 4.3. *Let* $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ *be an* M*-partition. Then*

$$max\{m - 2^n + 1, \lceil \frac{m - 2^{n-1} + 1}{2} \rceil\} \leq \lambda_n \leq \lceil \frac{m}{2} \rceil.$$

Furthermore, all three bounds are sharp.

Proof: If $\lambda_n > \lceil \frac{m}{2} \rceil$ then $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1} < m - \lceil \frac{m}{2} \rceil = \lfloor \frac{m}{2} \rfloor \leq \lceil \frac{m}{2} \rceil < \lambda_n$. This implies $\lfloor \frac{m}{2} \rfloor$ cannot be expressed as a subpartition of $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ which contradicts $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ being an M-partition. Hence $\lambda_n \leq \lceil \frac{m}{2} \rceil$. The lower bounds follow from Lemma 4.1 and Remark 4.2. Algorithm 2.8, Algorithm 3.1 and Algorithm 3.3 insure that all three bounds can be attained for any given m. □

Corollary 4.4. *Let* $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ *be an* M*-partition. Then*

$$\lfloor \frac{m}{2} \rfloor \leq \lambda_0 + \lambda_1 + \cdots + \lambda_{n-1} \leq min\{\lfloor \frac{m + 2^{n-1} - 1}{2} \rfloor, 2^n - 1\}.$$

For a given m, we can restate Corollary 4.4 in terms of the $m^{(1)}$'s and in turn for the $m^{(12)}$'s of each such $m^{(1)}$.

Corollary 4.5. *Let* $m \in \mathbb{Z}^+$. *Then*

$$\lfloor \frac{m}{2} \rfloor \leq m^{(1)} \leq min\{\lfloor \frac{m + 2^{n-1} - 1}{2} \rfloor, 2^n - 1\} \tag{1}$$

and for each such $m^{(1)}$ *we have*

$$\lfloor \frac{m^{(1)}}{2} \rfloor \leq m^{(12)} \leq min\{\lfloor \frac{m^{(1)} + 2^{n-2} - 1}{2} \rfloor, 2^{n-1} - 1\}. \tag{2}$$

Furthermore, all these bounds are attained.

Remark 4.6. The lower bound for $m^{(1)}$ is precisely the inequality $m - m^{(1)} \leq m^{(1)} + 1$. Similarly, $\lfloor \frac{m^{(1)}}{2} \rfloor \leq m^{(12)}$ is equivalent to $m^{(1)} - m^{(12)} \leq m^{(12)} + 1$.

Theorem 4.7. *Let* $m \in \mathbb{Z}^+$ *with* $2^n + 2^{n-1} - 1 \leq m < 2^{n+1}$. *Then*

$$a_m = \sum \{a_{m^{(1)}} : m^{(1)} \text{ satisfies inequality (1)}\} = \sum_{m^{(1)} = \lfloor \frac{m}{2} \rfloor}^{2^n - 1} a_{m^{(1)}}.$$

Proof: Let $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1}$ be an M-partition of any such $m^{(1)}$. Since $2^n + 2^{n-1} - 1 \leq m$ then $2^{n-1} \leq m - (2^n - 1) \leq m - m^{(1)}$. By Lemma 4.3, $\lambda_{n-1} \leq \lceil \frac{m^{(1)}}{2} \rceil \leq 2^{n-1} \leq m - m^{(1)}$. By Remark 4.6, $m - m^{(1)} \leq m^{(1)} + 1$. Therefore, all partitions of $m^{(1)}$ satisfying inequality (1) extend to an M-partition of m in the sense of Remark 2.15. □

Example 4.8. The M-partitions of 25 are extended from the M-partitions of $25^{(1)} = 12, 13, 14, 15$. Consequently, $a_{25} = a_{12} + a_{13} + a_{14} + a_{15}$. The M-partitions of 25 are listed here with $25 - 25^{(1)}$ in bold.

$$Mp(12) + \mathbf{13} = \{1 + 2 + 3 + 6 + \mathbf{13},\ 1 + 2 + 4 + 5 + \mathbf{13}\}$$
$$Mp(13) + \mathbf{12} = \{1 + 2 + 3 + 7 + \mathbf{12},\ 1 + 2 + 4 + 6 + \mathbf{12}\}$$
$$Mp(14) + \mathbf{11} = \{1 + 2 + 4 + 7 + \mathbf{11}\}$$
$$Mp(15) + \mathbf{10} = \{1 + 2 + 4 + 8 + \mathbf{10}\}$$

In general, for $2^n \le m < 2^{n+1}$, not every M-partition of an $m^{(1)}$ will have largest part no larger than $m - m^{(1)}$. As a result, the calculation of a_m may not be as straightforward as that of Theorem 4.7.

Example 2.11 continued. Let $m = 16$. By Corollary 4.5 we have $8 \le 16^{(1)} \le 11$. Thus the M-partitions are a subcollection of the following ordered compositions with $16 - 16^{(1)}$ in bold.

$$\begin{aligned} Mp(8) + \mathbf{8} &= \{1+1+2+4+\mathbf{8},\ 1+1+3+3+\mathbf{8},\ 1+2+2+3+\mathbf{8}\} \\ Mp(9) + \mathbf{7} &= \{1+1+2+5+\mathbf{7},\ 1+1+3+4+\mathbf{7},\ 1+2+2+4+\mathbf{7}, \\ &\quad\ 1+2+3+3+\mathbf{7}\} \\ Mp(10) + \mathbf{6} &= \{1+1+3+5+\mathbf{6},\ 1+2+2+5+\mathbf{6},\ 1+2+3+4+\mathbf{6}\} \\ Mp(11) + \mathbf{5} &= \{\underline{1+1+3+6+\mathbf{5}},\ \underline{1+2+2+6+\mathbf{5}},\ 1+2+3+5+\mathbf{5}, \\ &\quad\ 1+2+4+4+\mathbf{5}\} \end{aligned}$$

The two underlined compositions are not partitions because of the order on their parts but they do have the same parts as the compositions directly above them and these are M-partitions. Excluding the two underlined compositions, the remaining 12 ordered compositions are M-partitions and so $a_{16} = 12$.

In the proof of Theorem 4.7, $2^n + 2^{n-1} - 1 \le m$ was only required for $\lambda_{n-1} \le m - m^{(1)}$. All the other conditions of Remark 2.15 were honored by virtue of inequality (1). Keeping in mind that the M-partitions of $m^{(1)}$ are constructed on M-partitions of $m^{(12)}$ satisfying inequality (2), we can once again re-interpret Remark 2.15 as follows.

Remark 4.9. The number of M-partitions of m, a_m equals the cardinality of the set of partitions given by
$\mathcal{M}_1 := \{\, \lambda_0 + \lambda_1 + \cdots + \lambda_{n-2} + (m^{(1)} - m^{(12)}) \,:\, \lambda_{n-2} \le m^{(1)} - m^{(12)} \le m - m^{(1)}$ and $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-2} + (m^{(1)} - m^{(12)})$ is an M-partition of $m^{(1)}\}$.

Next we have a simple lemma that characterizes those partitions of $m^{(1)}$ that do not extend to M-partitions of m.

Lemma 4.10. *Let $m \in \mathbb{Z}^+$ with $2^n \le m < 2^{n+1}$ and assume that $m^{(1)} - m^{(12)} > m - m^{(1)}$. If $m^{(12)} = \lambda_0 + \lambda_1 + \cdots + \lambda_{n-2}$ is an M-partition then $\lambda_{n-2} < m^{(1)} - m^{(12)}$.*

Proof: Since $m^{(12)} < 2^{n-1}$ then $3m^{(12)} = 2m^{(12)} + m^{(12)} \le 2(2^{n-1} - 1) + m^{(12)} = 2^n + m^{(12)} - 2$. Also, $2^n \le m$ which implies that $3m^{(12)} \le m + m^{(12)} - 2$. By assumption we have $m + m^{(12)} < 2m^{(1)}$ and so $3m^{(12)} < 2m^{(1)} - 2$. Subtracting $2m^{(12)} - 2$ from both sides yields $\frac{m^{(12)}+2}{2} < m^{(1)} - m^{(12)}$. But $\lceil \frac{m^{(12)}}{2} \rceil < \frac{m^{(12)}+2}{2}$ and so $\lceil \frac{m^{(12)}}{2} \rceil < m^{(1)} - m^{(12)}$. Since λ_{n-2} is the largest part of an M-partition of $m^{(12)}$ then, by Lemma 4.3, we have $\lambda_{n-2} \le \lceil \frac{m^{(12)}}{2} \rceil$ and so $\lambda_{n-2} < m^{(1)} - m^{(12)}$. $\square$

We will now calculate the cardinality of the set $Mp(m)$ by determining the cardinality of the set $\mathcal{M}_1$ described in Remark 4.9. We will do so by a recurrence relation.

Theorem 4.11. *For any $m \in \mathbb{Z}^+$ there is a recurrence relation for a_m given by*

$$a_m = \sum_{m^{(1)} = \lfloor \frac{m}{2} \rfloor}^{min\{\lfloor \frac{m+2^{n-1}-1}{2} \rfloor, 2^n - 1\}} \{\, a_{m^{(1)}} - \sum_{m^{(12)} = \lfloor \frac{m^{(1)}}{2} \rfloor}^{2m^{(1)} - m - 1} a_{m^{(12)}} \,\}.$$

Proof: Let $\mathcal{M}$ equal the set $\{\, \lambda_0+\lambda_1+\cdots+\lambda_{n-2}+(m^{(1)}-m^{(12)}) \,:\, \lambda_0+\lambda_1+\cdots+\lambda_{n-2}+(m^{(1)}-m^{(12)})$ is an M-partition of $m^{(1)}\}$ and let $\mathcal{M}_2$ equal the subset of $\mathcal{M}$ given by $\{\, \lambda_0+\lambda_1+\cdots+\lambda_{n-2}+(m^{(1)}-m^{(12)}) \,:\, m^{(1)}-m^{(12)} > m-m^{(1)}$ and $\lambda_0+\lambda_1+\cdots+\lambda_{n-2}+(m^{(1)}-m^{(12)})$ is an M-partition of $m^{(1)}\}$. Then $\mathcal{M}=\mathcal{M}_1\cup\mathcal{M}_2$ is a disjoint union of the set $\mathcal{M}$ and so we must have $|\mathcal{M}_1| = |\mathcal{M}| - |\mathcal{M}_2|$.

The set $\mathcal{M}$ is the set of all M-partitions of $m^{(1)}$ satisfying inequality (1).

$$\mathcal{M} \;=\; \bigcup_{m^{(1)}=\lfloor\frac{m}{2}\rfloor}^{min\{\lfloor\frac{m+2^{n-1}-1}{2}\rfloor,2^n-1\}} Mp(m^{(1)})\,.$$

On the other hand, Lemma 4.10 says that $\mathcal{M}_2$ is in bijection with the set of all M-partitions of $m^{(12)}$ for all $m^{(12)}$ satisfying inequality (2) and with $m^{(1)}-m^{(12)} > m-m^{(1)}$. That is, $\mathcal{M}_2$ is in bijection with the set of all M-partitions of $m^{(12)}$ with $\lfloor\frac{m^{(1)}}{2}\rfloor \le m^{(12)} \le 2m^{(1)}-m-1$ where $m^{(1)}$ satisfies inequality (1) and thus we can write the cardinality of $\mathcal{M}_2$ as

$$|\,\mathcal{M}_2\,| \;=\; \Big|\bigcup_{m^{(1)}=\lfloor\frac{m}{2}\rfloor}^{min\{\lfloor\frac{m+2^{n-1}-1}{2}\rfloor,2^n-1\}} \;\bigcup_{m^{(12)}=\lfloor\frac{m^{(1)}}{2}\rfloor}^{2m^{(1)}-m-1} Mp(m^{(12)})\Big|\,.$$

Recalling that $a_m \;=\; |\,\mathcal{M}\,| - |\,\mathcal{M}_2\,|$, we can write

$$a_m \;=\; \sum_{m^{(1)}\,=\,\lfloor\frac{m}{2}\rfloor}^{min\{\lfloor\frac{m+2^{n-1}-1}{2}\rfloor,\,2^n-1\}} \{\, a_{m^{(1)}} \;-\; \sum_{m^{(12)}=\lfloor\frac{m^{(1)}}{2}\rfloor}^{2m^{(1)}-m-1} a_{m^{(12)}}\,\}.$$

□

m	a_m	m	a_m	m	a_m	m	a_m
1	1	17	15	33	91	49	26
2	1	18	13	34	82	50	20
3	1	19	14	35	89	51	20
4	1	20	11	36	77	52	14
5	2	21	12	37	80	53	14
6	1	22	9	38	70	54	10
7	1	23	10	39	73	55	10
8	3	24	6	40	60	56	6
9	4	25	6	41	63	57	6
10	3	26	4	42	53	58	4
11	4	27	4	43	54	59	4
12	2	28	2	44	43	60	2
13	2	29	2	45	44	61	2
14	1	30	1	46	35	62	1
15	1	31	1	47	36	63	1
16	12	32	84	48	26	64	908

TABLE 1. Values of a_m for $1 \le m \le 64$.

Remark 4.12. As we would expect, Theorem 4.7 follows as a special case of Theorem 4.11. The set $\mathcal{M}_2$ contains no elements precisely when $\lfloor \frac{m^{(1)}}{2} \rfloor > 2m^{(1)} - m - 1$. This occurs only if $2^n + 2^{n-1} - 1 \leq m < 2^{n+1}$.

Example 2.11 continued. Let $m = 16$. According to Theorem 4.11 $a_{16} = a_8 + a_9 + a_{10} + (a_{11} - a_5) = 3+4+3+(4-2)$. The only instance of $\lfloor \frac{m^{(1)}}{2} \rfloor \leq 2m^{(1)} - m - 1$ being satisfied is when $16^{(1)} = 11$. Looking at the M-partitions of 11 we see that there are two of them with largest part larger than $16 - 11 = 5$; $\underline{1+1+3+6}$ and $\underline{1+2+2+6}$. Both of these M-partitions have largest part 6 and so must be built upon all the M-partitions of 5. Hence, we subtract a_5 from a_{11}.

5. Simplifying the recurrence relation for $|Mp(m)|$

In this section we exhibit a generating function for $m \in \mathbb{Z}^+$ provided that $2^n + 2^{n-1} - 1 \leq m < 2^{n+1}$. In particular the recurrence relation of Theorem 4.7 has a generating function.

Lemma 5.1. *For even m with $2^n + 2^{n-1} \leq m < 2^{n+1}$ we have $a_m = a_{m+1}$.*

Proof: Since $2^n + 2^{n-1} \leq m \leq 2^{n+1} - 2$ then Theorem 4.7 will suffice to calculate both a_m and a_{m+1}. Since m is even then $\lfloor \frac{m}{2} \rfloor = \lfloor \frac{m+1}{2} \rfloor$ and so the recurrence relation of Theorem 4.7 is the same for both a_m and a_{m+1}. □

We prove another lemma which will play a crucial role in the proof of the main theorem of this section.

Lemma 5.2. *For any integer $j \geq 0$ define the recurrence relation $b_j = b_{j-1} + b_{\lfloor \frac{j}{2} \rfloor}$ with initial condition $b_0 = 1$. Then $b_j = \sum_{i=0}^{j} b_{\lfloor \frac{i}{2} \rfloor}$*

Proof: The lemma is true for $j = 0$ and $j = 1$. Utilizing an induction argument assume true for all $j < l$. Then

$$b_l = b_{l-1} + b_{\lfloor \frac{l}{2} \rfloor} = \sum_{i=0}^{l-1} b_{\lfloor \frac{i}{2} \rfloor} + b_{\lfloor \frac{l}{2} \rfloor} = \sum_{i=0}^{l} b_{\lfloor \frac{i}{2} \rfloor}.$$

The last two equalities follow by the inductive hypothesis and so our claim is true for any non negative integer. □

The recurrence relation of Lemma 5.2 provides a more efficient accounting of a_m than that in Theorem 4.7.

Lemma 5.3. *Let $m \in \mathbb{Z}^+$ satisfying $2^n + 2^{n-1} - 1 \leq m \leq 2^{n+1} - 1$ and write m in the form $m = 2^{n+1} - 1 - k$. Then $a_m = a_{2^{n+1}-1-k} = b_{\lfloor \frac{k}{2} \rfloor}$.*

Proof: This will be shown by induction on $n = \lfloor \log_2 m \rfloor$. From Table 1 we can see that our claim is true for $n = 0, 1, 2$ and so assume that our claim is true for all positive integers less than some n and pick an m such that $2^n + 2^{n-1} - 1 \leq m \leq 2^{n+1} - 1$.

Since $m = 2^{n+1} - 1 - k$ then $\lfloor \frac{m}{2} \rfloor = \lfloor \frac{2^{n+1}-2-(k-1)}{2} \rfloor = 2^n - 1 - \lfloor \frac{k}{2} \rfloor$. From Theorem 4.7 we have

$$a_m = \sum_{m^{(1)} = \lfloor \frac{m}{2} \rfloor}^{2^n - 1} a_{m^{(1)}} = \sum_{m^{(1)} = 2^n - 1 - \lfloor \frac{k}{2} \rfloor}^{2^n - 1} a_{m^{(1)}}.$$

Each $m^{(1)}$ satisfies $\lfloor \log_2 m^{(1)} \rfloor = n-1$ and so our inductive hypothesis says this last summand (after reversing the order of summation) can be expressed as follows

$$a_m = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} a_{2^n-1-i} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} b_{\lfloor \frac{i}{2} \rfloor} = b_{\lfloor \frac{k}{2} \rfloor}.$$

The last equality comes from Lemma 5.2. □

We say a partition is *binary* if all its parts are powers of 2. See [1] for results about such partitions. In [2], Knuth studied binary partitions whose parts were all distinct and, amongst other things, derived the following result.

Theorem 5.4. *(Knuth) The number of distinct binary partitions of* $2j$ *into powers of* 2 *equals* b_j *where* b_j *is the recurrence relation in Lemma 5.2. Furthermore, this recurrence relation has a generating function given by*

$$(1-x)^{-1} \prod_{j=0}^{\infty} (1-x^{2^j})^{-1}.$$

We consequently have our main result which is a generating function for a_m when $2^n + 2^{n-1} - 1 \leq m < 2^{n+1}$.

Corollary 5.5. *If* $2^n + 2^{n-1} - 1 \leq m \leq 2^{n+1} - 1$ *and* $m = 2^{n+1} - 1 - k$ *then* a_m *equals the coefficient of* $x^{\lfloor \frac{k}{2} \rfloor}$ *in the above generating function.*

For the case of $2^n \leq m \leq 2^n + 2^{n-1} - 2$ it appears that the best we can do is the following: If $2^n \leq m = 2^{n+1} - 1 - k \leq 2^n + 2^{n-1} - 2$ and $m' = 2^n - 1 - k'$ is such that $m = 2^{n-1} + 2^{n-2} + m'$ then $b_{\lfloor \frac{k}{2} \rfloor} - a_m = b_{\lfloor \frac{k'}{2} \rfloor} - a_{m'}$. However, it seems that no generating function can be arrived at for m in the interval $2^n \leq m \leq 2^n + 2^{n-1} - 2$. In other words, a generating function for the recurrence relation of Theorem 4.11 could not be arrived at.

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