Normal Order: Combinatorial Graphs

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Abstract. A conventional context for supersymmetric problems arises when we consider systems containing both boson and fermion operators. In this note we consider the normal ordering problem for a string of such operators. In the general case, upon which we touch briefly, this problem leads to combinatorial numbers, the so-called *Rook numbers*. Since we assume that the two species, bosons and fermions, commute, we subsequently restrict ourselves to consideration of a single species, single-mode boson monomials. This problem leads to elegant generalisations of well-known combinatorial numbers, specifically Bell and Stirling numbers. We explicitly give the generating functions for some classes of these numbers. In this note we concentrate on the combinatorial graph approach, showing how some important classical results of graph theory lead to transparent representations of the combinatorial numbers associated with the boson normal ordering problem.

1. Normal Ordering

In this note we give a brief review of some combinatorial graphs associated with the normal ordering of creation and annihilation operators.

The process of normally ordering a string of creation and annihilation operators simply means reordering the elements of the string so that all the annihilation operators appear on the right, taking into account the commutation (or anti-commutation) relations. The value of such a procedure is that, for example, the expectation value in a coherent state of such a normally ordered string may be immediately seen. Performing this operation on strings of bosons or fermions leads to classical combinatorial numbers and, in the analogous *quon* case, to q-variants of these numbers. In a "supersymmetric" context one may well have to consider a general string of bosons and fermions; but as the bosons and fermions are assumed to commute, the ordering reduces trivially to ordering the two species separately. We thus assume the basic commutation relations

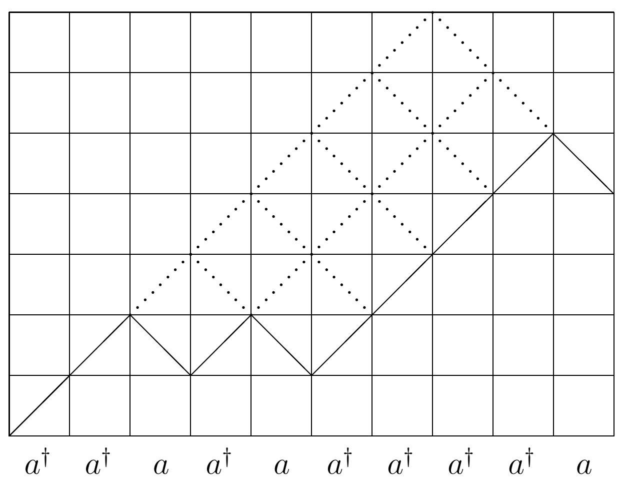
- Bosons $[a_i, {a_j}^\dagger] = \delta_{ij}$
- Fermions $\{f_i, {f_j}^\dagger\} = \delta_{ij}$
- Mixed $[a_i, {f_j}^\dagger] = 0$

We note that the boson ordering problem is equivalent to ordering a string of operators $a \equiv d/dx$ and $a^\dagger \equiv x$.

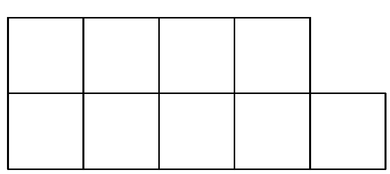
For both bosons and fermions, the ordering process of a general string results in combinatorial numbers called *Rook numbers*. We shall merely give a brief account of this approach here; further details, especially with respect to the fermion case, may be found in Ref. [1]. In this note we *illustrate* rather than *prove* the methods described.

2. Rook numbers and generalized Stirling numbers

The normal form of a bosonic string or "word", $w = w(a, a^\dagger)$, satisfies $\mathcal{N}(w) = w$. Normal forms of boson strings are connected to the so-called *Rook numbers* [2], [3] in the following way [4]. Draw a North-East ($\nearrow$) line over each creation operator $a^\dagger$ and a South-East ($\searrow$) line over each annihilation operator a so as to produce a continuous line. This gives the diagram below.



The picture is completed by the dotted lines as shown. The result of this process is a *Ferrers diagram* or *board* [3] representing, in this case, the partition $\{5, 4\}$ of 9 represented by the following diagram.



The k-th *rook number* $r_k(B)$ of a Ferrers board B is the number of ways of placing k non-capturing rooks on the board. For our example of the partition $\{5, 4\}$ one has

k	0	1	2	$k > 2$
$r_k(B)$	1	9	16	0

and thus the normal form of $w = a^\dagger a^\dagger a a^\dagger a a^\dagger a^\dagger a^\dagger a^\dagger a$ is given by

$$\mathcal{N}(w) = a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger aaa + 9\, a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger aa + 16\, a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a. \tag{1}$$

For $w = aa^\dagger a^2 (a^\dagger)^2$, the board represents the partition $[3, 2, 2]$ and

$$\begin{aligned} \mathcal{N}(aa^\dagger a^2 (a^\dagger)^2) &= a^\dagger a^\dagger a^\dagger aaa + 7\, a^\dagger a^\dagger aa + 10\, a^\dagger a + 2 \\ &= r_0(B)(a^\dagger)^3 a^3 + r_1(B)(a^\dagger)^2 a^2 + r_2(B)(a^\dagger)a + r_3(B). \end{aligned} \tag{2}$$

In fact Eqs.(1) and (2) are illustrations of the general formula expressing the normal form $\mathcal{N}(w)$ with the help of the rook numbers as

$$\mathcal{N}(w) = \sum_{k=0}^{\infty} r_k(B) : w^{(k)} : \tag{3}$$

In Eq.(3) $: w^{(k)} :$ means that in the word $w = w(a, a^\dagger)$ we cross out k a's and k $a^\dagger$'s and then order normally the result *without* taking into account the commutation relations. In fact for finite word w the sum in Eq.(3) has a finite number of non-vanishing terms. Note that coefficients of the normal monomials are the rook numbers of the board; it is possible to give a simple algorithm which computes these numbers.

Similar observations apply to the normal ordering of *quons* a_q (q-bosons) satisfying

$$[a_q, a_q^\dagger]_q \equiv a_q a_q^\dagger - q a_q^\dagger a_q = 1,$$

the q-Weyl algebra†.

If we restrict ourselves to simple recurring strings in one boson mode, which we shall henceforth do, we see that classical combinatorial numbers appear naturally [5], [6].

The normal ordering problem for canonical bosons $[a, a^\dagger] = 1$ is related to certain combinatorial numbers $S(n, k)$ called *Stirling numbers of the second kind* through [7]

$$(a^\dagger a)^n = \sum_{k=1}^{n} S(n, k)(a^\dagger)^k a^k, \tag{4}$$

with corresponding numbers $B(n) = \sum_{k=1}^{n} S(n, k)$ called *Bell numbers*. In fact, for physicists, these equations may be taken as the *definitions* of the Stirling and Bell numbers. For the quons a_q (q-bosons) mentioned above, a natural q-generalisation [8], [9] of these numbers is

$$(a_q^\dagger a_q)^n = \sum_{k=1}^{n} S_q(n, k)(a_q^\dagger)^k a_q^k. \tag{5}$$

In the *canonical* boson case, for integers $n, r, s > 0$ we define generalized Stirling numbers of the second kind $S_{r,s}(n, k)$ through ($r \geq s$):

$$[(a^\dagger)^r a^s]^n = (a^\dagger)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n, k)(a^\dagger)^k a^k, \tag{6}$$

as well as generalized Bell numbers $B_{r,s}(n)$

$$B_{r,s}(n) = \sum_{k=s}^{ns} S_{r,s}(n, k). \tag{7}$$

For both $S_{r,s}(n, k)$ and $B_{r,s}(n)$ exact and explicit formulas have been found [5], [6]. We refer the interested reader to these sources for further information on those extensions. However,

† An interesting, if as yet experimental, observation is that the resulting polynomials in q obtained by setting $a_q = 1 = a_q^\dagger$ are *unimodal*.

in this note we shall mainly deal with the classical Bell and Stirling numbers, corresponding to $B_{1,1}(n)$ and $S_{1,1}(n)$ in our notation, and the extension to $B_{2,1}(n)$ and $S_{2,1}(n)$.

The conventional and picturesque description of the classical Bell and Stirling Numbers of the second kind is in terms of the distribution of differently coloured balls among identical containers or, equivalently, the number of partitions of an n-element set [10], [11]. The relation of this classical definition to the normal order expansion of $(a^\dagger a)^n$ is via the contractions induced by the application of Wick's Theorem. In general, $S(n,m)$ gives the coefficient of the term $:(a^\dagger a)^m :\equiv (a^\dagger)^m a^m$ in the normal ordering expansion, directly from the (physicist's) definition of $S(n,m)$.

Taking the concrete example $(a^\dagger a)^3$,

- $S(3,1)=1$ is the coefficient of the term $: a^\dagger a :\equiv a^\dagger a$
- $S(3,2)=3$ is the coefficient of the term $: (a^\dagger a)^2 :\equiv (a^\dagger)^2 a^2$
- $S(3,3)=1$ is the coefficient of the term $: (a^\dagger a)^3 :\equiv (a^\dagger)^3 a^3$

In the next section we relate these combinatorial numbers to certain graphs. We shall elaborate on this graphical approach elsewhere [12].

3. Generating Functions and Graphs

In general, for combinatorial numbers $g(n)$ we may define an *exponential generating function* $G(x)$ through [13]

$$G(x)=\sum_{n=0}^{\infty} g(n)\frac{x^n}{n!}. \tag{8}$$

For the Bell numbers, this generating function takes the particularly nice form [11]

$$G(x)=\sum_{n=0}^{\infty} B(n)\frac{x^n}{n!}=\exp(\exp(x)-1). \tag{9}$$

We shall derive this exponential generating function by the use of a simple graph theory technique. Some initial terms of the sequence $\{B(n)\}$ are $\{1,2,5,15,52,203,877,\ldots\}$.

Another convenient way of representing combinatorial numbers is by means of *graphs*. To illustrate this, we now consider a graphical method for describing the combinatorial numbers $B(n)$ associated with the normal order expansion of $(a^\dagger a)^n$. For concreteness, take the cases $n=1,2,3$ (Figure 1). In this diagram the filled dots on the left represent the differently coloured balls, while the empty dots represent the identical containers. This pictorial approach gives the values for $S(n,m)$. By convention, $S(n,0)=\delta_{n,0}$.

One reason the graphical representation is useful is that there exist some rather powerful results which apply to graphs and their associated generating functions. We list and apply two such results, labelled A and B, in what follows.

A. Connected graph theorem

This states that if $C(x)=\sum_{n=1}^{\infty} c(n)x^n/n!$ is the exponential generating function of *labelled connected* graphs, *viz.* $c(n)$ counts the number of connected graphs of order n, then

$$A(x)=\exp(C(x)) \tag{10}$$

is the exponential generating function for *all* graphs.

We may apply this very simply to the case of the $B(n)$ graphs in Figure 1. For each order n, the *connected* graphs clearly consist of a single graph. Therefore for each n we have

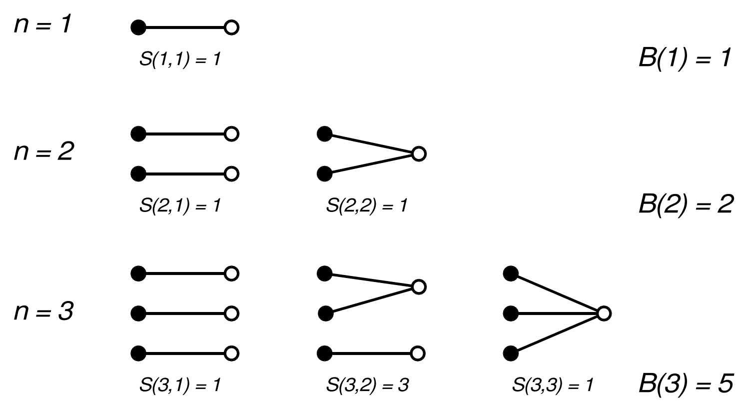


Figure 1. Graphs for $B(n), \quad n = 1, 2, 3.$

$c(n) = 1$; whence, $C(x) = exp(x) - 1$. It follows that the generating function for *all* the graphs $A(x)$ is given by

$$A(x) = \exp(\exp(x) - 1) \tag{11}$$

which is the generating function for the Bell numbers.

Such graphs may be generalised to give graphical representations for the extensions $B_{r,s}(n)$ [12]. We illustrate this by using the following powerful result on certain classes of graphs:

B. Generating function for a class of graphs

We generalize the graphical representation for the Bell numbers given in Figure 1. As before, we shall be counting labelled lines. A line starts from a black dot, the *origin*, and ends at a white dot, the *vertex*. What we refer to as *origin* and *vertex* is, of course, arbitrary. At this point there are no other rules, although we are at liberty to impose further restrictions; a black dot may be the origin of 1,2,3,... lines, and a white dot the vertex for 1,2,3,... lines. We may further associate *strengths* V_s with each vertex receiving s lines, and multipliers L_m with a black dot which is the origin of m lines. Again $\{V_s\}$ and $\{L_m\}$ play symmetric roles; in this note we shall only consider cases where the L_m are either 0 or 1.

In Figure 2 we illustrate these rules for four different graphs corresponding to $n = 4$.

There is an exponential generating function $G(x, V, L)$ which counts the number $g(n)$ of graphs with n lines arising from the above rules [14], [15]:

$$\begin{aligned} G(x, V, L) = & \exp\left(\sum_{m=1}^{\infty} L_m \frac{x^m}{m!} \frac{d^m}{dy^m}\right) \exp\left(\sum_{s=1}^{\infty} V_s \frac{y^s}{s!}\right)\Bigg|_{y=0} \\ \equiv & \sum_{n=0}^{\infty} g(n) \frac{x^n}{n!} \end{aligned} \tag{12}$$

Example 1: The exponential generating function corresponding to Figure 1 is obtained by putting $L_m = 0$ for $m \neq 1$ and $V_s = 1$ for all s. This immediately gives the exponential

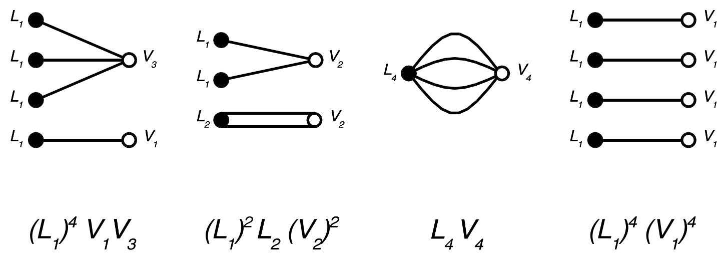


Figure 2. Some examples of 4-line graphs.

generating function for the Bell numbers $B_{1,1}(n) \equiv B(n)$ as $\exp(\exp(x-1))$, a result we have already obtained through use of the Connected Graph Theorem above.
Example 2: The diagrams of Figure 1, that is corresponding to $L_m = 0$ for $m \neq 1$, are in a sense generic for the Wick contractions occurring in expressions such as $((a^\dagger)^r a)^n$ [12]. However, the coefficients V_s depend on the exponent r. We obtain the exponential generating function for $B_{2,1}(n)$, corresponding to the normal ordering of $((a^\dagger)^2 a)^n$, by putting $V_s = s!$. This immediately leads to the formal expression

$$\begin{aligned} G(x) = & \exp\left(x\frac{d}{dy}\right)\exp\left(\sum_{s=1}^{\infty} y^s\right)\Bigg|_{y=0} \\ \equiv & \exp\left(\frac{x}{1-x}\right) \end{aligned} \tag{13}$$

which is the exponential generating function for $B_{2,1}(n)$ [5], [6].

Many other applications and extensions of the ideas sketched in this note will be found in [12].

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