FREE QUASI-SYMMETRIC FUNCTIONS OF ARBITRARY LEVEL

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Abstract. We introduce analogues of the Hopf algebra of Free quasi-symmetric functions with bases labelled by colored permutations. As applications, we recover in a simple way the descent algebras associated with wreath products $\Gamma \wr \mathfrak{S}_n$ and the corresponding generalizations of quasi-symmetric functions. Finally, we obtain Hopf algebras of colored parking functions, colored non-crossing partitions and parking functions of type B.

1. Introduction

The Hopf algebra of Free quasi-symmetric functions **FQSym** [3] is a certain algebra of noncommutative polynomials associated with the sequence $(\mathfrak{S}_n)_{n\geq 0}$ of all symmetric groups. It is connected by Hopf homomorphisms to several other important algebras associated with the same sequence of groups : Free symmetric functions (or coplactic algebra) **FSym** [19, 3], Non-commutative symmetric functions (or descent algebras) **Sym** [4], Quasi-Symmetric functions *QSym* [5], Symmetric functions *Sym*, and also, Planar binary trees **PBT** [11, 7], Matrix quasi-symmetric functions **MQSym** [3, 6], Parking functions **PQSym** [9, 16], and so on.

Among the many possible interpretations of *Sym*, we can mention the identification as the representation ring of the tower of algebras

$$\mathbb{C} \to \mathbb{C}\,\mathfrak{S}_1 \to \mathbb{C}\,\mathfrak{S}_2 \to \cdots \to \mathbb{C}\,\mathfrak{S}_n \to \cdots\ , \tag{1}$$

that is

$$Sym \simeq \oplus_{n\geq 0} R(\mathbb{C}\,\mathfrak{S}_n), \tag{2}$$

where $R(\mathbb{C}\,\mathfrak{S}_n)$ is the vector space spanned by isomorphism classes of irreducible representations of $\mathfrak{S}_n$, the ring operations being induced by direct sum and outer tensor product of representations [13].

Another important interpretation of *Sym* is as the support of Fock space representations of various infinite dimensional Lie algebras, in particular as the level 1 irreducible highest weight representations of $\widehat{\mathfrak{gl}}_\infty$ (the infinite rank Kac-Moody algebra of type A_∞, with Dynkin diagram $\mathbb{Z}$, see [8]).

The analogous level l representations of this algebra can also be naturally realized with products of l copies of *Sym*, or as symmetric functions in l independent sets of variables

$$(Sym)^{\otimes l} \simeq Sym(X_0; \ldots ; X_{l-1}) =: Sym^{(l)}, \tag{3}$$

and these algebras are themselves the representation rings of wreath product towers $(\Gamma \wr \mathfrak{S}_n)_{n\geq 0}$, Γ being a group with l conjugacy classes [13] (see also [26, 25]).

We shall therefore call for short $Sym(X_0; \ldots; X_{l-1})$ the algebra of symmetric functions of level l. Our aim is to associate with $Sym^{(l)}$ analogues of the various Hopf algebras mentionned at the beginning of this introduction.

We shall start with a level l analogue of **FQSym**, whose bases are labelled by l-colored permutations. Imitating the embedding of **Sym** in **FQSym**, we obtain a Hopf subalgebra of level l called $\mathbf{Sym}^{(l)}$, which turns out to be dual to Poirier's quasi-symmetric functions, and whose homogenous components can be endowed with an internal product, providing an analogue of Solomon's descent algebras for wreath products.

The Mantaci-Reutenauer descent algebra arises as a natural Hopf subalgebra of $\mathbf{Sym}^{(l)}$ and its dual is computed in a straightforward way by means of an appropriate Cauchy formula.

Finally, we introduce a Hopf algebra of colored parking functions, and use it to define Hopf algebras structures on parking functions and non-crossing partitions of type B.

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2. Free quasi-symmetric functions of level l

2.1. l-colored standardization. We shall start with an l-colored alphabet

$$A = A^0 \sqcup A^1 \sqcup \cdots \sqcup A^{l-1}, \tag{4}$$

such that all A^i are of the same cardinality N, which will be assumed to be infinite in the sequel. Let C be the alphabet $\{c_0, \ldots, c_{l-1}\}$ and B be the auxiliary ordered alphabet $\{1, 2, \ldots, N\}$ (the letter C stands for *colors* and B for *basic*) so that A can be identified to the cartesian product $B \times C$:

$$A \simeq B \times C = \{(b, c), b \in B,\ c \in C\}. \tag{5}$$

Let w be a word in A, represented as (v, u) with $v \in B^*$ and $u \in C^*$. Then the *colored standardized word* $\mathbf{Std}(w)$ of w is

$$\mathbf{Std}(w) := (\mathrm{Std}(v), u), \tag{6}$$

where $\mathrm{Std}(v)$ is the usual standardization on words.

Recall that the standardization process sends a word w of length n to a permutation $\mathrm{Std}(w) \in \mathfrak{S}_n$ called the *standardized* of w defined as the permutation obtained by iteratively scanning w from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\mathrm{Std}(w)$ is the permutation having the same inversions as w.

2.2. $\mathbf{FQSym}^{(l)}$ and $\mathbf{FQSym}^{(\Gamma)}$. A *colored permutation* is a pair (σ, u), with $\sigma \in \mathfrak{S}_n$ and $u \in C^n$, the integer n being the *size* of this permutation.

Definition 2.1. *The* dual free l-quasi-ribbon $\mathbf{G}_{\sigma,u}$ *labelled by a colored permutation* (σ, u) *of size* n *is the noncommutative polynomial*

$$\mathbf{G}_{\sigma,u} := \sum_{w \in A^n; \mathbf{Std}(w) = (\sigma,u)} w \quad \in \mathbb{Z}\langle A \rangle. \tag{7}$$

Recall that the *convolution* of two permutations σ and μ is the set $\sigma*\mu$ (identified with the formal sum of its elements) of permutations τ such that the standardized word of the $|\sigma|$ first letters of τ is σ and the standardized word of the remaining letters of τ is μ (see [21]).

Theorem 2.2. *Let (σ', u') and (σ'', u'') be colored permutations. Then*

$$\mathbf{G}_{\sigma',u'}\ \mathbf{G}_{\sigma'',u''} = \sum_{\sigma\in\sigma'*\sigma''} \mathbf{G}_{\sigma,u'\cdot u''}, \tag{8}$$

where $w_1 \cdot w_2$ is the word obtained by concatenating w_1 and w_2. Therefore, the dual free l-quasi-ribbons span a $\mathbb{Z}$-subalgebra of the free associative algebra.

Moreover, one defines a coproduct on the $\mathbf{G}$ functions by

$$\Delta\mathbf{G}_{\sigma,u} := \sum_{i=0}^{n} \mathbf{G}_{(\sigma,u)_{[1,i]}} \otimes \mathbf{G}_{(\sigma,u)_{[i+1,n]}}, \tag{9}$$

where n is the size of σ and $(\sigma, u)_{[a,b]}$ is the standardized colored permutation of the pair (σ', u') where σ' is the subword of σ containing the letters of the interval $[a, b]$, and u' the corresponding subword of u.

For example,

$$\begin{aligned}\Delta\mathbf{G}_{3142,2412} = 1 \otimes \mathbf{G}_{3142,2412} + \mathbf{G}_{1,4} \otimes \mathbf{G}_{231,212} + \mathbf{G}_{12,42} \otimes \mathbf{G}_{12,21} \\ + \mathbf{G}_{312,242} \otimes \mathbf{G}_{1,1} + \mathbf{G}_{3142,2412} \otimes 1.\end{aligned} \tag{10}$$

Theorem 2.3. *The coproduct is an algebra homomorphism, so that* $\mathbf{FQSym}^{(l)}$ *is a graded bialgebra. Moreover, it is a Hopf algebra.*

Definition 2.4. *The* free l-quasi-ribbon $\mathbf{F}_{\sigma,u}$ *labelled by a colored permutation (σ, u) is the noncommutative polynomial*

$$\mathbf{F}_{\sigma,u} := \mathbf{G}_{\sigma^{-1},u\cdot\sigma^{-1}}, \tag{11}$$

where the action of a permutation on the right of a word permutes the positions of the letters of the word.

For example,

$$\mathbf{F}_{3142,2142} = \mathbf{G}_{2413,1422}\,. \tag{12}$$

The product and coproduct of the $\mathbf{F}_{\sigma,u}$ can be easily described in terms of shifted shuffle and deconcatenation of colored permutations.

Let us define a scalar product on $\mathbf{FQSym}^{(l)}$ by

$$\langle \mathbf{F}_{\sigma,u}, \mathbf{G}_{\sigma',u'}\rangle := \delta_{\sigma,\sigma'}\delta_{u,u'}, \tag{13}$$

where δ is the Kronecker symbol.

Theorem 2.5. *For any $U, V, W \in$* $\mathbf{FQSym}^{(l)}$*,*

$$\langle \Delta U, V \otimes W\rangle = \langle U, VW\rangle, \tag{14}$$

so that, $\mathbf{FQSym}^{(l)}$ *is self-dual: the map $\mathbf{F}_{\sigma,u} \mapsto {\mathbf{G}_{\sigma,u}}^*$ is an isomorphism from* $\mathbf{FQSym}^{(l)}$ *to its graded dual.*

Note 2.6. Let ϕ be any bijection from C to C, extended to words by concatenation. Then if one defines the free l-quasi-ribbon as

$$\mathbf{F}_{\sigma,u} := \mathbf{G}_{\sigma^{-1},\phi(u)\cdot\sigma^{-1}}, \tag{15}$$

the previous theorems remain valid since one only permutes the labels of the basis $(\mathbf{F}_{\sigma,u})$.

Moreover, if C has a group structure, the colored permutations $(\sigma, u) \in \mathfrak{S}_n \times C^n$ can be interpreted as elements of the semi-direct product $H_n := \mathfrak{S}_n \ltimes C^n$ with multiplication rule

$$(\sigma; c_1, \ldots, c_n) \cdot (\tau; d_1, \ldots, d_n) := (\sigma\tau; c_{\tau(1)}d_1, \ldots, c_{\tau(n)}d_n). \tag{16}$$

In this case, one can choose $\phi(\gamma) := \gamma^{-1}$ and define the scalar product as before, so that the adjoint basis of the $(\mathbf{G}_h)$ becomes $\mathbf{F}_h := \mathbf{G}_{h^{-1}}$. In the sequel, we will be mainly interested in the case $C := \mathbb{Z}/l\,\mathbb{Z}$, and we will indeed make that choice for ϕ.

2.3. **Algebraic structure.** Recall that a permutation σ of size n is *connected* [15, 3] if, for any $i < n$, the set $\{\sigma(1), \ldots, \sigma(i)\}$ is different from $\{1, \ldots, i\}$.

We denote by $\mathcal{C}$ the set of connected permutations, and by $c_n := |\mathcal{C}_n|$ the number of such permutations in $\mathfrak{S}_n$. For later reference, we recall that the generating series of c_n is

$$c(t) := \sum_{n\geq 1} c_n t^n = 1 - \left(\sum_{n\geq 0} n!t^n\right)^{-1} = t + t^2 + 3\,t^3 + 13\,t^4 + 71\,t^5 + 461\,t^6 + O(t^7)\,.$$

Let the *connected colored permutations* be the (σ, u) with σ connected and u arbitrary. Their generating series is given by $c(lt)$.

It follows from [3] that $\mathbf{FQSym}^{(l)}$ is free over the set $\mathbf{G}_{\sigma,u}$ (or $\mathbf{F}_{\sigma,u}$), where (σ, u) is connected.

Since $\mathbf{FQSym}^{(l)}$ is self-dual, it is also cofree.

2.4. **Primitive elements.** Let $\mathcal{L}^{(l)}$ be the primitive Lie algebra of $\mathbf{FQSym}^{(l)}$. Since Δ is not cocommutative, $\mathbf{FQSym}^{(l)}$ cannot be the universal enveloping algebra of $\mathcal{L}^{(l)}$. But since it is cofree, it is, according to [12], the universal enveloping dipterous algebra of its primitive part $\mathcal{L}^{(l)}$. Let $d_n = \dim \mathcal{L}_n^{(l)}$.

Recall that the *shifted concatenation* $w \bullet w'$ of two elements w and w' of $\mathbb{N}^*$, is the word obtained by concatenating to w the word obtained by shifting all letters of w' by the length of w. We extend it to colored permutations by simply concatenating the colors and concatenating *with shift* the permutations. Let $\mathbf{G}^{\sigma,u}$ be the multiplicative basis defined by $\mathbf{G}^{\sigma,u} = \mathbf{G}_{\sigma_1,u_1} \cdots \mathbf{G}_{\sigma_r,u_r}$ where $(\sigma, u) = (\sigma_1, u_1) \bullet \cdots \bullet (\sigma_r, u_r)$ is the unique maximal factorization of $(\sigma, u) \in \mathfrak{S}_n \times C^n$ into connected colored permutations.

Proposition 2.7. *Let $\mathbf{V}_{\sigma,u}$ be the adjoint basis of $\mathbf{G}^{\sigma,u}$. Then, the family $(\mathbf{V}_{\alpha,u})_{\alpha\in\mathcal{C}}$ is a basis of $\mathcal{L}^{(l)}$. In particular, we have $d_n = l^n c_n$.*

As in [3], we conjecture that $\mathcal{L}^{(l)}$ is free.

3. Non-commutative symmetric functions of level l

Following McMahon [14], we define an *l-partite number* $\mathbf{n}$ as a column vector in $\mathbb{N}^l$, and a *vector composition of* $\mathbf{n}$ of weight $|\mathbf{n}| := \sum_1^l n_i$ and length m as a $l \times m$ matrix $\mathbf{I}$ of nonnegative integers, with row sums vector $\mathbf{n}$ and no zero column.

For example,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 1 \\ 4 & 2 & 1 & 3 \end{pmatrix} \tag{17}$$

is a vector composition (or a 3-composition, for short) of the 3-partite number $\begin{pmatrix} 4 \\ 5 \\ 10 \end{pmatrix}$ of weight 19 and length 4.

For each $\mathbf{n} \in \mathbb{N}^l$ of weight $|\mathbf{n}| = n$, we define a level l *complete homogeneous symmetric function* as

$$S_{\mathbf{n}} := \sum_{u;|u|_i=n_i} \mathbf{G}_{1\cdots n,u}. \tag{18}$$

It is the sum of all possible colorings of the identity permutation with n_i occurrences of color i for each i.

Let $\mathbf{Sym}^{(l)}$ be the subalgebra of $\mathbf{FQSym}^{(l)}$ generated by the $S_{\mathbf{n}}$ (with the convention $S_{\mathbf{0}} = 1$). The Hilbert series of $\mathbf{Sym}^{(l)}$ is easily found to be

$$S_l(t) := \sum_n \dim \mathbf{Sym}_n^{(l)} t^n = \frac{(1-t)^l}{2(1-t)^l - 1}. \tag{19}$$

Theorem 3.1. $\mathbf{Sym}^{(l)}$ *is free over the set* $\{S_{\mathbf{n}}, |\mathbf{n}| > 0\}$*. Moreover,* $\mathbf{Sym}^{(l)}$ *is a Hopf subalgebra of* $\mathbf{FQSym}^{(l)}$*.*

The coproduct of the generators is given by

$$\Delta S_{\mathbf{n}} = \sum_{\mathbf{i}+\mathbf{j}=\mathbf{n}} S_{\mathbf{i}} \otimes S_{\mathbf{j}}, \tag{20}$$

where the sum $\mathbf{i}+\mathbf{j}$ *is taken in the space* $\mathbb{N}^l$*. In particular,* $\mathbf{Sym}^{(l)}$ *is cocommutative.*

We can therefore introduce the basis of products of level l complete function, labelled by l-compositions

$$S^{\mathbf{I}} = S_{\mathbf{i}_1} \cdots S_{\mathbf{i}_m}, \tag{21}$$

where $\mathbf{i}_1, \cdots, \mathbf{i}_m$ are the columns of $\mathbf{I}$.

Theorem 3.2. *If* C *has a group structure,* $\mathbf{Sym}_n^{(l)}$ *becomes a subalgebra of* $\mathbb{C}[C \wr \mathfrak{S}_n]$ *under the identification* $\mathbf{G}_h \mapsto h$*.*

This provides an analogue of Solomon's descent algebra for the wreath product $C \wr \mathfrak{S}_n$. The proof amounts to check that the Patras descent algebra of a graded bialgebra [17] can be adapted to $\mathbb{N}^l$-graded bialgebras.

As in the case $l = 1$, we define the *internal product* $*$ as being opposite to the law induced by the group algebra. It can be computed by the following splitting formula, which is a straightforward generalization of the level 1 version.

Proposition 3.3. *Let $\mu_r : (\mathbf{Sym}^{(l)})^{\otimes r} \to \mathbf{Sym}^{(l)}$ be the product map. Let $\Delta^{(r)} : (\mathbf{Sym}^{(l)}) \to (\mathbf{Sym}^{(l)})^{\otimes r}$ be the r-fold coproduct, and $*_r$ be the extension of the internal product to $(\mathbf{Sym}^{(l)})^{\otimes r}$. Then, for $F_1, \dots, F_r$, and $G \in \mathbf{Sym}^{(l)}$,*

$$(F_1 \cdots F_r) * G = \mu_r[(F_1 \otimes \cdots \otimes F_r) *_r \Delta^{(r)} G]. \tag{22}$$

The group law of C is needed only for the evaluation of the product of one-part complete functions $S_{\mathbf{m}} * S_{\mathbf{n}}$.

Example 3.4. *With $l = 2$ and $C = \mathbb{Z}/2\mathbb{Z}$,*

$$\begin{aligned} S^{\begin{pmatrix}1&0\\1&1\end{pmatrix}} * S^{\begin{pmatrix}0&2\\1&0\end{pmatrix}} &= \mu_2\left[\left(S^{\begin{pmatrix}1\\1\end{pmatrix}} \otimes S^{\begin{pmatrix}0\\1\end{pmatrix}}\right) *_2 \Delta S^{\begin{pmatrix}0&2\\1&0\end{pmatrix}}\right] \\ &= \left(S^{\begin{pmatrix}1\\1\end{pmatrix}} * S^{\begin{pmatrix}2\\0\end{pmatrix}}\right)\left(S^{\begin{pmatrix}0\\1\end{pmatrix}} * S^{\begin{pmatrix}1\\0\end{pmatrix}}\right) + \left(S^{\begin{pmatrix}1\\1\end{pmatrix}} * S^{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\right)\left(S^{\begin{pmatrix}0\\1\end{pmatrix}} * S^{\begin{pmatrix}1\\0\end{pmatrix}}\right) \\ &= S^{\begin{pmatrix}1&1\\1&0\end{pmatrix}} + S^{\begin{pmatrix}1&1&0\\0&0&1\end{pmatrix}} + S^{\begin{pmatrix}0&0&0\\1&1&1\end{pmatrix}}. \end{aligned}$$

Recall that the underlying colored alphabet A can be seen as $A^0 \sqcup \cdots \sqcup A^{l-1}$, with $A^i = \{a_j^{(i)}, j \geq 1\}$. Let $\mathbf{x} = (x^{(0)}, \dots, x^{(l-1)})$, where the $x^{(i)}$ are l commuting variables. In terms of A, the generating function of the complete functions can be written as

$$\sigma_{\mathbf{x}}(A) = \prod_{i\geq 0}^{\rightarrow} \left(1 - \sum_{0\leq j\leq l-1} x^{(j)} a_i^{(j)}\right)^{-1} = \sum_{\mathbf{n}} S_{\mathbf{n}}(A)\mathbf{x}^{\mathbf{n}}, \tag{23}$$

where $\mathbf{x}^{\mathbf{n}} = (x^{(0)})^{n_0} \cdots (x^{(l-1)})^{n_{l-1}}$.

This realization gives rise to a Cauchy formula (see [10] for the $l = 1$ case), which in turn allows one to identify the dual of $\mathbf{Sym}^{(l)}$ with an algebra introduced by S. Poirier in [18].

4. Quasi-symmetric functions of level l

4.1. Cauchy formula of level l.

Let $X = X^0 \sqcup \cdots \sqcup X^{l-1}$, where $X^i = \{x_j^{(i)}, j \geq 1\}$ be an l-colored alphabet of commutative variables, also commuting with A. Imitating the level 1 case (see [3]), we define the Cauchy kernel

$$K(X, A) = \prod_{j\geq 1}^{\rightarrow} \sigma_{\left(x_j^{(0)}, \dots, x_j^{(l-1)}\right)}(A). \tag{24}$$

Expanding on the basis $S^{\mathbf{I}}$ of $\mathbf{Sym}^{(l)}$, we get as coefficients what can be called the *level l monomial quasi-symmetric functions* $M_{\mathbf{I}}(X)$

$$K(X,A) = \sum_{\mathbf{I}} M_{\mathbf{I}}(X) S^{\mathbf{I}}(A), \tag{25}$$

defined by

$$M_{\mathbf{I}}(X) = \sum_{j_1<\cdots<j_m} \mathbf{x}_{j_1}^{\mathbf{i}_1} \cdots \mathbf{x}_{j_m}^{\mathbf{i}_m}, \tag{26}$$

with $\mathbf{I} = (\mathbf{i}_1, \ldots, \mathbf{i}_m)$.

These last functions form a basis of a subalgebra $QSym^{(l)}$ of $\mathbb{C}[X]$, which we shall call the *algebra of quasi-symmetric functions of level l*.

4.2. **Poirier's Quasi-symmetric functions.** The functions $M_{\mathbf{I}}(X)$ can be recognized as a basis of one of the algebras introduced by Poirier: the $M_{\mathbf{I}}$ coincide with the $M_{(C,v)}$ defined in [18], p. 324, formula (1), up to indexation.

Following Poirier, we introduce the level l quasi-ribbon functions by summing over an order on l-compositions: an l-composition C is finer than C', and we write $C \leq C'$, if C' can be obtained by repeatedly summing up two consecutive columns of C such that no non-zero element of the left one is strictly below a non-zero element of the right one.

This order can be described in a much easier and natural way if one recodes an l-composition $\mathbf{I}$ as a pair of words, the first one $d(\mathbf{I})$ being the set of sums of the elements of the first k columns of $\mathbf{I}$, the second one $c(\mathbf{I})$ being obtained by concatenating the words $i^{\mathbf{I}_{i,j}}$ while reading of $\mathbf{I}$ by columns, from top to bottom and from left to right. For example, the 3-composition of Equation (17) satisfies

$$d(\mathbf{I}) = \{5, 10, 14, 19\} \quad \text{and} \quad c(\mathbf{I}) = 13333\,22233\,1123\,12333\,. \tag{27}$$

Moreover, this recoding is a bijection if the two words $d(\mathbf{I})$ and $c(\mathbf{I})$ are such that the descent set of $c(\mathbf{I})$ is a subset of $d(\mathbf{I})$. The order previously defined on l-compositions is in this context the inclusion order on sets d: $(d',c) \leq (d,c)$ iff $d' \subseteq d$.

It allows us to define the *level l quasi-ribbon functions* $F_{\mathbf{I}}$ by

$$F_{\mathbf{I}} = \sum_{\mathbf{I}' \leq \mathbf{I}} M_{\mathbf{I}'}. \tag{28}$$

Notice that this last description of the order $\leq$ is reminiscent of the order $\leq'$ on descent sets used in the context of quasi-symmetric functions and non-commutative symmetric functions: more precisely, since it does not modify the word $c(\mathbf{I})$, the order $\leq$ restricted to l-compositions of weight n amounts for l^n copies of the order $\leq'$. The computation of its Möbius function is therefore straightforward.

Moreover, one can directly obtain the $F_{\mathbf{I}}$ as the commutative image of certain $\mathbf{F}_{\sigma,u}$: any pair (σ, u) such that σ has descent set $d(\mathbf{I})$ and $u = c(\mathbf{I})$ will do.

5. The Mantaci-Reutenauer algebra

Let $\mathbf{e}_i$ be the canonical basis of $\mathbb{N}^l$. For $n \geq 1$, let

$$S_n^{(i)} = S_{n\cdot\mathbf{e}_i} \in \mathbf{Sym}^{(l)}, \tag{29}$$

be the *monochromatic complete symmetric functions.*

Proposition 5.1. *The $S_n^{(i)}$ generate a Hopf-subalgebra $\mathrm{MR}^{(l)}$ of $\mathbf{Sym}^{(l)}$, which is isomorphic to the Mantaci-Reutenauer descent algebra of the wreath products $\mathfrak{S}_n^{(l)} = (\mathbb{Z}/l\,\mathbb{Z}) \wr \mathfrak{S}_n$.*

It follows in particular that $\mathrm{MR}^{(l)}$ is stable under the composition product induced by the group structure of $\mathfrak{S}_n^{(l)}$. The bases of $\mathrm{MR}^{(l)}$ are labelled by colored compositions (see below).

The duality is easily worked out by means of the appropriate Cauchy kernel. The generating function of the complete functions is

$$\sigma_{\mathbf{x}}^{\mathrm{MR}}(A) := 1 + \sum_{j=0}^{l-1}\sum_{n\geq 1} S_n^{(j)}.(x^{(j)})^n, \tag{30}$$

and the Cauchy kernel is as usual

$$K^{\mathrm{MR}}(X,A) = \overrightarrow{\prod_{i\geq 1}} \sigma_{\mathbf{x}_i}^{\mathrm{MR}}(A) = \sum_{(I,u)} M_{(I,u)}(X) S^{(I,u)}(A), \tag{31}$$

where (I,u) runs over colored compositions $(I,u) = ((i_1,\ldots,i_m),(u_1,\ldots,u_m))$ that is, pairs formed with a composition and a color vector of the same length. The $M_{I,u}$ are called the *monochromatic monomial quasi-symmetric functions* and satisfy

$$M_{(I,u)}(X) = \sum_{j_1<\cdots<j_m} (x_{j_1}^{(u_1)})^{i_1}\cdots(x_{j_m}^{(u_m)})^{i_m}. \tag{32}$$

Proposition 5.2. *The $M_{(I,u)}$ span a subalgebra of $\mathbb{C}[X]$ which can be identified with the graded dual of $\mathrm{MR}^{(l)}$ through the pairing*

$$\langle M_{(I,u)}, S^{(J,v)}\rangle = \delta_{I,J}\delta_{u,v}, \tag{33}$$

where δ is the Kronecker symbol.

Note that this algebra can also be obtained by assuming the relations

$$x_i^{(p)}x_i^{(q)} = 0, \text{ for } p \neq q \tag{34}$$

on the variables of $QSym^{(l)}$.

Baumann and Hohlweg have another construction of the dual of $\mathrm{MR}^{(l)}$ [2] (implicitly defined in [18], Lemma 11).

6. Level l parking quasi-symmetric functions

6.1. **Usual parking functions.** Recall that a *parking function* on $[n] = \{1, 2, \dots, n\}$ is a word $\mathbf{a} = a_1 a_2 \cdots a_n$ of length n on $[n]$ whose nondecreasing rearrangement $\mathbf{a}^{\uparrow} = a'_1 a'_2 \cdots a'_n$ satisfies $a'_i \leq i$ for all i. Let PF_n be the set of such words. It is well-known that $|\mathrm{PF}_n| = (n+1)^{n-1}$.

Gessel introduced in 1997 (see [24]) the notion of *prime parking function*. One says that $\mathbf{a}$ has a *breakpoint* at b if $|\{\mathbf{a}_i \leq b\}| = b$. The set of all breakpoints of $\mathbf{a}$ is denoted by $BP(\mathbf{a})$. Then, $\mathbf{a} \in \mathrm{PF}_n$ is prime if $BP(\mathbf{a}) = \{n\}$.

Let $\mathrm{PPF}_n \subset \mathrm{PF}_n$ be the set of prime parking functions on $[n]$. It can easily be shown that $|\mathrm{PPF}_n| = (n-1)^{n-1}$ (see [24]).

We will finally need one last notion: $\mathbf{a}$ has a *match* at b if $|\{\mathbf{a}_i < b\}| = b-1$ and $|\{\mathbf{a}_i \leq b\}| \geq b$. The set of all matches of $\mathbf{a}$ is denoted by $Ma(\mathbf{a})$.

We will now define generalizations of the usual parking functions to any level in such a way that they build up a Hopf algebra in the same way as in [16].

6.2. **Colored parking functions.** Let l be an integer, representing the number of allowed colors. A *colored parking function* of level l and size n is a pair composed of a parking function of length n and a word on $[l]$ of length l.

Since there is no restriction on the coloring, it is obvious that there are $l^n(n+1)^{n-1}$ colored parking functions of level l and size n.

It is known that the convolution of two parking functions contains only parking functions, so one easily builds as in [16] an algebra $\mathbf{PQSym}^{(l)}$ indexed by colored parking functions:

$$\mathbf{G}_{(\mathbf{a}',u')}\mathbf{G}_{(\mathbf{a}'',u'')} = \sum_{\mathbf{a}\in\mathbf{a}'*\mathbf{a}''} \mathbf{G}_{(\mathbf{a},u'\cdot u'')}. \tag{35}$$

Moreover, one defines a coproduct on the $\mathbf{G}$ functions by

$$\Delta\mathbf{G}_{(\mathbf{a},u)} = \sum_{i\in BP(\mathbf{a})} \mathbf{G}_{(\mathbf{a},u)_{[1,i]}} \otimes \mathbf{G}_{(\mathbf{a},u)_{[i+1,n]}} \tag{36}$$

where n is the size of $\mathbf{a}$ and $(\mathbf{a}, u)_{[a,b]}$ is the parkized colored parking function of the pair $(\mathbf{a}', u')$ where $\mathbf{a}'$ is the subword of $\mathbf{a}$ containing the letters of the interval $[a, b]$, and u' the corresponding subword of u.

Theorem 6.1. *The coproduct is an algebra homomorphism, so that* $\mathbf{PQSym}^{(l)}$ *is a graded bialgebra. Moreover, it is a Hopf algebra.*

6.3. **Parking functions of type B.** In [20], Reiner defined non-crossing partitions of type B by analogy to the classical case. In our context, he defined the level 2 case. It allowed him to derive, by analogy with a simple representation theoretical result, a definition of parking functions of type B as the words on $[n]$ of size n.

We shall build another set of words, also enumerated by n^n that sheds light on the relation between type-A and type-B parking functions and provides a natural Hopf algebra structure on the latter.

First, fix two colors $0 < 1$. We say that a pair of words $(\mathbf{a}, u)$ composed of a parking function and a binary colored word is a *level* 2 *parking function* if

- the only elements of $\mathbf{a}$ that can have color 1 are the matches of $\mathbf{a}$.
- for all element of $\mathbf{a}$ of color 1, all letters equal to it and to its left also have color 1,
- all elements of $\mathbf{a}$ have at least once the color 0.

For example, there are 27 level 2 parking functions of size 3: there are the 16 usual ones all with full color 0, and the eleven new elements

$$\begin{aligned}&(111,100),(111,110),(112,100),(121,100),(211,010),\\&(113,100),(131,100),(311,010),(122,010),(212,100),(221,100).\end{aligned} \tag{37}$$

The first time the first rule applies is with $n = 4$, where one has to discard the words $(1122, 0010)$ and $(1122, 1010)$ since 2 is not a match of 1122. On the other hand, both words $(1133, 0010)$ and $(1133, 1010)$ are B_4-parking functions since 1 and 3 are matches of 1133.

Theorem 6.2. *The restriction of* $\mathbf{PQSym}^{(2)}$ *to the* $\mathbf{G}$ *elements indexed by level* 2 *parking functions is a Hopf subalgebra of* $\mathbf{PQSym}^{(2)}$*.*

6.4. **Non-crossing partitions of type** B**.** Remark that in the level 1 case, non-crossing partitions are in bijection with non-decreasing parking functions. To extend this correspondence to type B, let us start with a non-decreasing parking function $\mathbf{b}$ (with no color). We factor it into the maximal shifted concatenation of prime non-decreasing parking functions, and we choose a color, here 0 or 1, for each factor. We obtain in this way $\binom{2n}{n}$ words π, which can be identified with *type* B *non-crossing partitions*.

Let

$$\mathbf{P}^{\pi} = \sum_{\mathbf{a}\sim\pi} \mathbf{F}_{\mathbf{a}} \tag{38}$$

where $\sim$ denotes equality up to rearrangement of the letters. Then,

Theorem 6.3. *The* $\mathbf{P}^{\pi}$*, where* π *runs over the above set of non-decreasing signed parking functions, form the basis of a cocommutative Hopf subalgebra of* $\mathbf{PQSym}^{(2)}$*.*

All this can be extended to higher levels in a straightforward way: allow each prime non-decreasing parking function to choose any color among l and use the factorization as above. Since non-decreasing parking functions are in bijection with Dyck words, the choice can be described as: each block of a Dyck word with no return-to-zero, chooses one color among l. In this version, the generating series is obviously given by

$$\frac{1}{1 - l\frac{1-\sqrt{1-4t}}{2}}. \tag{39}$$

For $l = 3$, we obtain the sequence A007854 of [22].

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