THE REFLEXIVE DIMENSION OF A LATTICE POLYTOPE

CHRISTIAN HAASE AND ILARION V. MELNIKOV

ABSTRACT. The reflexive dimension refldim(P) of a lattice polytope P is the minimal d so that P is the face of some d-dimensional reflexive polytope.

We show that refldim(P) is finite for every P, and give bounds for refldim(kP) in terms of refldim(P) and k.

1. INTRODUCTION

Reflexive polytopes were originally defined with theoretical physics applications in mind. In string theory, reflexive polytopes and the associated toric varieties play a crucial role in the most quantitatively predictive form of the mirror symmetry conjecture [2, 10]. Aside from such physical uses, we want to advertise their study as interesting combinatorial objects. They enjoy a variety of interesting combinatorial properties [4, 11] which are not well understood. In this note we explore how restrictive the condition of reflexivity turns out to be. We introduce the notion of reflexive dimension of an arbitrary lattice polytope which could be a starting point to study the questions like

- Which polytopes are reflexive?
- Can we get a handle on possible or impossible combinatorial types?
- Can we find upper or lower bounds for the f-vector or Ehrhart coefficients?

A lattice polytope P is the convex hull in $\mathbb{R}^d$ of a finite set of lattice points, i.e., points in $\mathbb{Z}^d$. Its dimension $\dim(P)$ is the dimension of its affine span $\mathrm{aff}(P)$ as an affine space. We will identify lattice equivalent lattice polytopes, where two lattice polytopes P and P' are *lattice*

equivalent if there exists an affine map $\mathrm{aff}(P) \to \mathrm{aff}(P')$ that maps $\mathbb{Z}^d \cap \mathrm{aff}(P)$ bijectively onto $\mathbb{Z}^{d'} \cap \mathrm{aff}(P')$, and which maps P to P'. Every lattice polytope is lattice equivalent to a full-dimensional one, and a full-dimensional lattice polytope has a unique presentation

$$P = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{y}_i, \mathbf{x} \rangle \geq c_i \text{ for } i = 1, \ldots, k\},$$

where the $\mathbf{y}_i$ are primitive elements of the dual lattice $(\mathbb{Z}^d)^\vee$, the c_i are integers, and k is minimal. This system of inequalities will also be referred to as $\mathbf{A}\mathbf{x} \geq \mathbf{c}$.

1.1. DEFINITION. A lattice polytope $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} \geq \mathbf{c}\}$ with interior lattice point $\mathbf{x}_0$ is *reflexive* if $\mathbf{A}\mathbf{x}_0 - \mathbf{c} = \mathbb{1}$, where $\mathbb{1}$ is the all-one vector $(1, \ldots, 1)^t$.

It follows that reflexive polytopes have precisely one interior lattice point which lies in an adjacent lattice hyperplane to any facet. This is sometimes described as "all facets are distance one from the interior lattice point."

The existence of a unique interior lattice point implies that there is only a finite number of equivalence classes of reflexive polytopes in any given dimension [9]. The only one dimensional reflexive polytope is a segment of length 2. In two dimensions there are the 16 reflexive polygons given in Figure 1. Three and four dimensional reflexive polytopes have been classified by Maximilian Kreuzer and Harald Skarke [6, 7]. There are $4,319$ in dimension 3 and $473,800,776$ in dimension 4.
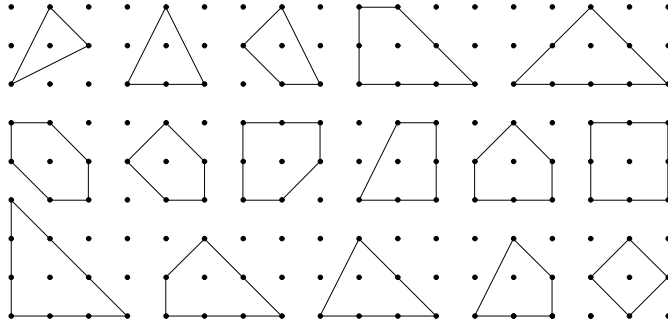

FIGURE 1. All reflexive polygons (up to equivalence).

2. REFLEXIVE POLYTOPES AND REFLEXIVE DIMENSION

The condition that a polytope be reflexive has some rather remarkable consequences, as we now discuss. We begin with some definitions.

- If P is a full-dimensional polytope (not necessarily lattice) with the origin $\mathbf{0}$ in the interior, then the polar dual

$$P^\vee = \{\mathbf{y} \in (\mathbb{R}^d)^* : \langle \mathbf{y}, \mathbf{x} \rangle \geq -1 \text{ for all } \mathbf{x} \in P\}$$

  is again a full-dimensional polytope with $\mathbf{0}$ in the interior (compare [1, Section IV.1]).
- The volume $\mathrm{vol}(P)$ of a lattice polytope is always normalized with respect to the unimodular $(\dim P)$–simplex in $\mathrm{aff}(P) \cap \mathbb{Z}^d$.
- The Ehrhart polynomial of a lattice polytope P, counts lattice points in dilations of P (compare [1, Section VIII.5]).
  $\mathbf{Ehr}(P,k) := |k\,P \cap \mathbb{Z}^d|$, $\mathbf{Ehr}^\circ(P,k) := |(k\,P)^\circ \cap \mathbb{Z}^d|$ for non-negative integers k. These are, in fact, polynomials so that we can evaluate for negative k: $\mathbf{Ehr}^\circ(P,k) = (-1)^{\dim P}\mathbf{Ehr}(P,-k)$. Because $\mathbf{Ehr}$ is a polynomial, its generating function can be written as a rational function

$$\sum_{k \geq 0} \mathbf{Ehr}(P,k)\, t^k \;\; =: \;\; \frac{\mathfrak{Ehr}(P,t)}{(1-t)^{\dim P + 1}}$$

- A lattice polytope P defines an ample line bundle L_P on a projective toric variety X_P. (See, e.g., [3, Section 3.4].) If L_P is very ample, it provides an embedding $X_P \hookrightarrow \mathbb{P}^{r-1}$, where $r = |P \cap \mathbb{Z}^d|$. So we can think of X_P canonically sitting in projective space.

The reader is invited to add some more equivalences to the following.

2.1. THEOREM [2, 5]. *Let P be a full–dimensional lattice polytope with unique interior lattice point $\mathbf{0}$. Then the following conditions are equivalent:*

- (i) *P is reflexive.*
- (ii) *The polar dual $P^\vee$ is a lattice polytope.*
- (iii) $\mathrm{vol}(P) = \sum \mathrm{vol}(F)$*, the sum ranging over all facets (codimension one faces) F of P.*
- (iv) $\mathbf{Ehr}(P,k) = \mathbf{Ehr}^\circ(P,k+1)$ *for all k.*
- (v) $\mathfrak{Ehr}(P,1/t) = (-1)^{d+1}t\,\mathfrak{Ehr}(P,t)$
- (vi) *The projective toric variety X_P defined by P is Fano.*
- (vii) *Every generic hyperplane section of X_P is Calabi–Yau.[1]*

---

[1]If the line bundle L_P is not very ample, one can still define "P-generic hypersurfaces" of X_P to generalize the notion of generic hyperplane section.

These conditions are not as restrictive as one might expect based on the fact that in each dimension there is a finite number of reflexive polytopes. More precisely, we have the following proposition.

2.2. PROPOSITION. *Every lattice polytope is lattice equivalent to a face of some reflexive polytope.*

Proposition 2.2 motivates us to define the reflexive dimension of a lattice polytope.

2.3. DEFINITION. Let P be a lattice polytope. Then its *reflexive dimension* is the smallest d such that P is lattice equivalent to a face of a reflexive d–polytope.

Proof of Proposition 2.2. Let $P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{Ax} \geq \mathbf{c}\}$ be a lattice polytope defined by k inequalities. Suppose that $\mathbf{0}$ is an interior point of P. Then, $c_i \leq -1$, $i = 1, \ldots, k$. If equality holds then P is already reflexive. Otherwise, we will construct a $(d+1)$–polytope P' with strictly bigger $\mathbf{c}'$.

Suppose $c_k < -1$, and introduce one more variable x_{d+1}. Then consider the polytope

$$P' = \{(\mathbf{x}, x_{d+1}) \in \mathbb{R}^{d+1} : \langle \mathbf{y}_i, \mathbf{x} \rangle \geq c_i \text{ for } i = 1, \ldots, k-1, \text{ and}$$
$$\langle \mathbf{y}_k, \mathbf{x} \rangle - x_{d+1} \geq c_k + 1, \text{ and}$$
$$x_{d+1} \geq -1\}.$$

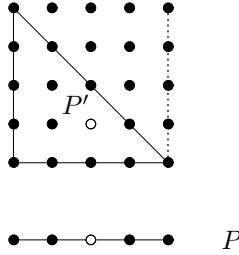Combinatorially, as seen in Figure 2, P' is the wedge of P over the facet

FIGURE 2. P and P'

$\langle \mathbf{y}_m, \mathbf{x} \rangle = c_k$. It has $P \times \{-1\}$ as a facet. Iterating this construction, we add $\|\mathbb{1} + \mathbf{c}\|_1$ dimensions to finally obtain a reflexive polytope.

If P does not have an interior point, we can first embed P as a face of a lattice polytope P' such that P' has an interior point and then apply the procedure above to P'. □

3. The reflexive dimension of a line segment

What is the reflexive dimension of a given polytope P? To answer this question it is natural to start by determining the reflexive dimension of some sample polytopes. While the 0-1-cube and the standard simplex are facets of reflexive polytopes, the question quickly gets subtle, when it comes to, e.g., multiples of these. In this section we will give bounds for the simplest of all polytopes, the segment of length ℓ.

3.1. **Small d.** The unique reflexive one dimensional polytope has edge length 2, and by inspection of Figure 1, it is clear that the edge lengths realized in 2 dimensions are $\{1, 2, 3, 4\}$. A search of the lists of three and four dimensional reflexive polytopes has yielded the following edge lengths.[2]

3.1. PROPOSITION. *There is a reflexive three-polytope which has an edge of length ℓ if and only if $\ell \in \{1, \ldots, 10, 12\}$.*

There is a reflexive four-polytope which has an edge of length ℓ if and only if $\ell \in \{1, \ldots, 54, 56, 57, 58, 60, 63, 64, 66, 70, 72, 78, 84\}$.

So in particular, refldim($[0, 11]$) = 4 > 3 = refldim($[0, 12]$), and the reflexive dimension of line segments is not monotone in ℓ, as could also be seen from refldim($[0, 1]$) = 2 > 1 = refldim($[0, 2]$). There is precisely one reflexive three-polytope with an edge of length 12, and precisely one reflexive four-polytope with an edge of length 84.

3.2. **Large d.** Having examined the low dimensional case, we would now like to find a bound on refldim($[0, \ell]$) for large ℓ.

In order to obtain an asymptotic lower bound, we use a bound for the volume of a lattice polytope which contains exactly one interior lattice point.

3.2. THEOREM [9, 13]. *If Q is a D-polytope which contains exactly one interior lattice point, then*

$$\mathrm{vol}(Q) \leq 14^{\left(2^{D+1}D\right)} \cdot D\,!$$

---

[2]The data is available at `http://tph16.tuwien.ac.at/~kreuzer/CY/`. The four-dimensional list can be searched by using the program PALP [8].

If the d-polytope P is a face of the reflexive D–polytope Q, then $\operatorname{vol}(P) \leq \operatorname{vol}(Q)$, and using the above bound we immediately obtain a (crude) lower bound:

3.3. COROLLARY. *There is a universal constant m such that the reflexive dimension of the segment of length ℓ is at least $m \log \log \ell$.*

To find an upper bound, we need to construct reflexive polytopes with long edges. For a sequence $\mathbf{a} = (a_1, \ldots, a_d)$ of positive integers, consider the simplex $\mathbf{S}(\mathbf{a})$ which is the convex hull of the origin and multiples $a_i \mathbf{e}_i$ of the standard unit vectors. Particularly efficient are the simplices of Micha Perles, Jörg Wills, and Joseph Zaks [12]. Let $t_1 = 2$ and $t_{i+1} = t_i^2 - t_i + 1$.[3] Then the simplex $\mathbf{S}(t_1, \ldots, t_d)$ is reflexive with interior lattice point $\mathbb{1}$. It has a segment of length $t_d > 2^{2^{d-2}}$ as a face. In fact, the slight modification $\mathbf{S}(t_1, \ldots, t_{d-1}, 2t_d - 2)$ is also reflexive. Note that in dimensions $1, 2, 3, 4$, these modified simplices are the unique polytopes that realize the longest edge length.

This is not enough to establish an upper bound $M \log \log \ell$, since, as seen from the computational results above, the reflexive dimension is not monotone in ℓ. The method presented in the proof of Proposition 2.2 provides the upper bound $\ell - 1$: it realizes the segment as a face of the ℓ times dilated standard $(\ell - 1)$–simplex. A better bound is obtained as follows.

3.4. PROPOSITION. *There is a universal constant M such that the reflexive dimension of the segment of length ℓ is at most $M \sqrt{\log \ell}$.*

Luckily for us, Michael Vose has provided a theorem that reduces the proof to a standard argument.

3.5. THEOREM [15, Theorem 1]. *There exists an increasing sequence N_k of positive integers such that any integer $1 < m < N_k$ is the sum of not more than $O(\sqrt{\log N_{k-1}})$ distinct divisors of N_k.*

Proof of Proposition 3.4. Given ℓ, choose k so that $N_{k-1} < \ell \leq N_k$. Now, write

$$N_k \ell - N_k - 1 = m\ell + n$$

---

[3] The sequence A000058 in the OEIS [14]

with $0 \leq n < \ell$. Then both m and n are smaller than N_k, and we can write

$$m = \sum_{i=1}^{r} m_i \quad \text{and} \quad n = \sum_{j=1}^{s} n_j,$$

where the m_i and the n_j divide N_k, and $r+s = O(\sqrt{\log \ell})$. Using these decompositions, we can define integers $a_i = N_k/m_i$ and $b_j = N_k\ell/n_j$. The claim is, that the simplex $\mathbf{S}(\mathbf{a}, \mathbf{b}, \ell)$ is reflexive with interior point $\mathbb{1}$. (It is an $(r + s + 1)$-dimensional simplex.)

The only facet in question is the one which does not contain the origin. The integral functional $\mathbf{f} = \sum_{i=1}^{r} m_i\ell\mathbf{e}_i^* + \sum_{j=1}^{s} n_j\mathbf{e}_{r+j}^* + N_k\mathbf{e}_{r+s+1}^*$ evaluates to $N_k\ell$ on the vertices $a_i\mathbf{e}_i$, on $b_j\mathbf{e}_{r+j}$, and on $\ell\mathbf{e}_{r+s+1}$. That same functional evaluates on $\mathbb{1}$ to

$$\sum_{i=1}^{r} m_i\ell + \sum_{j=1}^{s} n_j + N_k = m\ell + n + N_k = N_k\ell - 1.$$

$\square$

It is conceivable that both bounds are sharp, in the sense that we do know of a sequence of lengths whose reflexive dimension behaves like $\log \log \ell$, and there might be a different sequence of lengths whose reflexive dimension behaves like $\sqrt{\log \ell}$.

4. Products, Joins, Dilations

In this section, we collect some more and some less trivial observations how the reflexive dimension behaves with respect to standard operations on polytopes.

If $P \subset \mathbb{R}^d$ and $P' \subset \mathbb{R}^{d'}$ are polytopes, their join $P * P'$ is the convex hull in $\mathbb{R}^{d+d'+1}$ of $P \times \{\mathbf{0}\} \times 0$ and $\{\mathbf{0}\} \times P' \times 1$.

4.1. Proposition.

$$\text{refldim}(P \times P') \leq \text{refldim}(P) + \text{refldim}(P')$$
$$\text{refldim}(P * P') \leq \text{refldim}(P) + \text{refldim}(P')$$

Proof. If P, P' are faces of reflexive polytopes Q, Q' respectively, then $P \times P'$ is a face of $Q \times Q'$, and (unless $P = Q$ or $P = Q'$) $P * P'$ is a face of $\text{conv}(Q \times \{\mathbf{0}\} \cup \{\mathbf{0}\} \times Q')$. $\square$

We can employ the same method we used for $[0, \ell] = \ell\,[0, 1]$ to deal with dilations of polytopes in general.

4.2. PROPOSITION.   *Suppose that the simplex $\mathbf{S}(a_1, \ldots, a_r)$ is reflexive, and that P is a reflexive s-polytope with interior point $\mathbf{0}$. Then the convex hull of $(a_1\mathbf{e}_1, \mathbf{0}), \ldots, (a_{r-1}\mathbf{e}_{r-1}, \mathbf{0})$, and $(a_r\mathbf{e}_r, a_rP)$ is a reflexive $(r + s)$-polytope which contains a face (equivalent to) a_rP.*

Proof. Suppose P is given by $\{\mathbf{y} : \mathbf{Ay} \geq -\mathbb{1}\}$. Let Q_1 be the convex hull of $(a_1\mathbf{e}_1, \mathbf{0}), \ldots, (a_{r-1}\mathbf{e}_{r-1}, \mathbf{0})$, and $(a_r\mathbf{e}_r, a_rP)$, and let Q_2 be given by $\{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in \mathbf{S}(\mathbf{a}) \text{ and } x_r\mathbb{1} + \mathbf{Ay} \geq 0\}$.

It is easy to see that $Q_1 = Q_2$. The point $(\mathbb{1}, \mathbf{0})$ is an interior point of Q_2, and all facets are at distance one. Since Q_1 has integral vertices, $Q_1 = Q_2$ is reflexive. $\square$

From this the final result follows.

4.3. COROLLARY.   *Given a polytope P the reflexive dimension of kP is bounded from above by $\mathrm{refldim}(kP) \leq \mathrm{refldim}(P) + M\sqrt{\log k}$.*

Proof. From the proof of Proposition 3.4, we know that the simplex $\mathbf{S}(a_1, \ldots, a_{r-1}, k)$ is a reflexive polytope of dimension r bounded by $r \leq M\sqrt{\log k}$, where M is a universal constant. Let Q be a reflexive polytope of dimension $\mathrm{refldim}(P)$ that contains P as a face. Given $\mathbf{S}$ and Q, it follows from Proposition 4.2 that kQ is a face of a reflexive polytope of dimension $\mathrm{refldim}(P) + r \leq \mathrm{refldim}(P) + M\sqrt{\log k}$. $\square$

Let us conclude with a question. Is the reflexive dimension of the Minkowski sum $P + P'$ bounded by $\mathrm{refldim}(P) + \mathrm{refldim}(P') + c$?

REFERENCES

[1] Alexander Barvinok. *A course in convexity*, volume 54 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[2] Victor V. Batyrev. Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties. *J. Alg. Geom.*, 3:493–535, 1994.

[3] William Fulton. *Introduction to Toric Varieties*, volume 131 of *Annals of Math. Studies*. Princeton University Press, 1993.

[4] Christian Haase. Reflexive polytopes in dimension 2 and 3, and the numbers 12 and 24. In Matthias Beck and Christian Haase, editors, *Integer Points in Polyhedra*, Cont. Math. AMS, 2004. To appear.

[5] Takayuki Hibi. Dual polytopes of rational convex polytopes. *Combinatorica*, 12(2):237–240, 1992.

[6] Maximilian Kreuzer and Harald Skarke. Classification of reflexive polyhedra in three dimensions. *Adv. Theor. Math. Phys.*, 2(4):853–871, 1998.

[7] Maximilian Kreuzer and Harald Skarke. Complete classification of reflexive polyhedra in four dimensions. *Adv. Theor. Math. Phys.*, 4(6):1209–1230, 2000.

[8] Maximilian Kreuzer and Harald Skarke. PALP: a package for analysing lattice polytopes with applications to toric geometry. *Comput. Phys. Comm.*, 157(1):87–106, 2004. `math.SC/0204356`, available at `http://tph16.tuwien.ac.at/~kreuzer/CY/`.

[9] Jeffrey C. Lagarias and Günter M. Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Canadian J. Math.*, 43(5):1022–1035, 1991.

[10] David R. Morrison and M. Ronen Plesser. Summing the instantons: quantum cohomology and mirror symmetry in toric varieties. *Nuclear Phys. B*, 440(1-2):279–354, 1995.

[11] Benjamin Nill. Gorenstein toric Fano varieties. Preprint `math.AG/0405448`, 2004.

[12] Micha A. Perles, Jörg M. Wills, and Joseph Zaks. On lattice polytopes having interior lattice points. *Elemente der Math.*, 37:44–46, 1982.

[13] Oleg Pikhurko. Lattice points in lattice polytopes. *Mathematika*, 48(1-2):15–24 (2003), 2001.

[14] Neil J. A. Sloane. The on-line encyclopedia of integer sequences. `http://www.research.att.com/~njas/sequences/`, 2004.

[15] Michael D. Vose. Egyptian fractions. *Bull. London Math. Soc.*, 17(1):21–24, 1985.

Duke Math, Durham, NC 27708-0320
E-mail address: `haase@math.duke.edu`

Duke Physics, Durham, NC 27708-0305
E-mail address: `lmel@phy.duke.edu`