Proper Partitions of a Polygon and k-Catalan Numbers

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Abstract

Let P be a polygon whose vertices have been colored (labeled) cyclically with the numbers $1, 2, \ldots, c$. Motivated by conjectures of Propp, we are led to consider partitions of P into k-gons which are proper in the sense that each k-gon contains all c colors on its vertices. Counting the number of proper partitions involves a generalization of the k-Catalan numbers. We also show that in certain cases, any proper partition can be obtained from another by a sequence of moves called flips.

1 Introduction

Let $\mathbb{N}$ denote the nonnegative integers. In September of 2003, James Propp [8] proposed a series of related problems to the Domino List, an email group discussing matters related to tiling. One of the problems was as follows.

Conjecture 1.1 *Suppose the vertices of a convex polygon P are labeled cyclically $1, 2, 1, 2, \ldots$ Call a triangulation of P proper if no triangle is monochromatic and let a_N be the number of such triangulations if P has $N+2$ vertices. Then*

$$a_N = \begin{cases} \dfrac{2^n}{2n+1}\dbinom{3n}{n} & \text{if } N = 2n \text{ where } n \in \mathbb{N}, \\[2ex] \dfrac{2^{n+1}}{2n+2}\dbinom{3n+1}{n} & \text{if } N = 2n+1 \text{ where } n \in \mathbb{N}. \end{cases}$$

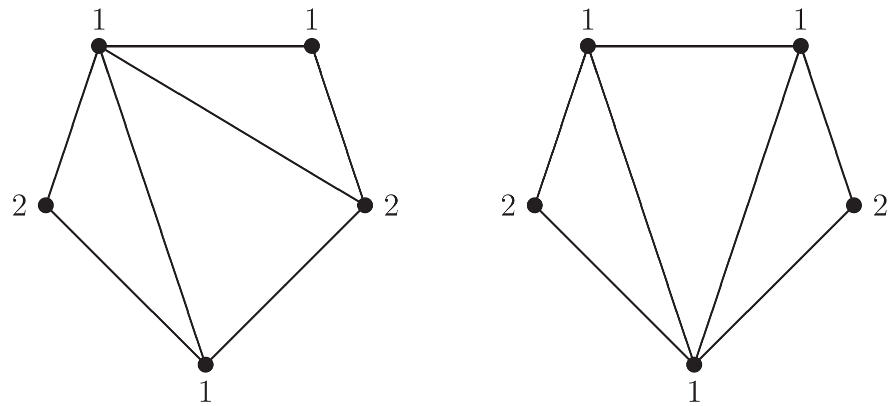

Figure 1: Two triangulations

Note that these counts are closely connected with the *k-Catalan numbers* defined by

$$C_{n,k} = \frac{1}{(k-1)n+1}\binom{kn}{n}$$

for $n, k \in \mathbb{N}$. The ordinary Catalan numbers are obtained when $k = 2$. More information about $C_{n,k}$ can be found in Stanley's text [11, pp. 168-173].

We will prove Propp's conjectures below. We will also generalize them to partitions of P involving k-gons for $k \geq 4$. First, however, we need some terminology. Let P be a convex polygon whose vertices have been colored (labeled) counterclockwise with the sequence $1, 2, \ldots, c, 1, 2, \ldots, c, \ldots$ We will always draw P with a horizontal edge at the top and start the coloring with the left endpoint of that edge.

A *partition of* P is the graph π obtained by drawing some straight line segments (chords) between vertices of P in a plane fashion, i.e., so that no two chords intersect in P's interior. If all the bounded regions of this graph are k-gons then it will be called a *k-partition.* A 3-partition will be referred to as a *triangulation.* If a k-gon contains the top edge, then its *standard reading* will be the sequence of its vertices read counterclockwise starting with the left vertex of the top edge.

A k-partition is *proper* if each k-gon contains all of the c colors among its vertices. In the case of a triangulation with two colors, this means that no triangle is monochromatic. Two triangulations of a pentagon are shown in Figure 1. The one on the left is proper but the one on the right is not.

In the next section we will prove Propp's triangulation conjectures. In fact, in all cases we will give two proofs. One will involve generating functions and the Lagrange Inversion Formula [11, Section 5.4]. The other will be combinatorial, using objects counted by a generalization of the k-Catalan numbers. In Section 3, we will derive analogous formulae for partitions of P into k-gons for $k \geq 4$.

Section 4 concerns flips. Suppose that two k-gons in a partition π share an edge so that their union is a $(2k-2)$-gon, Q. Then a partition $\overline{\pi}$ is connected to π by a *flip*, written $\pi \sim \overline{\pi}$, if it agrees with π everywhere except that the chord of Q has been replaced by another chord connecting two opposite vertices of Q. The two triangulations in Figure 1 are connected by a flip where Q is the quadrilateral with standard reading $1, 1, 2, 1$. We will say that π and $\overline{\pi}$ are *connected by a sequence of flips* if there is a sequence $\pi = \pi_0 \sim \pi_1 \sim \ldots \sim \pi_l = \overline{\pi}$. A well-known theorem of K. Wagner [15] states that any two triangulations of a polygon (uncolored) are connected by flips. In fact, Wagner's theorem applies to the more general case where one allows the set of vertices of the triangulations to include points interior to the polygon. In Section 4 we show that any two k-partitions of P are connected by a sequence of flips. However, if we insist that all the partitions in the sequence be proper, called a *proper flip sequence*, only triangulations with two colors can necessarily be connected. This answers a question of Propp [8]. We should note that D. Thurston [14] has considered flips of two hexagons sharing *two* edges which is equivalent to flipping a pair of dominos in a domino tiling.

The final section is devoted to comments and open questions.

2 Triangulations

We first prove Conjecture 1.1 which we restate here for convenience.

Theorem 2.1 *Let a_N be the number of proper triangulations of an $(N+2)$-gon, P, whose vertices have been colored cyclically with 1 and 2. Then*

$$a_N = \begin{cases} \dfrac{2^n}{2n+1}\dbinom{3n}{n} & \text{if } N = 2n, \\[2ex] \dfrac{2^{n+1}}{2n+2}\dbinom{3n+1}{n} & \text{if } N = 2n+1. \end{cases}$$

Proof We consider a single edge as a proper partition of itself so $a_0 = 1$. Now suppose $N = 2n+1 > 0$ and consider a proper triangulation π of P. The top edge of P is labeled 11. So for π to be proper, that edge must be in a triangle with one of the vertices labeled 2. Say this is the ith 2 in the standard reading of P, where $i \geq 0$ (so we start numbering with zero). Then the two sides of the triangle split P into a $(2i+2)$-gon and a $(2n-2i+2)$-gon which are properly triangulated by π. This gives us the recursion

$$a_{2n+1} = \sum_{i=0}^{n} a_{2i} a_{2n-2i}.$$

Similarly if $N = 2n > 0$ we get

$$a_{2n} = \sum_{i=0}^{2n-1} a_i a_{2n-1-i}.$$

Let x be a variable and consider the generating functions

$$A_0 = A_0(x) = \sum_{n \geq 1} a_{2n} x^n,$$
$$A_1 = A_1(x) = \sum_{n \geq 0} a_{2n+1} x^n.$$

Converting the two recursions into generating function equations gives

$$A_0 = 2x(1 + A_0)A_1,$$
$$A_1 = (1 + A_0)^2.$$

Plugging the second equation into the first we obtain $A_0 = 2x(1+A_0)^3$ which is easy to solve by Lagrange Inversion. We use the notation $[x^n]A(x)$ for the coefficient of x^n in the generating function $A(x)$. Then, for $n \geq 1$, we get

$$a_{2n} = [x^n]A_0 = \frac{1}{n}[x^{n-1}]2^n(1+x)^{3n} = \frac{2^n}{n}\binom{3n}{n-1}$$

which is equivalent to the first formula in the statement of the theorem. Similarly, we can now use Lagrange Inversion on the formula for A_1 in terms of A_0 to obtain

$$a_{2n+1} = [x^n]A_1 = \frac{1}{n}[x^{n-1}]2^n(1+x)^{3n} \cdot 2(1+x) = \frac{2^{n+1}}{n}\binom{3n+1}{n-1}$$

which again can be manipulated into the form given above. ■

When there are three colors, one can also compute the number of proper triangulations. However, if the number of vertices of P is congruent to one modulo three, then the cyclical labeling will result in the top edge being labeled 11 and so there can be no proper triangulations. So in that case, we modify the labeling so that the last vertex in the standard reading of P is labeled 2. The proof of the next result is so similar to the one just given, we omit it.

Theorem 2.2 *Let b_N be the number of proper triangulations of an $(N+2)$-gon, P, whose vertices have been colored cyclically with 1, 2, and 3 (with the last vertex* *colored 2 if $N+2$ is congruent to one modulo 3). Then*

$$b_N = \begin{cases} \dfrac{1}{3n+1}\dbinom{4n}{n} & \text{if } N = 3n, \\[2ex] \dfrac{2}{3n+2}\dbinom{4n+1}{n} & \text{if } N = 3n+1, \\[2ex] \dfrac{3}{3n+3}\dbinom{4n+2}{n} & \text{if } N = 3n+2. \end{cases} \qquad \blacksquare$$

We would now like to give combinatorial proofs of these results. To do this, we recall one of the standard combinatorial interpretations of the k-Catalan numbers. If P is a polygon with $N+2$ uncolored vertices then $C_{n,k}$ is just the number of partitions of P into n polygons each having $k+1$ vertices provided such a partition is possible, i.e., when $N = n(k-1)$. We now show that certain uncolored partitions are related to proper partitions. (Trivially, uncolored partitions are just proper partitions with only one color, but we seek something more substantial.) This proof in the case $k=3$ was discovered independently by Yuliy Baryshnikov (as communicated by Propp [8]).

Theorem 2.3 *We have*

$$\begin{aligned} a_{2n} &= 2^n C_{n,3}, \\ b_{3n} &= C_{n,4}. \end{aligned}$$

Proof Of course these results follow immediately from the previous two theorems, but we wish to give a combinatorial proof.

First consider the statement abou a_{2n}. It suffices to give a 2^n-to-1 map from proper triangulations π of a 2-colored N-gon P, where $N = 2n+2$, to partitions of P into quadrilaterals. Since π is proper, every triangle has exactly one edge whose endpoints are the same color. It follows that if we remove these edges then the result is a partition π' of P into n quadrilaterals.

Now take an arbitrary 4-partition π' of P. To show that π' occurs 2^n times in the image of our map, note that any quadrilateral Q appearing in P must have the colors on its vertices alternate. This is because if some edge of Q had both endpoints of the same color, then that chord would cut off a subpolygon of P with an odd number of vertices and it would be impossible to partition that part of P into quadrilaterals. It follows that the inverse image of π' consists of all π which can be obtained by adding back either of the two diagonals in each quadrilateral. Since there are n quadrilaterals, the map is 2^n-to-1 as claimed.

To obtain the formula for b_{3n} we need a bijection between proper triangulations π of a 3-colored N-gon P, where $N = 3n + 2$, to partitions of P into pentagons. Given π, consider the triangle T containing the top edge which is colored 12. Then the third vertex of T must be colored 3. Now there is a unique second triangle T' containing the 13 edge and a unique third triangle T'' containing the 23 edge. The union of these three triangles forms a pentagon whose standard reading is $1, 2, 3, 1, 2$. Furthermore, each of the subpolygons of P outside this pentagon have $3n'+2$ vertices for some n' (depending on the subpolygon) and are cyclically labeled in the same way as P up to a permutation of the colors. It follows that we can iterate this construction to find a partition π' of P into pentagons.

To construct the inverse map, suppose we are given a pentagon partition π'. Then in each pentagon R of π' will have its vertices colored cyclically as $i, i+1, i+2, i+3, i+4$ for some $1 \leq i \leq 3$ where we are adding modulo three. It follows that there will be a single color j which appears only once among the vertices of R and the other two colors will both appear twice. So there is a unique way of making a proper triangulation of R, namely by adding the two chords containing the vertex colored j. Doing this in each pentagon, produces the inverse map. ■

We would also like to have noncolored analogues of the a_N's and b_N's which do not correspond to k-Catalan numbers. Let $d \in \mathbb{N}$. Let P be a polygon rooted at an edge which we will always take to be the top edge. A *(k, d)-partition of P* is a partition such that all the regions are k-gons except for the one containing the root edge which is a d-gon. By convention if $d = 2$ then, since the root edge is the only edge containing itself, we just have an ordinary k-partition of P. Define the *(k, d)-Catalan number* to be

$$C_{n,k,d} = \frac{d}{(k-1)n+d}\binom{kn+d-1}{n}.$$

Note that $C_{n,k,1} = C_{n,k}$. The numbers $C_{n,3,d}$ have appeared in the work of Brown on nonseparable planar maps [2]; Deutsch, Feretic and Noy on directed polyominoes [3]; and of Noy on noncrossing trees [7]. As far as we know, combinatorial interpretations have not been given to the other $C_{n,k,d}$.

The following result generalizes the k-partition interpretation of $C_{n,k}$. Similar generalizations can be given for other interpretations of the k-Catalan numbers.

Theorem 2.4 *For $n \geq 0$, $d \geq 1$ and $k \geq 2$, let P be a rooted polygon with $n(k-1)+d+1$ uncolored vertices. Then*

$C_{n,k,d}$ = *number of $(k+1, d+1)$-partitions of P into n regions which are k-gons.*

Proof The proof is much like that of Theorem 2.1 so we will just sketch it. Considering the way the $(d+1)$-gon splits P leads to a recursion for $e_{n,k,d}$ which is defined to be the right side of the above equation. Letting

$$E_1 = E_1(x) = \sum_{n\geq 1} e_{n(k-1)+2}x^n,$$

$$E_d = E_d(x) = \sum_{n\geq 0} e_{n(k-1)+d+1}x^n,$$

for $d \geq 2$ we get functional equations

$$\begin{aligned} E_1 &= x(1+E_1)^k \\ E_d &= (1+E_1)^d. \end{aligned}$$

Using Lagrange Inversion completes the proof. ■

Now we can give a more definitive version of Theorem 2.3

Theorem 2.5 *For $d = 1, 2$ we have*

$$a_{2n+d-1} = 2^n C_{n,3,d}.$$

For $d = 1, 2, 3$ we have

$$b_{3n+d-1} = C_{n,4,d}.$$

Proof As before, we are done if we appeal to our previous theorems but we wish to give a combinatorial proof. The proof is similar to that of Theorem 2.3. The only difference for a_{2n+1} is that there are now an odd number of triangles. So the triangle containing the root edge is not paired with anything, becoming the triangle in the rooted partition counted by $C_{n,3,2}$.

The same idea works for b_{3n+1} and $C_{n,4,2}$. In the case of b_{3n+2}, one notes that the top edge is labeled 12 so that the triangle containing it has 13 as a chord of P. Pairing this triangle with the one on the opposite side of the 13 chord gives the necessary quadrilateral for $C_{n,4,3}$. Note that this quadrilateral must have vertices $1, 2, 3, 2$ in the standard reading and the remaining triangles can be grouped in triples to form pentagons as in the proof of Theorem 2.3. Now to construct the inverse, the labeling of P forces the quadrilateral in the rooted partition to have the standard reading just given in order for the rest of P to be partitionable into pentagons. Finally, each pentagon can be dissected into triangles, again as in the proof of Theorem 2.3. ■

3 Partitions with $k \geq 4$

Throughout this section we will assume that $c = k \geq 4$. It will also simplify notation to write the k-Catalan numbers as

$$C_{n,k} = \frac{1}{n}\binom{kn}{n-1}.$$

This is equivalent to the original definition except when $n = 0$ in which case the latter is not well defined.

Theorem 3.1 *Let c_N be the number of proper k-partitions of an $(N+2)$-gon, P, whose vertices have been colored cyclically with $1, 2, \ldots, k$ where $k \geq 4$. Then $c_0 = 1$ and for $N \geq 1$*

$$c_N = \begin{cases} \dfrac{1}{n}\dbinom{(k-1)^2 n}{n-1} & \text{if } N = (k-2)kn, \\[2ex] \dfrac{k-1}{n}\dbinom{(k-1)^2 n + (k-2)}{n-1} & \text{if } N = (k-2)(kn+1), \\[2ex] 0 & \text{else.} \end{cases}$$

Proof There does not exist any k-partition of P if $k-2$ does not divide N, so clearly $c_N = 0$ in this case. Thus we may assume that $M = N/(k-2)$ is an integer. Dividing M by k we can write $M = kn + r$ for some $n \geq 0$ and $0 \leq r < k$.

We claim that $c_N = 0$ if $r \neq 0, 1$. We prove this by induction. Proceeding as in the proof of Theorem 2.1 we have

$$c_N = \sum_{N_1 + \cdots N_{k-1} = N-(k-2)} c_{N_1} \cdots c_{N_{k-1}}.$$

Suppose a term in the sum is nonzero, forcing N_i to be divisible by $k-2$ for $1 \leq i \leq k-1$. So we write $N_i/(k-2) = M_i = kn_i + r_i$ for each i. Also we may assume that $r_i = 0$ or 1 for each i, either by induction or by direct inspection in the base case $N = 2(k-2)$. If we have both an $r_i = 0$ and an $r_j = 1$ then in the sequence $r_1, \ldots, r_{k-1}$ we must have a zero followed by a one or vice-versa. But then in the k-gon containing the top edge, the edges corresponding to these two c_{N_i} form a path of length two whose endpoints have the same color because they are at a distance which is a multiple of k counterclockwise along P. So the partition is not proper, contradicting the fact that the term is nonzero. So the only other possibility is that $r_i = 0$ for all i or $r_i = 1$ for all i which correspond to $r = 1$ or $r = 0$ since the N_i sum to $N - (k-2)$.

The rest of the proof proceeds as in Theorem 2.1. One defines generating functions

$$C_0 = C_0(x) = \sum_{n\geq 1} c_{(k-2)kn} x^n,$$

$$C_1 = C_1(x) = \sum_{n\geq 0} c_{(k-2)(kn+1)} x^n$$

which satisfy functional equations

$$\begin{aligned} C_0 &= xC_1^{k-1}, \\ C_1 &= (C_0+1)^{k-1}. \end{aligned}$$

Lagrange Inversion completes the proof. ■

Again, we can give a combinatorial proof of the portion of the previous theorem related to the (k,d)-Catalan numbers.

Theorem 3.2 *We have*

$$\begin{aligned} c_{(k-2)kn} &= C_{n,(k-1)^2,1}, \\ c_{(k-2)(kn+1)} &= C_{n,(k-1)^2,k-1}. \end{aligned}$$

Proof For the first equality, it suffices to find a bijection between proper k-partitions π of a polygon P with $(k-2)kn+2$ vertices and uncolored partitions π' of P into subpolygons with $(k-1)^2+1 = (k-2)k+2$ vertices. Given π, consider the k-gon, Q, containing the top edge. From the combinatorial part of the proof of the previous theorem, c_N is a sum of products of c_{N_i} where the associated remainders satisfy $r_i = 1$ for all i. It follows that the vertices of Q read counterclockwise are $1, k, k-1, \ldots, 2$. Now glue the k-gons sharing an edge with Q onto Q to form a polygon R with $(k-2)k+2$ vertices. Similar considerations show that R's vertices read counterclockwise will be the same as the usual color ordering we use for polygons. So we can remove R from P and iterate this construction. The collection of R's obtained form the desired partition π'.

To obtain the inverse map, consider a $[(k-2)k+2]$-partition π' of P. Then each subpolygon R will be labeled in the usual coloring order up to a permutation of the colors. So there is a unique proper k-partition of R, namely the one obtained by drawing a chord from the 1 of the top edge to the first k going counterclockwise, then another chord from that k to the next possible $k-1$ going in the same direction, and so forth (assuming for the sake of the description that the color permutation is the identity). Once all of the R's have been partitioned in this manner, one obtains a proper k-partition π of P. It is easy to see that this is indeed the inverse, so we are done.

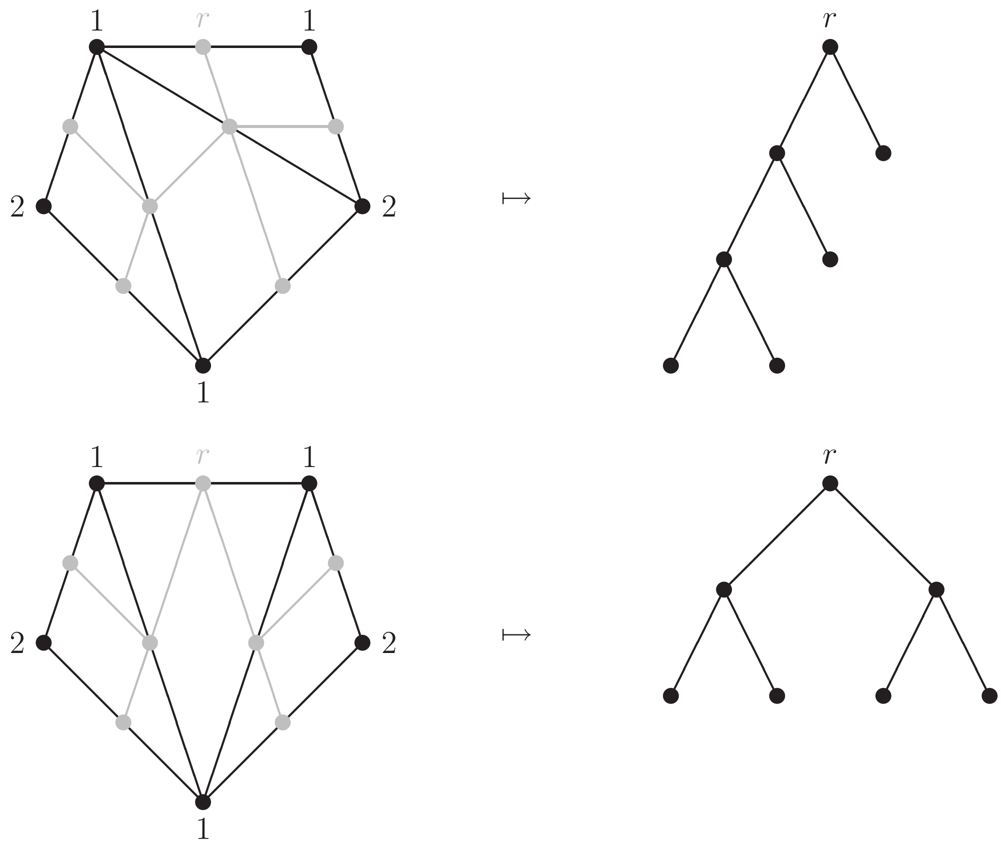

Figure 2: From partitions to trees

For the second inequality, note that the number of k-gons in π will be one more than a multiple of k. So we will be able to glue them together as before except that one, the root polygon, will be left over. In other regards, we have essentially the same bijection. ■

4 Flips

We will first consider uncolored partitions. It will be useful to use one of the other combinatorial interpretations of $C_{n,k}$ in terms of k-ary trees [11]. A *k-ary tree*, T, is a rooted, plane tree where each vertex has either k children or no children. The former vertices are called *internal* and the latter *leaves*. The subtree T_v of T *generated* by a vertex v consists of v and all its descendants. If v is an internal vertex then we let $v', v'', \ldots, v^{(k)}$ be its children listed left to right and let $T_v', T_v'', \ldots, T_v^{(k)}$ denote the trees the trees they generate, respectively. Vertex v' is called the *first* or *leftmost* child of v while $v^{(k)}$ is the *last* or *rightmost*.

It is well-known that $C_{n,k}$ counts the number of k-ary trees with n internal vertices. In fact, there is a bijection between the partitions and trees counted by $C_{n,k}$ which we will need. Given at partition π of polygon P, put a tree vertex in

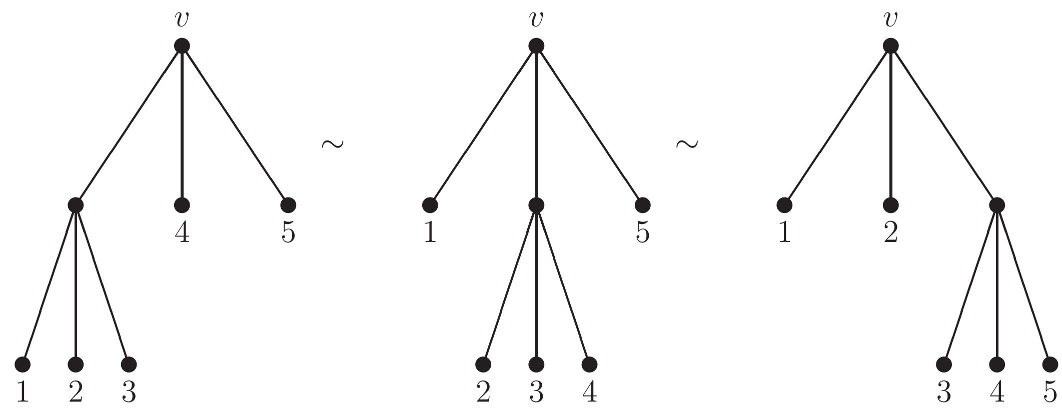

Figure 3: Flips when $k = 3$

every edge of π, including the edges of P. Now pick an edge of P to contain the root vertex r of T. We will always pick the top edge. Start to build T by connecting r to each of the vertices in the other edges bounding the face containing the root edge of P. This process can be iterated, using the vertices currently adjacent to r as roots of subtrees of T. An example of this construction applied to the partitions of Figure 1 will be found in Figure 2. When the tree is superimposed on the partition, it is shown in gray. It is not hard to construct the inverse for this map and thus show it is a bijection.

We need to see what a flip does when translated into the language of trees via this bijection. Let T be a tree and select a vertex v and one of its children $x = v^{(i)}$. Consider the pairwise disjoint subtrees

$$T'_v, T''_v, \ldots, T_v^{(i-1)}, T'_x, T''_x, \ldots, T_x^{(k)}, T_v^{(i+1)}, T_v^{(i+2)}, \ldots, T_v^{(k)}$$

listed left to right in the order in which they are encountered in T (i.e., in depth-first order). Then a tree $\overline{T}$ is a flip of T, written $T \sim \overline{T}$ if it is isomorphic to T outside of T_v and there is some child y of v such that when one makes the list in $\overline{T}$ for y corresponding to the above list in T for x, then corresponding trees in the two lists are isomorphic. For example, Figure 3 shows the situation when $k = 3$. Notice that the vertices labeled $1, 2, 3, 4, 5$ actually stand for the subtrees generated by those vertices.

In order to show that all k-ary trees with n internal vertices are connected by flips, we will need the following statistic on trees. The *left path* P of T will be the unique path starting at r and continuing by always taking the leftmost child. Let $l(T)$ denote the length of this path. The *left comb*, C, is the unique tree on n internal vertices such that $l(C) = n$. The first tree in Figure 2 is the left comb when $n = 3$.

Theorem 4.1 *Let $T, \overline{T}$ be two k-ary trees with n internal vertices. Then T and $\overline{T}$ are connected by a sequence of flips.*

Proof It suffices to show that any T can be connected to the left comb C by a sequence of flips. We induct on n. If $n = 1$ there is nothing to prove. Notice that $l(T) \leq l(C)$ for all k-ary T with n internal vertices, with equality if and only if $T = C$. So it suffices to prove that if $T \neq C$ then there is a flip such that the resulting $\overline{T}$ has $l(\overline{T}) > l(T)$. Since $T \neq C$ there is some vertex v on the left path of T having a child x such that $x \neq v'$ and x is internal. Using $y = v'$ for the flip creates the desired $\overline{T}$. ∎

We will now show that when $c = 2$ then any two proper triangulations of P are connect by a proper sequence of flips. This can be done by using the previous result and our interpretation of colored triangulations in terms of noncolored ones. But we prefer a direct proof which will entail a nice characterization of the corresponding proper trees. Let a binary tree T be *proper* if it corresponds to a proper triangulation under the bijection between all triangulations and all binary trees. Then the following result is easy to prove by induction on the number of internal nodes, so it's proof is omitted. In it, $m(T)$ stands for the number of edges of T.

Lemma 4.2 *A binary tree T is proper if and only if for each internal vertex v either $m(T_v')$ or $m(T_v'')$ is divisible by four.*

We now get a flip connection result for proper binary trees.

Theorem 4.3 *Let $T, \overline{T}$ be proper binary trees with n internal nodes. Then there is a proper sequence of flips connecting them.*

One can prove this by combining the ideas behind Theorems 2.5 and 4.1. Here we will present an alternative direct proof. As in the demonstration of Theorem 4.1, it suffices to show that given $T \neq C$ then we can connect it by a proper sequence to some tree U where $l(U) > l(T)$. Let x and y be the right and left children of the root r, respectively. By induction, we can turn T_x and T_y into combs by a proper sequence. Call the result V. If $l(V) > l(T)$ then we are done.

Otherwise, note that x is internal and V_x'' is a single vertex. If $m(V_y)$ or $m(V_x')$ is divisible by four then, by the previous lemma, we can apply a flip with $v = r$ and x, y playing the same roles they did in the definition to obtain a proper tree U with $l(U) > l(T)$. If both $m(V_y)$ and $m(V_x')$ have remainder two on division by four, then do a flip with x, x' and x'' taking the roles of v, x and y, respectively. The resulting tree W is proper and now doing the flip with $v = r$ and x, y as usual gives the desired tree V. ∎

Connectivity by a proper sequence of flips breaks down for $c = k \geq 3$. For example, $c_{(k-2)(k+1)}$ counts the $(k-1)$-ary trees with $k+1$ internal vertices where the root has exactly one internal child and that child has k internal children. Clearly none of these are connected by a flip.

5 Comments and open problems

5.1 Other labelings

Propp [8] also conjectured a formula for the number of proper triangulations of a polygon colored so that the standard reading is m ones followed by n twos, denoted $1^m, 2^n$. We prove it now.

Proposition 5.1 *Let $d_{m,n}$ be the number of proper triangulations of a polygon P colored $1^m, 2^n$. Then*

$$d_{m,n} = \binom{m+n-2}{m-1}.$$

Proof If the triangle containing the top edge does not have one of the two nodes adjacent to that edge as its third vertex, then it will split P into two parts one of which will be monochromatic making further subdivision impossible. This observation leads to the recursion $d_{m,n} = d_{m-1,n} + d_{m,n-1}$ which, in conjunction with the boundary values $d_{1,n} = d_{m,1} = 1$, yields the result. ■

This raises the possibility that there may be other colorings of P which will lead to nice enumerations of the corresponding proper partitions. One can not generalize the previous proposition directly because for $c \geq 3$ colors arranged in c blocks it is easy to see that there are no possible proper partitions. But it would be interesting to find other arrangements of colors which do yield nice formulae. Note that we had to modify the cyclical labeling to get b_{3n+2} to be nonzero in Theorem 2.2. Perhaps there are also modifications which will do away with the zero values in Theorem 3.1.

5.2 The case $c < k$

The reader will have noticed that, while we permit $c < k$ in the definition of proper, we only stated any results for this case when $k = 3$. This is because other values lead to sequences which do not seem to be tractable. By way of illustration, suppose $c = 3$ and $k = 4$. Then the recursions for the corresponding sequence do not appear to translate into simple expressions for the associated generatiing functions. Furthermore, the sequence is not in Sloane's Encyclopedia of Integer Sequences [10]. So this avenue does not look promising.

5.3 Other definitions

Our definition of proper was carefully chosen to cover all cases found so far where enumeration in closed form is possible. But it is conceivable that other definitions would also yield interesting results. For example, one might try defining proper to mean that no k-gon is monochromatic. Unfortunately, this does not seem to bear fruit. For example, suppose that $c = k = 3$ and that P is an $(N+2)$-gon with the usual cyclic coloring. Let

$$b'_N = \text{number of triangulations of } P \text{ with no triangle monochromatic.}$$

Then proceeding in the usual way using recursions, one is led to solving the following system of generating function equations

$$\begin{aligned} B_0 &= 2x(1+B_0)B_2 + xB_1^2, \\ B_1 &= (1+B_0)^2 + 2xB_1B_2, \\ B_2 &= 2B_1(1+B_0). \end{aligned}$$

Handing the problem to Mathematica results in an output where the solution depends on solving a quintic equation. And the sequence b'_N is not in Sloane.

Another approach to obtaining more results would be to extend the definition of proper to $c > k$ by saying that in this case each k-gon needs to have k different colors on its vertices. We have checked the case $c = 4$ and $k = 3$, but run up against the same problem as in the previous paragraph. However, it seems that there should be some definition of proper which would give colored versions of all the (k, d)-Catalan numbers and not just those with parameters $((k-1)^2, 1)$ or $((k-1)^2, k-1)$.

5.4 Eliminating induction

In the proof of Theorem 2.3 the proof that $a_{2n} = 2^n C_{n,3}$ was a global construction involving flipping the diagonals of quadrilaterals. By contrast the proof of $b_{3n} = C_{n,4}$, while still combinatorial, was inductive. It would be pleasing to have a noninductive proof of the later result. The same applies to the identities in Theorem 3.2.

5.5 Proper flip sequences

It is disappointing that two proper trees can only be connected by a sequence of proper flips in the case $c = 2$, $k = 3$. But perhaps there are some other simple moves which would suffice to connect proper trees in more cases. The trees in the counterexample at the end of the previous section are all connected by rotations about the root. There are still examples where even flipping and rotation are not enough to connect all pairs of proper trees. But maybe a careful analysis would lead to a small set of moves which would work.

5.6 Tamari lattices

One can put a partial order on the set of binary trees with a given number of nodes by using the flips as the covering relations where T is covered by U if the flip taking T to U has $x = v'$ and $y = v''$ (in the notation of the flip definition). These posets are in fact lattices and have have been the object of study of a number of authors, including Blass and Sagan [1], Edelman and Reiner [4], Friedman and Tamari [5], Geyer [6], Reading [9], and Thomas [12]. Thomas and Armstrong [13] have been looking at the analogous structure for k-ary trees.

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