Transformations among large c conformal field theories

Marcin Jankiewicz[*] and Thomas W. Kephart[†]

Department of Physics and Astronomy, Vanderbilt University, Nashville, TN 37235, USA

We show that there is a set of transformations that relates all of the 24 dimensional even self-dual (Niemeier) lattices, and also leads to non-lattice objects that however cannot be interpreted as a basis for the construction of holomorphic conformal field theory. In the second part of this paper, we extend our observations to higher dimensional conformal field theories build on extremal partition functions, where we generate $c = 24k$ theories with spectra decomposable into the irreducible representations of the Fischer-Griess Monster. We observe interesting periodicities in the coefficients of extremal partition functions and characters of the extremal vertex operator algebras.

I. INTRODUCTION

It was proven by Dixon et al. [1] that the partition function $\mathcal{Z}$ of an arbitrary $c = 24$ holomorphic conformal field theory based on $\mathbb{R}^{24}/\Lambda$, where Λ could be any of the 24 even self-dual Niemeier lattices in 24 dimensions, can be written as follows

$$\mathcal{Z} = J + 24(h+1). \tag{1}$$

Here J is the weight zero modular function (with constant term set equal to zero), and h is a Coxeter number for a lattice solution. We will show that any Niemeier lattice Λ_1, represented in terms of the Θ-series related to the partition function (1), can be obtained from another Niemeier Λ_2. We accomplish this using projections that rearrange points in the lattice to form a new lattice. Only for the particular combination of the projection parameters corresponding to the Coxeter numbers of the Niemeier lattices, do we have a lattice as a solution. For other combinations non-lattice solutions are obtained.

Any Niemeier lattice can be used as a starting point, i.e., any Θ-function corresponding to a lattice can be used for the initial Θ-function Z_0^0. The role of the transformation parameters is simple, they either rotate or rescale vectors in a lattice, moving some to different layers of the lattice. The number of transformation parameters depends strictly on an initial choice of the Θ-function Z_0^0, and hence on the number of different conjugacy classes or the number of canonical sublattices in a lattice. For example, for E_8^3, which is one of the Niemeier lattices, we have initially three parameters, one for each $SO(16)$ spinor conjugacy class. In the case of $D_{16} \times E_8$ initially we have 4 parameters, one for the $SO(16)$ vector, spinor and conjugate spinor that build up the D_{16} sublattice, and one parameter for the $SO(16)$ spinor of E_8. However, in these examples, upon constructing the new Θ-series (or partition function), the initial number of parameters can be reduced to a single independent parameter, leading to the Θ-function related to (1) with h being represented by the last free transformation parameter.

Parametrization of the twisted sector has been used to obtain new theories from 16 dimensional even self-dual lattices [2]. These include theories that were already known, like supersymmetric $E_8 \times E_8$, nonsupersymmetric but non-tachyonic $SO(16) \times SO(16)$, and previously unknown theories like nonsupersymmetric and tachyonic $E_8 \times SO(16)$, etc.

We generalize the analysis of [2], to 24 dimensional lattices and also relax the constraints on the transformation parameters, i.e., we will no longer be working with $\mathbb{Z}_2$ actions acting on the conjugacy classes, but rather with more complicated actions.

Since Θ-functions of the 24 dimensional lattices are modular forms of weight 12, and partition functions are modular

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[*]Electronic address: m.jankiewicz@vanderbilt.edu
[†]Electronic address: thomas.w.kephart@vanderbilt.edu

invariant (weight zero), a natural question to ask is, which object is (more) physical? The answer to this question depends on the system being investigated.

II. MODULAR TRANSFORMATIONS AND A 16 DIMENSIONAL EXAMPLE

We start with some basic definitions and a classic example already present in the literature [2]. A formal definition of a Θ-series of an even self-dual lattice Λ is

$$Z_\Lambda = \sum_{x \in \Lambda} N(m) q^m, \tag{2}$$

where $N(m)$ is a number of vectors with length squared equal to an even number m. From a mathematical point of view, Z_Λ is a modular form of weight $\dim(\Lambda)/2$. Let us recall the difference between Z_Λ and a partition function $\mathcal{Z}$. Namely, a partition function $\mathcal{Z}$ is a modular function (a modular form of weight zero). By looking at the modular properties of Z_Λ we conclude that it is related to $\mathcal{Z}$ by $\mathcal{Z} = Z_\Lambda / \eta^{\dim(\Lambda)}$, where

$$\eta(q) = q^{1/12} \prod_{m=1}^{\infty} (1 - q^{2m}) \tag{3}$$

is a modular form of weight $1/2$ called Dedekind η-function with $q = e^{\pi i \tau}$ and τ is a modular parameter. For most of the time, we will focus on the lattices, hence we will work with Θ-functions. However we will make some remarks about partition functions as well.

Let us investigate the relationship between $SO(32)$ and $E_8 \times E_8$ compactification lattices. The Θ-function of both can be expressed in terms of different conjugacy classes of the $SO(16) \times SO(16)$ lattice, which is a maximal common subgroup of both $SO(32)$ and $E_8 \times E_8$. $SO(2N)$ groups have four conjugacy classes namely, the *adjoint* (I_N), the *vector* (V_N), the *spinor* (S_N), and the *conjugate spinor* (C_N). They can be expressed in terms of Jacobi-θ functions as follows [3]:

$$I_N \equiv \frac{1}{2}(\theta_3^N + \theta_4^N), \ V_N \equiv \frac{1}{2}(\theta_3^N - \theta_4^N), \tag{4}$$

$$S_N \equiv \frac{1}{2}\theta_2^N, \ C_N \equiv \frac{1}{2}\theta_2^N, \tag{5}$$

where N is the rank of $SO(2N)$. Both *spinor* and *conjugate spinor* have the same Θ-expansions.

Before going further, let us recall some of the properties of the modular group $\Gamma \simeq SL(2, \mathbb{Z})$. It is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

hence any element in this group can be written as $T^{n_k} S T^{n_{k-1}} S...S T^{n_1}$ where the n_i are integers [4]. The transformation rules for Jacobi-θ functions are given in (A2)-(A7), and from them we obtain **S**-transformed conjugacy classes

$$\mathbf{S}(I_N) = \frac{1}{2}(I_N + V_N + S_N + C_N), \ \mathbf{S}(V_N) = \frac{1}{2}(I_N + V_N - S_N - C_N), \tag{7}$$

$$\mathbf{S}(S_N) = \frac{1}{2}(I_N - V_N), \ \mathbf{S}(C_N) = \frac{1}{2}(I_N - V_N), \tag{8}$$

and **T**-transformations acting on **S**-transformed conjugacy classes of $SO(2N)$ give

$$\mathbf{TS}(I_N) = \frac{1}{2}(I_N - V_N + S_N + C_N), \ \mathbf{TS}(V_N) = \frac{1}{2}(I_N - V_N - S_N - C_N), \tag{9}$$

$$\mathbf{TS}(S_N) = \frac{1}{2}(I_N + V_N), \ \mathbf{TS}(C_N) = \frac{1}{2}(I_N + V_N). \tag{10}$$

We are going to use the **TS** rather than the **T** transformation in order to restore the modular invariance of the new partition functions resulting from the construction presented below.

Now consider the Θ-function of the $SO(32)$ lattice, which is given in terms of $SO(16) \times SO(16)$ conjugacy classes by

$$Z_{SO(32)} = (I_8^2 + V_8^2 + S_8^2 + C_8^2), \tag{11}$$

where squares are just a short-hand notation for (I_8, I_8), etc. A $\mathbb{Z}_2$ action on the conjugacy classes means we have to multiply a given conjugacy class by ± 1. If one chooses to act with transformations that flip the sign of the vector and conjugate spinor representations of the second $SO(16)$ [2], as shown in table I

	I_8	V_8	S_8	C_8
First $SO(16)$	+	+	+	+
Second $SO(16)$	+	−	+	−

TABLE I: A $\mathbb{Z}_2$ transformation applied to $Z_{SO(32)}$.

one gets a new Θ-series

$$Z_1 = (I_8^2 - V_8^2 + S_8^2 - C_8^2). \tag{12}$$

It is obvious that Z_1 is not modular invariant as can be seen from its q-expansion. I.e., some of the coefficients are negative. In order to restore wanted modular properties, one has to add the **S** and the **TS** transformed forms of Z_1:

$$\mathbf{S}Z_1 = I_8 S_8 + S_8 I_8 + C_8 V_8 + V_8 C_8, \tag{13}$$

$$\mathbf{TS}Z_1 = I_8 S_8 + S_8 I_8 - C_8 V_8 - V_8 C_8. \tag{14}$$

Adding partition functions (11)-(14), and also taking into account the overall normalization, one obtains the Θ-function of the dual $E_8 \times E_8$ theory, namely

$$\frac{1}{2} \left(Z_{SO(32)} + Z_1 + \mathbf{S}Z_1 + \mathbf{TS}Z_1 \right) = Z_{E_8 \times E_8} = (I_8 + S_8)^2. \tag{15}$$

III.   TRANSFORMATION MODEL IN 24 DIMENSIONS

Now we move on to 24 dimensional lattices. We begin by writing the $D_{16} \times E_8$ Niemeier lattice represented in terms of $D_8 \times D_8 \times D_8$ conjugacy classes[1]. We follow the standard notation (see for example [1]), where Z_0^0 is the initial untwisted sector, Z_0^1 is a projection, $Z_1^0 = \mathbf{S}Z_0^1$ and $Z_1^1 = \mathbf{T}Z_1^0$ are modular transformed (twisted) sectors. Therefore

$$Z_0^0 = (I^2 + V^2 + S^2 + C^2)(I + S), \tag{16}$$

which explicitly, after evaluation of conjugacy classes in terms of Jacobi-θ functions can be expanded into a q-series

$$Z_0^0 = 1 + 720q^2 + 179280q^4 + \mathcal{O}(q^6). \tag{17}$$

Now let us make a projection, which in the most general way, can be written as

$$Z_0^1 = (I^2 + a\, V^2 + b\, S^2 + c\, C^2)(I + e\, S), \tag{18}$$

where a, b, c and e are projection parameters, that change the signs and/or scale the conjugacy classes. We also evaluate the q-expansion of Z_0^1 to see the influence of the projection we have just made

$$Z_0^1 = 1 + 16(21 + 16a + 8e)q^2 + \mathcal{O}(q^4). \tag{19}$$

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[1] We omit subscript "8" in the notation.

This series is still even in powers of q, however self-duality may have been lost. Using transformation properties (7)-(8) we get twisted sectors

$$
\begin{aligned}
Z_1^0 \equiv \mathbf{S}Z_0^1 \; = \; &\frac{1}{8} \left[(I + V + S + C)^2 + a(I + V - S - C)^2 + b(I - V)^2 + c(I - V)^2 \right] \times \\
&\times \; [I + V + S + C + e \; (I - V)] \,,
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
Z_1^1 \equiv \mathbf{TS}Z_0^1 \; = \; &\frac{1}{8} \left[(I - V + S + C)^2 + a(I - V - S - C)^2 + b(I + V)^2 + c(I + V)^2 \right] \\
&\times \; [I - V + S + C + e \; (I + V)] \,.
\end{aligned}
\tag{21}
$$

New transformed (orbifolded under certain circumstances) theories can be represented in the following general form

$$
Z_{\text{new}} = \frac{1}{2} \left(Z_0^0 + Z_0^1 + Z_1^0 + Z_1^1 \right) \,.
\tag{22}
$$

We have chosen $\mathbf{S}$ and $\mathbf{TS}$ transformations to restore modular invariance of a Θ-function. Neither Z_1^0 nor Z_1^1 are even. We can see this directly from the q-expansion of Z_1^0:

$$
Z_1^0 = \frac{1}{8}(1 + a + b + c)(1 + e) + 2 \left[2 - b - c + e - 3be - 3ce + a(3 + e) \right] q + \mathcal{O}(q^2) \,.
\tag{23}
$$

However, when added to Z_1^0, the $\mathbf{TS}$ transformed Θ-function eliminates terms with odd powers in q, i.e. its q-expansion is:

$$
Z_1^1 = \frac{1}{8}(1 + a + b + c)(1 + e) - 2 \left[2 - b - c + e - 3be - 3ce + a(3 + e) \right] q + \mathcal{O}(q^2) \,.
\tag{24}
$$

An even self-dual lattice scaled by the appropriate power of Dedekind η-function is by definition a modular function, but modular invariance does not necessarily imply we have a lattice; therefore modular invariance is a more general concept. This is important since only even lattices can be used as a basis for a compactification torus. But as we see we are safe; the odd terms in the twisted sectors have cancelled.

Note, Z_{new} is modular invariant regardless of the values of the transformation parameters. However, a q-expansion of a theory transformed in a way described above has to be properly normalized. Normalization is not yet guaranteed, since the zeroth order term in the q-expansion is given by

$$
1 + \frac{1}{8}(1 + a + b + c)(1 + e) \,.
\tag{25}
$$

For $e = 1$, we are left with only one combination of the parameters a, b and c, such that normalization of the zeroth order term is fixed to 1 i.e., $a + b + c = -1$. As a result of this fixing, one is left with

$$
Z_{\text{new}} = (I + S)^3 \,.
\tag{26}
$$

The resulting lattice corresponds to the $E_8 \times E_8 \times E_8$ isospectral[2] partner of $D_{16} \times E_8$. But there is a more interesting case. For $e = -1$ the normalization condition is already fixed, but this puts no constraints on the rest of the parameters. Moreover, the parameters (a, b, c) are found in a specific combination[3] in every order in q

$$
x = 3a - b - c \,.
\tag{27}
$$

We show this by explicitly evaluating Z_{new}

$$
Z_{\text{new}} = 1 + 48(13 + 2x)q^2 - 144(-1261 + 16x)q^4 + 4032(4199 + 6x)q^6 + \dots \,,
\tag{28}
$$

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[2] A pair of lattices is said to be isospectral if they have the same Θ-expansion.
[3] For other values of e the analysis gets more complicated.

Lattice	h	x	a	b	c
D_{24}	46	5	1	-1	-1
E_8^3	30	1	1	1	1
D_{12}^2	22	-1	-1	-1	-1
$E_7^2 D_{10}$	18	-2	0	1	1
D_8^3	14	-3	-1	1	-1

TABLE II: Relatives of $D_{16} \times E_8$ lattice, and their parametrization.

which can be rewritten

$$Z_{\text{new}} = \left[1 + 624q^2 + 181584q^4 + 16930368q^6 + ...\right] + 96x \left[q^2 - 24q^4 + 252q^6 + ...\right] , \tag{29}$$

where we have divided Z_{new} into x-dependent and x-independent parts. The number of independent parameters (after one fixes e) is reduced to one. This parameter, x, can be related to the Coxeter number h of a given lattice. The relation is model dependent and depends on an initial choice of Z_0^0. For the case at hand

$$x \equiv \frac{h - 26}{4} . \tag{30}$$

We observe that the Θ-series in the first square bracket in (29) is an even self-dual (i.e. invariant under **S** and **T**) function. It does not correspond to a lattice solution. However it can be related to the J-invariant (see (A8) in Appendix-A). Terms in the second square bracket in (29) form a unique cusp form of weight 12 [4], which can be written as the 24th power of the Dedekind η-function. Using this knowledge we can write our solution in a more compact form

$$Z_{\text{new}} = [J + 24(h + 1)] \eta^{24} , \tag{31}$$

where h is a positive integer and is equal to the Coxeter number of the 24 dimensional even self-dual lattice if Z_{new} forms a Niemeier lattice. Only for specific values of x does one get a solution that corresponds to a lattice. However, for the majority of cases representation in terms of group lattices is not possible. Physically this means that the gauge fields of the string theory (or CFT) do not transform under any gauge group, hence one is left with $24 \cdot h$ singlets. Here we assume that the kissing number (or coefficient of the first nonvanishing q term for non-lattice cases, which we will also call kissing number) is a multiple of 24.

We now classify all solutions that can be derived by this technique. Using (30) in (31) one finds the allowed values of x form a set of 8191 elements[4]. This is true under two conditions, first we assume that all the coefficients in a q-expansion are positive integers [5], [7]. This assumption is not only reasonable but also physical, since these coefficients give us the number of states at each string mass level from the partition function point of view, and from the lattice point of view they correspond to the number of sites in each layer. The second assumption is that the "kissing number" for both lattices and non-lattices is an integer number which can change by one[5].

Remembering that our starting point was the $D_{16} \times E_8$ lattice, one can immediately see that integer values of (a, b, c) parameters correspond to "relatives"[6] of this lattice [5]. Table II shows the transformation parameters for all five relatives of $D_{16} \times E_8$. All except $E_7^2 D_{10}$, can be gotten from the action of a $\mathbb{Z}_2$. Since E_8^3 and $D_{16} \times E_8$ are isospectral, we can set $a = b = c = 1$ in that case. There are three other possible "integer combinations" for x, corresponding to D_6^4, D_4^6 and A_1^{24} (table III), that are maximal subgroups of $D_{16} \times E_8$. Other integer values for x that reach Niemeier lattices cannot be expressed in terms of integer values of (a, b, c), which means that more complicated

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[4] For what it is worth 8191 is a Marsenne prime.

[5] One could assume that a kissing number for a non-lattice solution is still a multiple of 24, as with lattices. Then modular properties of the Θ-series are preserved, but the number of solutions is reduced to 342.

[6] We call a pair of group lattices relatives if they share a common maximal subgroup.

Lattice	h	x	a	b	c
D_6^4	10	-4	-1	1	0
D_4^6	6	-5	-1	1	1
A_1^{24}	2	-6	-2	1	1

TABLE III: Maximal subgroups of $D_{16} \times E_8$ lattice, and their parametrization.

actions are needed. We list all the lattice solutions and the corresponding x parametrization in table IV. Other values of (a, b, c) give the other Niemeier lattices and non-lattices solutions.

As a second example, we choose another Niemeier lattice E_8^3 with a Θ-function

$$Z_0^0 = (I + S)^3. \tag{32}$$

where the projected sector is

$$Z_0^1 = (I + a\ S)(I + b\ S)(I + c\ S). \tag{33}$$

Again to restore modular invariance we introduce twisted sectors Z_1^0 and Z_1^1. As a result a new modular invariant Θ-function is obtained, and can be written in the form (31), except that the relation between Coxeter number and transformation parameters is different. For example, after fixing $b = c = -1$ we are left with one free parameter $a = (h - 22)/8$, which tells us that the space of the values of the a parameters reaches all 8191 solutions.

If one restricts parameters (a, b, c) to ± 1 one gets a family of simple $\mathbb{Z}_2$ actions transforming the original lattice into the relatives $D_{16} \times E_8$ and $D_8 \times D_8 \times D_8$ of E_8^3.

Let us finish this section with a following simple observation. The $\mathbb{Z}_2$ orbifold actions are the actions which break/restore symmetry in a special way. If Λ_1 and Λ_2 have a common maximal subgroup then there is a $\mathbb{Z}_2$ action that transforms Λ_1 into Λ_2. This is not the case when Λ_1 and Λ_2 do not have a common maximal subgroup. Therefore, this is possible for only a few pairs of Niemeier lattices (see table V). This result follows the lines of the procedure discussed by Dolan et al. in [6].

$$
\begin{array}{ccccccccc}
E_8 D_{16} & \to & D_8^3 & & & & & & \\
& & \downarrow & & & & & & \\
E_8^3 & \to & D_8^3 & \to & D_4^6 & \to & A_1^{24} & \to & \text{Leech} \\
D_{24} & \to & D_{12}^2 & \to & D_6^4 & \to & A_3^8 & & \\
E_7^2 D_{10} & \to & A_7^2 D_5^2 & & & & & &
\end{array}
$$

TABLE V: Patterns of lattices obtained by $\mathbb{Z}_2$ actions.

We want to emphasize that the main point of this section was to introduce transformations between existent lattices, and hence between known partition functions of their holomorphic conformal field theories. We do not claim that the other class of solutions we found (namely non-lattices) can be seen as new CFTs. They are however modular functions and can be seen as a left-over from the choice of the projection parameter which corresponds to the analog of a Coxeter number of the non-existent Niemeier lattice. In the next section we discuss the generalization of these results to $k > 1$. These examples will serve as a basis for construction of potential higher dimensional CFTs based on extremal Θ-functions or partition functions. The list of physical requirements which has to be imposed on a one-loop partition function of a conformal field theory was given for example in [7] or in [8]. All of these constraints are satisfied by any 'candidate CFT' build on even self-dual lattice presented in this section and known earlier in the literature [1]. Following this line of reasoning we proceed to discuss classes of CFTs build on extremal even self-dual lattices in dimensions $24k$ if they exist, or if not, on extremal even self-dual Θ-series with $c = 24k$. This avoids possible (or almost certain) complications of the construction based on non-lattice objects.

IV. CFT WITH $c = 24k$

In this section we generalize our procedure to higher dimensions. We concentrate on lattices in $24k$ dimensions, since their Θ-functions can be expressed in terms of positive integer[7] powers of Z_{new} given in (31). For example, one can use them in the construction of the lattices with dense packing in 48 dimensions and the highest packing in 24. These lattices are build on the so called extremal Θ-functions.

For $k = 1$ the kissing number (so the first non-zero coefficient in the q-expansion of the lattice) of the lattice with the highest packing is obtained as follows. The coefficients a_2 and a_4 in the q-expansion $1 + a_2 q^2 + a_4 q^4 + ...$ are constrained by the equation $24a_2 + a_4 = 196560$ (see below). From this we see that the maximum packing corresponds to the choice $a_2 = 0$ and $a_4 = 196560$.

In general, the equivalent of a 24 dimensional even self-dual, i.e. modular invariant lattice, can be obtained from

$$\eta^{24}(J + 24 + a_2) = 1 + a_2 q^2 + (196560 - 24a_2)q^4 + 252(66560 + a_2)q^6 + ... \tag{34}$$

where a_2 is a positive integer. The extremal $a_2 = 0$ case is a Leech lattice. In order to preserve wanted properties we have to put constraints on values of the integer a_2. It is easy to see that $a_2 \in [0, 8190]$, generates q-expansion with positive entries. The same kind of constraint can be imposed in 48 dimensions, where a modular invariant Θ-series is written in the general form

$$\eta^{48}(J + 24 + a)(J + 24 + b) = 1 + (a + b)q^2 + [2 \cdot 196560 - 24(a + b) + ab]\, q^4 + $$
$$ + \; 12 \, [2795520 + 16401(a + b) - 4ab]\, q^6 + ... \tag{35}$$

By rewriting the expression as

$$1 + a_2 q^2 + a_4 q^4 + (52416000 + 195660a_2 - 48a_4)q^6 + ... \tag{36}$$

we see that a dense packing with kissing number 52416000 is obtained if one chooses $a_2 = a_4 = 0$. It is the P_{48} lattice [5]. In 72 dimensions we have

$$1 + a_2 q^2 + a_4 q^4 + a_6 q^6 + (6218175600 + 57091612a_2 + 195660a_4 - 72a_6)q^8 + ... \tag{37}$$

In this case a lattice corresponding to the extremal Θ-series would be obtained by setting a_2, a_4 and a_6 to zero so that the corresponding kissing number would be 6218175600 except for the fact that this Θ-series is not known to correspond to a lattice [9]. The extremal Θ-series in $24k$ dimensions obtained from this procedure is gotten by the requirement that all of the coefficients $a_2, ..., a_{2k}$ vanish[8]. We can find in principle the number of solutions with this parametrization, i.e., sensible Θ-functions in $24k$ dimensions. In 24 dimensions the values of a_2 were constrained. In the rest of the cases, i.e., $k > 1$, the number of independent parameters is k. However again the parameter space is finite. Using this information one can calculate the number of possible Θ-functions in any dimension. For example in 48 dimensions we find 806022416786149 Θ-series. In this plethora of possibilities only some fraction can be interpreted as lattices, which can be related to CFT[9].

Finally the partition function for any $24k$ dimensional theory contains a finite number of tachyons. For $k = 1$ there is a single tachyon with $(m)^2 = -1$, for $k > 1$ we have k tachyon levels in the spectrum. The most general formula of a partition function in $24k$ dimensions is

$$\mathcal{J}_k(\vec{x}) \equiv \prod_{m=1}^{k} (J + 24 + x_m) = \frac{1}{q^{2k}} \left[1 + \sum_{m=(k-1)}^{\infty} f_{2m}(x_1, ..., x_k) q^{2m-2k} \right], \tag{38}$$

where $\vec{x} = (x_1, ..., x_k)$ and $f_{2m} \geq 0$ are polynomials in the x_i. The lowest (tachyonic) state with $(m)^2 = -k$ is always populated by a single tachyon, and higher states are functions of $(x_1, x_2, ..., x_k)$. The x_is can be chosen in such a

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[7] However, one can generalize this procedure to other dimensions ($8k$) as well.

[8] Alternatively, we can use the bound known in from the lattice theory [10], that the minimal norm of n-dimensional unimodular lattice is $\mu \leq \left[\frac{n}{24} \right] + 2$.

[9] For example there are 24^2 even self-dual CFTs constructed out of 24 dimensional Niemeier lattices.

way that all tachyon levels above the lowest level are absent, hence the next populated level would be occupied by massless states. This choice involves the elimination of $k-1$ parameters. The series would then depend on a single parameter x_k, more precisely the massless level is a polynomial in x_k of the order k. The remainder of the spectrum does not depend on the choice of x_k, in analogy with the 24 dimensional case. What is appealing in these models is that for $k \gg 1$ we can have a single tachyonic state with arbitrarily large negative mass square that could potentially decouple from the spectrum leaving only states with $(m^2) \geq 0$, and the partition function of such a theory is still a well defined modular function. This may be an alternative to tachyon condensation [11].

Let us evaluate a few of examples with only a single tachyon coupled to the identity at q^{2k} level for $k = 1, 2, 3$ and 4 which we define as $\mathcal{G}_k = \mathcal{J}_k|_{extremal}$. These are:

$$
\begin{aligned}
\mathcal{G}_1(x_1) &= \frac{1}{q^2} + (24 + x_1) + 196884q^2 + ... \\
\mathcal{G}_2(x_2) &= \frac{1}{q^4} + (393192 - 48x_2 - x_2^2) + 42987520q^2 + ... \\
\mathcal{G}_3(x_3) &= \frac{1}{q^6} + (50319456 - 588924x_3 + 72x_3^2 + x_3^3) + 2592899910q^2 + ... \\
\mathcal{G}_4(x_4) &= \frac{1}{q^8} + (-75679531032 - 48228608x_4 + 784080x_4^2 - 96x_4^3 - x_4^4) + 80983425024q^2 + ...
\end{aligned}
\tag{39}
$$

The allowed values of the polynomial coefficient of q^0 are integers that run from zero to the value of the q^2 coefficient. If one changes k, then the q^2 coefficients are Fourier coefficients of the unique weight-2 normalized meromorphic modular form for $SL(2, \mathbb{Z})$ with all poles at infinity [12]. The partition functions (39) are examples of the replication formulas [13].

There exists an interesting alternative set of CFTs with partition functions [14] that we will call $\mathcal{H}_k$ where for different values of k we can have

$$
\begin{aligned}
\mathcal{H}_1 &= \frac{1}{q^2} + 196884q^2 + ... \\
\mathcal{H}_2 &= \frac{1}{q^4} + 1 + 42987520q^2 + ... \\
\mathcal{H}_3 &= \frac{1}{q^6} + \frac{1}{q^2} + 1 + 2593096794q^2 + ... \\
\mathcal{H}_4 &= \frac{1}{q^8} + \frac{1}{q^4} + \frac{1}{q^2} + 1 + 81026609428q^2 + ...
\end{aligned}
\tag{40}
$$

Again we fix the tachyon levels by appropriate choices of the xs. Note that the q^2 coefficients for $k = 1$ and 2 coincide in (39) and (40) but not for larger k. These are characters of the extremal vertex operator algebra of rank $24k$ (if it exists) [14]. These characters were obtained by requiring $24k + \sum_{m=1}^{k} x_m = 0$ (so that the $-(k-1)$ state is empty), all other coefficients of tachyon levels up the to massless states are fixed to one.

The extremal 24 dimensional case has been shown to be related to the Fischer-Griess monster group. In fact $\mathcal{G}_1(x_1)$ is the modular function j when $x_1 = -24$. j has the expansion

$$
j = \frac{1}{q^2} + 196884q^2 + 21493760q^4 + 864299970q^6 + 20245856256q^8 + ...
\tag{41}
$$

and the coefficients of this expansion decompose into dimensions of the irreducible representations of the monster (see table VI), where we use the notation

$$
j = \frac{1}{q^2} + j_2q^2 + j_4q^4 + ...
\tag{42}
$$

For $24k$ one can expand the q^{2n} coefficients of $\mathcal{G}_k$ in terms of j coefficients which in turn can be expanded in terms of the dimensions of irreducible representations of the monster. Table VII demonstrates explicitly how the coefficients of the extremal $24k$ partition functions are decomposed into the coefficients of j.

Observe that the pattern of the g_{2n} coefficients in the kth row in Table VII is periodic with period k. The first k rows

	$J-invariant$	$Monster$
j_2	196884	$1 + 196883$
j_4	21493760	$1 + 196883 + 21296876$
j_6	864299970	$2 \cdot 1 + 2 \cdot 196883 + 21296876 + 842609326$
j_8	20245856256	$3 \cdot 1 + 3 \cdot 196883 + 21296876 + 2 \cdot 842609326 + 18538750076$
j_{10}	333202640600	$4 \cdot 1 + 5 \cdot 196883 + 3 \cdot 21296876 + 2 \cdot 842609326 + 18538750076$ $+19360062527 + 293553734298$
j_{12}	4252023300096	$3 \cdot 1 + 7 \cdot 196883 + 6 \cdot 21296876 + 2 \cdot 842609326 + 4 \cdot 19360062527$ $+293553734298 + 3879214937598$

TABLE VI: Decomposition of the coefficients of j into irreducible representations of the Monster group (for more see [15], [16]).

of the table of g coefficients is overall $k!$ periodic. We conjecture that this periodicity also continues to hold for all k. The polynomial conditions to be satisfied to find the extremal partition functions for large k become increasingly more difficult to solve with increasing k, so we do not have results for $k > 6$. Table VIII give the general periodicity.

These results are somewhat reminiscent of Bott periodicity for the stable homotopy of the classical groups. Here we are dealing with (the equivalent of) increasing level algebras.

To summarize, when $k = 1$ it is known via standard Monster Moonshine that the coefficients of j decompose into Monster representations [13]. The related extremal lattices are Leech in 24 and P_{48} in 48 dimensions. The fact that all the higher k coefficients also decompose into Monster representations indicates that they have large symmetries containing the Monster and the fact that they have these symmetries may indicate that they are related to $24k$ dimensional lattices.

V. OPEN QUESTIONS/PROBLEMS

In the discussion section of [1] it is argued that by using $\mathbb{Z}_2$ twists of the 23 Lie type Niemeier lattices one gets holomorphic conformal field theories that are not graded isomorphic to any of the untwisted theories based on Niemeier lattices. What we have shown is that there exists a family of transformations, not necessarily of the simple $\mathbb{Z}_2$ form, that connects the class of holomorphic conformal field theories the Niemeier lattices. Furthermore these transformations connect non-Niemeier even self-dual $c = 24$ Θ-functions. These results generalize to any $c = 24k$ case (in particular, when the resulting parametrization corresponds to $24k$ dimensional lattice). We also found interesting patterns of periodicity related to the Monster moonshine.
There exists a possible application of conformal field theories (with high central charge) to cosmology, since for $k \to \infty$, $(m_{tachyon})^2 \to -k \cdot M_{Planck}^2 \to -\infty$, which suggests this tachyon maybe the single tachyon of $\mathcal{J}(x_k)$ and lead to some variant of a tachyon condensation [11]. Also in the case of $k \to \infty$ (hence divergent central charge [17], [18]) we get, for example, a theory with a gauge lattice G^k. Depending on the representation content of the gauge group one could potentially partially deconstruct G^k [19] to go from $2D$ CFT to a $4D$ theory.
It is important to mention that some of the results obtained here (related to the transformations between $c = 24$ CFTs) were partially present in the literature [6], [20]. For example the $\mathbb{Z}_2$ patterns (in table V) were explained in [20]. Also it was shown that from any even-self dual lattice it is always possible to construct one untwisted and twisted conformal field theory. However in our case it is enough to start from just a single even self-dual lattice to obtain all other lattice solutions by a proper choice of projection parameters. Finally there are a number of interesting open questions. For instance, is there any relation between solutions found here (including non-lattice solutions) and other solutions corresponding to higher level Kac-Moody algebras classified by Schellekens in [21], and others [22] and [23]? Is there an interpretation of non-lattice solutions? Can one interpret them as lattices on spaces with curvature?
The natural place for theories with $c > 24$ is in condensed matter systems, since there are not enough ghosts to cancel

k	g_2	g_4	g_6	g_8	g_{10}	g_{12}
2	$2j_4$	$2j_8 + j_2$	$2j_{12}$	$2j_{16} + j_4$	$2j_{20}$	$2j_{24} + j_6$
3	$3j_6$	$3j_{12}$	$3j_{18} + j_2$	$3j_{24}$	$3j_{30}$	$3j_{36} + j_4$
4	$4j_8$	$4j_{16} + 2j_4$	$4j_{24}$	$4j_{32} + 2j_8 + j_2$	$4j_{40}$	$4j_{48} + 2j_{12}$
5	$5j_{10}$	$5j_{20}$	$5j_{30}$	$5j_{40}$	$5j_{50} + j_2$	$5j_{60}$
6	$6j_{12}$	$6j_{24} + 3j_6$	$6j_{36} + 2j_4$	$6j_{48} + 3j_{12}$	$6j_{60}$	$6j_{72} + 3j_{18} + 2j_8 + j_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

k	g_{14}	g_{16}	g_{18}	g_{20}	g_{22}	g_{24}
2	$2j_{28}$	$2j_{32} + j_8$	$2j_{36}$	$2j_{40} + j_{10}$	$2j_{44}$	$2j_{48} + j_{12}$
3	$3j_{42}$	$3j_{48}$	$3j_{54} + j_6$	$3j_{60}$	$3j_{66}$	$3j_{72} + j_8$
4	$4j_{56}$	$4j_{64} + 2j_{16} + j_4$	$4j_{72}$	$4j_{80} + 2j_{20}$	$4j_{88}$	$4j_{96} + 2j_{24} + j_6$
5	$5j_{70}$	$5j_{80}$	$5j_{90}$	$5j_{100} + j_4$	$5j_{110}$	$5j_{120}$
6	$6j_{84}$	$6j_{96} + 3j_{24}$	$6j_{108} + 2j_{12}$	$6j_{120} + 3j_{30}$	$6j_{132}$	$6j_{144} + 3j_{36} + 2j_{16} + j_4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

k	g_{26}	g_{28}	g_{30}	g_{32}	g_{34}	g_{36}
2	$2j_{52}$	$2j_{56} + j_{14}$	$2j_{60}$	$2j_{64} + j_{16}$	$2j_{68}$	$2j_{72} + j_{18}$
3	$3j_{78}$	$3j_{84}$	$3j_{90} + j_{10}$	$3j_{96}$	$3j_{102}$	$3j_{108} + j_{12}$
4	$4j_{104}$	$4j_{112} + 2j_{28}$	$4j_{120}$	$4j_{128} + 2j_{32} + j_8$	$4j_{136}$	$4j_{144} + 2j_{36}$
5	$5j_{130}$	$5j_{140}$	$5j_{150} + j_6$	$5j_{160}$	$5j_{170}$	$5j_{180}$
6	$6j_{156}$	$6j_{168} + 3j_{42}$	$6j_{180} + 2j_{20}$	$6j_{192} + 3j_{48}$	$6j_{204}$	$6j_{216} + 3j_{54} + 2j_{24} + j_6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

k	g_{38}	g_{40}	g_{42}	g_{44}	g_{46}	g_{48}
2	$2j_{76}$	$2j_{80} + j_{20}$	$2j_{84}$	$2j_{88} + j_{22}$	$2j_{92}$	$2g_{96} + j_{24}$
3	$3j_{114}$	$3j_{120}$	$3j_{126} + j_{14}$	$3j_{132}$	$3j_{138}$	$3j_{144} + j_{16}$
4	$4j_{152}$	$4j_{160} + 2j_{40} + j_{10}$	$4j_{168}$	$4j_{176} + 2j_{44}$	$4j_{184}$	$4j_{192} + 2j_{48} + j_{12}$
5	$5j_{190}$	$5j_{200} + j_8$	$5j_{210}$	$5j_{220}$	$5j_{230}$	$5j_{240}$
6	$6j_{228}$	$6j_{240} + 3j_{60}$	$6j_{252} + 2j_{28}$	$6j_{264} + 3j_{66}$	$6j_{276}$	$6j_{288} + 3j_{72} + 2j_{32} + j_8$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

k	g_{50}	g_{52}	g_{54}	g_{56}	g_{58}	g_{60}
2	$2j_{100}$	$2j_{104} + j_{26}$	$2j_{108}$	$2j_{112} + j_{28}$	$2j_{116}$	$2j_{120} + j_{30}$
3	$3j_{150}$	$3j_{156}$	$3j_{162} + j_{18}$	$3j_{168}$	$3j_{174}$	$3j_{180} + j_{20}$
4	$4j_{200}$	$4j_{208} + 2j_{52}$	$4j_{216}$	$4j_{224} + 2j_{56} + j_{14}$	$4j_{232}$	$4j_{240} + 2j_{60}$
5	$5j_{250} + j_{10}$	$5j_{260}$	$5j_{270}$	$5j_{280}$	$5j_{290}$	$5j_{300} + j_{12}$
6	$6j_{300}$	$6j_{312} + 3j_{78}$	$6j_{324} + 2j_{36}$	$6j_{336} + 3j_{84}$	$6j_{348}$	$6j_{360} + 3j_{90} + 2j_{40} + j_{10}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE VII: Coefficients of $24k$ dimensional extremal partition functions $\mathcal{G}_k$ in terms of coefficients j_{2n} of modular function j.

a conformal anomaly. Nevertheless, we hope applications of the transformation techniques investigated here may lead to further/deeper understanding of dualities relating $\mathcal{N} = 2$ heterotic string theories in $2D$ [24].

Acknowledgments

$k=2$	$k=2$	$k=2$	$k=2$
g_{4i+2}	$2j_{2(4i+2)}$	h_{4i+2}	$2j_{2(4i+2)}$
g_{4i+4}	$2j_{2(4i+4)} + j_{2(2i+2)}$	h_{4i+4}	$2j_{2(4i+4)} + j_{2(2i+2)}$
$k=3$	$k=3$	$k=3$	$k=3$
g_{6i+2}	$3j_{3(6i+2)}$	h_{6i+2}	$3j_{3(6i+2)} + j_{6i+2}$
g_{6i+4}	$3j_{3(6i+4)}$	h_{6i+4}	$3j_{3(6i+4)} + j_{6i+4}$
g_{6i+6}	$3j_{3(6i+6)} + j_{2i+2}$	h_{6i+6}	$3j_{3(6i+6)} + j_{2i+2} + j_{6i+6}$
$k=4$	$k=4$	$k=4$	$k=4$
g_{8i+2}	$4j_{4(8i+2)}$	h_{8i+2}	$4j_{4(8i+2)} + 2j_{2(8i+2)} + j_{8i+2}$
g_{8i+4}	$4j_{4(8i+4)} + 2j_{2(2i+4)}$	h_{8i+4}	$4j_{4(8i+4)} + 2j_{2(2i+4)} + 2j_{2(8i+4)} + j_{8i+4} + j_{4i+2}$
g_{8i+6}	$4j_{4(8i+6)}$	h_{8i+6}	$4j_{4(8i+6)} + 2j_{2(8i+6)} + j_{8i+6}$
g_{8i+8}	$4j_{4(8i+8)} + 2j_{(8i+8)} + j_{2i+2}$	h_{8i+8}	$4j_{4(8i+8)} + 2j_{(8i+8)} + j_{2i+2} + 2j_{2(8i+8)} + j_{8i+8} + j_{4i+4}$
$k=5$	$k=5$	$k=5$	$k=5$
g_{10i+2}	$5j_{5(10i+2)}$	h_{12i+2}	$g_{12i+2} + 3j_{3(12i+2)} + 2j_{2(12i+2)} + j_{12i+2}$
g_{10i+4}	$5j_{5(10i+4)}$	h_{12i+4}	$g_{12i+4} + 3j_{3(12i+4)} + 2j_{2(12i+4)} + j_{12i+4} + j_{6i+2}$
g_{10i+6}	$5j_{5(10i+6)}$	h_{12i+6}	$g_{12i+6} + 3j_{3(12i+6)} + 2j_{2(12i+6)} + j_{12i+6} + j_{4i+2}$
g_{10i+8}	$5j_{5(10i+8)}$	h_{12i+8}	$g_{12i+8} + 3j_{3(12i+8)} + 2j_{2(12i+8)} + j_{12i+8} + j_{6i+4}$
g_{10i+10}	$5j_{5(10i+10)} + j_{2i+2}$	h_{12i+10}	$g_{12i+10} + 3j_{3(12i+10)} + 2j_{2(12i+10)} + j_{12i+10}$
		h_{12i+12}	$g_{10i+12} + 3j_{3(12i+12)} + 2j_{2(12i+12)} + j_{12i+12} + j_{6i+6} + j_{4i+4}$
$k=6$	$k=6$		
g_{12i+2}	$6j_{6(12i+2)}$		
g_{12i+4}	$6j_{6(12i+4)} + 3j_{3(6i+2)}$		
g_{12i+6}	$6j_{6(12i+6)} + 2j_{2(4i+2)}$		
g_{12i+8}	$6j_{6(12i+8)} + 3j_{3(6i+4)}$		
g_{12i+10}	$6j_{6(12i+10)}$		
g_{12i+12}	$6j_{6(12i+12)} + 3j_{3(6i+6)}$ $+2j_{2(4i+4)} + j_{2i+2}$		

TABLE VIII: Periodicity of the coefficients g_n for $c=24k$ extremal partition functions $\mathcal{G}_k$, and for h_n coefficients of characters of the extremal vertex operator algebras $\mathcal{H}_k$ in terms of coefficients the j_{2n} of the modular function j ($k=6$ case for h_n is not displayed since it is long but it has period 24).

APPENDIX A: RELATIONS BETWEEN MODULAR FUNCTIONS AND LATTICES

The Z_Λ of any lattice Λ can be expressed in terms of the Jacobi θ-functions:

$$\theta_3(\xi|\tau) \equiv \sum_{m=-\infty}^{\infty} e^{2mi\xi + \pi i \tau m^2} \,.$$

For us it will be enough to work with the simpler theta functions, often called θ constants, defined as:

$$\theta_2(\tau) \equiv e^{\pi i \tau/4}\theta_3(\frac{\pi\tau}{2}|\tau)\,,\ \theta_3(\tau) \equiv \theta_3(0|\tau)\,,$$
$$\theta_4(\tau) \equiv \theta_3(\tau+1)\,. \tag{A1}$$

These functions have fantastic properties including a basically infinite web of identities which will be used later on. Most important for us, is that they have very simple modular transformation properties. Here we show their behavior under the generators of the modular group, namely under **S** we have

$$\theta_2(-1/\tau) = \left(\frac{\tau}{i}\right)^{\frac{1}{2}} \theta_4(\tau)\,, \tag{A2}$$

$$\theta_3(-1/\tau) = \left(\frac{\tau}{i}\right)^{\frac{1}{2}} \theta_3(\tau)\,, \tag{A3}$$

$$\theta_4(-1/\tau) = \left(\frac{\tau}{i}\right)^{\frac{1}{2}} \theta_2(\tau)\,, \tag{A4}$$

and under **T**:

$$\theta_2(\tau + 1) = \sqrt{i}\,\theta_2(\tau)\,, \tag{A5}$$

$$\theta_3(\tau + 1) = \theta_4(\tau)\,, \tag{A6}$$

$$\theta_4(\tau + 1) = \theta_3(\tau)\,. \tag{A7}$$

These transformation rules are easily derived using the Poisson resummation formula [5]. Finally, we introduce the modular invariant function J, which plays a very important role in our considerations

$$\begin{aligned} J &\equiv \frac{1}{\eta^{24}}\left[(\theta_3(\tau)\theta_4(\tau))^{12} + (\theta_2(\tau)\theta_3(\tau))^{12} - (\theta_2(\tau)\theta_4(\tau))^{12}\right] + 24 \\ &= \frac{1}{q^2} + 196884q^2 + 21493760q^4 + \mathcal{O}(q^6) + \dots. \end{aligned} \tag{A8}$$

This function (sometimes called the J-invariant related to weight-zero modular function j by $J = j - 744$) is the modular form of weight zero, as can be easily proved using transformations (A2)-(A7). In the denominator of (A8) we have the 24th power of Dedekind η-function

$$\eta^{24}(\tau) = \left[q^{1/12}\prod_{m=1}^{\infty}(1 - q^{2m})\right]^{24} = q^2 - 24q^4 + 252q^6 - 1472q^8 + \dots\,, \tag{A9}$$

which is the unique form of weight 12, with the following transformation rules under **S** and **T**

$$\eta^{24}(-1/\tau) = (\sqrt{-i\tau}\,\eta(\tau))^{24}\,,\ \ \eta^{24}(\tau + 1) = (e^{i\pi/12}\eta(\tau))^{24}\,. \tag{A10}$$

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Lattice	Coxeter	x	Glue Code	a_2	a_4	a_6
E_8	30		+	240	2160	6720
E_8^2	30		++	480	61920	1050240
D_{16}	30		+	480	61920	1050240
D_{24}	46	5	+	1104	170064	17051328
$D_{16}E_8$	30	1	++	720	179280	16954560
E_8^3	30	1	+++	720	179280	16954560
A_{24}	25	$-1/4$	[0] + 2([5] + [10])	600	182160	16924320
D_{12}^2	22	-1	[00] + 2[12] + [11]	528	183888	16906176
$A_{17}E_7$	18	-2	[00] + 2[31] + 2[60] + [91]	432	186192	16881984
$D_{10}E_7^2$	18	-2	[000] + [110] + [310] + [211]	432	186192	16881984
$A_{15}D_9$	16	$-5/2$	[00] + 2([12] + [16] + [24]) + [08]	384	187344	16869888
D_8^3	14	-3	[000] + 3([011] + [122]) + [111]	336	188496	16857792
A_{12}^2	13	$-13/4$	[00] + 4([15] + [23] + [46])	312	189072	16851744
$A_{11}D_7E_6$	12	$-7/2$	[000] + 2[111] + 2[310] + 2[401] + 2[511] + [620]	288	189648	16845696
E_6^4	12	$-7/2$	[0000] + 8[1110]	288	189648	16845696
$A_9^2D_6$	10	-4	[000] + 4([121] + [132] + [240] + [341]) + 2[051] + [552]	240	190800	16833600
D_6^4	10	-4	[0000] + 12[0123] + [1111] + [2222] + [3333]	240	190800	16833600
A_8^3	9	$-17/4$	[000] + 6([114] + [330] + [122] + [244]) + 2[333]	216	191376	16827552
$A_7^2D_5^2$	8	$-9/2$	[0000] + 4([0211] + [1112] + [2202] + [3312] + [2411]) + 8[1301] + 2[0422] + [4400]	192	191952	16821504
A_6^4	7	$-19/4$	[0000] + 24[0123] + 8([1112] + [1222] + [2223])	168	192528	16815456
$A_5^4D_4$	6	-5	[00000] + 6[00331] + 24([00121] + [12231]) + 8([02220] + [11130]) + [33330]	144	193104	16809408
D_4^6	6	-5	[000000] + 45[000011] + 18[111111]	144	193104	16809408
A_4^6	5	$-21/4$	[000000] + 60[001122] + 40[111222] + 12[011111] + 12[022222]	120	193680	16803360
A_3^8	4	$-11/2$	[3(2001011)] + *cyclic permutations of* (2001011)	96	194256	16797312
A_2^{12}	3	$-23/4$	[2(11211122212)] + *cyclic permutations of* (11211122212)	72	194832	16791264
A_1^{24}	2	-6	[1(00000101001100110101111)] + *cyclic permutations of* (00000101001100110101111)	48	195408	16785216
Leech	0	$-13/2$	*None*	0	196560	16773120

TABLE IV: All lattice solutions in 24 dimensions. The second column represents the Coxeter number h. The third column shows x parametrization when $Z_0^0 = Z_{D_{16}E_8}$. The glue code in the explicit form is given for most of the lattices and in a generator form for A_3^8, A_2^{12}, A_1^{24} as in [5]. Numbers in this column represent conjugacy classes, + means combination of adjoint and spinor conjugacy classes, 0 stands for the adjoint, $1, 2, 3$ are vector, spinor and conjugate spinor for D_n lattices (similarly for A_n with $n-1$, and E_6, E_7 with two conjugacy classes). In the last three columns the first three coefficients in the Θ-series are listed with a_2 being a kissing number for a given lattice (except for Leech).