Matchings Avoiding Partial Patterns

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ABSTRACT

We show that matchings avoiding certain partial patterns are counted by the 3-Catalan numbers. We give a characterization of 12312-avoiding matchings in terms of restrictions on the corresponding oscillating tableaux. We also find a bijection between Schröder paths without peaks at level one and matchings avoiding both patterns 12312 and 121323. Such objects are counted by the super-Catalan numbers or the little Schröder numbers. A refinement of the super-Catalan numbers is obtained by fixing the number of crossings in the matchings. In the sense of Wilf-equivalence, we find that the patterns 12132, 12123, 12321, 12231, 12213 are equivalent to 12312.

Key words: Generating function, generating tree, matching, ternary tree, super-Catalan number, oscillating tableau.

AMS Mathematical Subject Classifications: 05A05, 05C30.

1. Introduction

A *matching* on a set $[2n] = \{1, 2, \ldots, 2n\}$ is a partition of $[2n]$ in which every block contains exactly two elements, or equivalently a graph on $[2n]$ in which every vertex has degree one. There are many ways to represent a matching. It can be displayed by drawing the $2n$ points on a horizontal line in the increasing order. This is called the *linear representation* of a matching [5]. An edge (i, j) is drawn as an arc between the nodes i and j above the horizontal line, where the vertices i and j are called the initial point and the end point, respectively. An edge $e = (i, j)$ is always written in such a way that $i < j$. Let $e = (i, j)$ and $e' = (i', j')$ be two edges of a matching M, we say that e *crosses* e' if they intersect with each other, in other words, if $i < i' < j < j'$. In this case, the pair of edges e and e' is called a *crossing* of the matching. Otherwise, e and e' are said to be *noncrossing*. The set of matchings on $[2n]$ is denoted by $\mathcal{M}_n$. Note that $|\mathcal{M}_n| = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

In this paper, we also use the representation of a matching M of n edges by a sequence of length $2n$ on the set $\{1, 2, \ldots, n\}$ such that each element i

$(1 \le i \le n)$ appears exactly twice, and the first occurrence of the element i precedes that of j if $i < j$. Such a representation is called the *Davenport-Schinzel sequence* [8, 23] or the *canonical sequential form* [20]. In fact, the canonical sequential representation of a matching is the sequence obtained from its linear representation by labeling the endpoints of each arc in the order of the appearance of its initial point such that the endpoints of each arc have the same label. For example, the matching in Figure 1 can be represented by 123123.

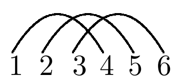


Figure 1. The matching 123123.

Let $\pi = \pi_1\pi_2\ldots\pi_k$ and $\tau = \tau_1\tau_2\ldots\tau_k$ be two sequences. If for any $1 \le i, j \le k$ we have $\pi_i < \pi_j$ if and only if $\tau_i < \tau_j$, then we say π and τ are *order-isomorphic*. The matching π contains an occurrence of τ if there is a subsequence in the canonical sequential form of π which is order-isomorphic to τ. In such a context τ is usually called a *pattern*. When a pattern forms a representation of a small matching, we say that it is complete; otherwise, we say that it is partial. In this paper we are mainly concerned with the partial pattern 12312. We say that π *avoids* τ, or τ*-avoiding*, if there is no occurrence of the pattern τ in the matching π. The set of all τ-avoiding matchings on $[2n]$ is denoted $\mathcal{M}_n(\tau)$. Denote by $M_n(\tau_1, \tau_2, \ldots, \tau_k)$ the set of matchings on $[2n]$ which avoid the patterns $\tau_1, \tau_2, \ldots, \tau_k$. Pattern avoiding matchings have been studied by de Médicis and Viennot [24], de Sainte-Catherine [28], Gessel and Viennot [15], Gouyou-Beauchamps [17, 18], Stein [32], Touchard [35], and recently by Klazar [20, 21, 22], Chen, Deng, Du, Stanley and Yan [6].

The *k-Catalan numbers*, or generalized Catalan numbers are defined by

$$C_{n,k} = \frac{1}{(k-1)n+1}\binom{kn}{n}$$

for $n \ge 1$ (see [19]). For $k = 2$, the 2-Catalan numbers are the usual Catalan numbers.

In this paper we show that 12312-avoiding matchings on $[2n]$ are counted by the 3-Catalan number, namely,

$$|\mathcal{M}_n(12312)| = \frac{1}{2n+1}\binom{3n}{n}.$$

We note that the following objects are also counted by the 3-Catalan numbers:

- complete ternary trees with n internal nodes, or $3n$ edges [25],
- even trees with $2n$ edges [4, 11],
- noncrossing trees with n edges [12, 25],
- the set of lattice paths from $(0,0)$ to $(2n, n)$ using steps $E = (1,0)$ and $N = (0,1)$ and never lying above the line $y = x/2$ [19],

- dissections of a convex $2n+2$-gon into n quadrilaterals by drawing $n-1$ diagonals, no two of which intersect in its interior [19],
- two line arrays $\binom{\alpha}{\beta}$, where $\alpha = \{a_1, a_2, \ldots, a_n\}$ and $\beta = \{b_1, b_2, \ldots, b_n\}$ such that $1 = b_1 = a_1 \leq b_2 \leq a_2 \ldots \leq b_n \leq a_n$ and $a_i \leq i$ [3].

The relations between ternary trees, even trees, and noncrossing trees have been studied by Chen [4], Feretic and Svrtan [13], Noy [14], and Panholzer and Prodinger [25]. Stanley discussed several of these families in [31, Problems $5.45-5.47$].

By using generating functions, we derive a formula for the number of matchings in $\mathcal{M}_n(12312)$ having exactly m crossings. We also show that the cardinality of $\mathcal{M}_{n-1}(12312, 121323)$ is the n-th super-Catalan number or the little Schröder number for $n \geq 1$ (see [26, Sequence A001003]). By considering the number of matchings in $\mathcal{M}_{n-1}(12312, 121323)$ having exactly m crossings we obtain a closed expression for a refinement of the super-Catalan numbers. The n-th super-Catalan number also counts the number of Schröder paths of semilength $n-1$ (i.e. lattice paths from $(0,0)$ to $(2n-2,0)$, with steps $H = (2,0)$, $U = (1,1)$, and $D = (1,-1)$ and not going below the x-axis) with no peaks at level one, as well as certain Dyck paths (see [26, Sequence A001003] and references therein). We find a bijection between Schröder paths of semilength n without peaks at level one and matchings on $[2n]$ avoiding both patterns 12312 and 121323.

Following the approach of Chen, Deng, Du, Stanley and Yan [6], we use oscillating tableaux to study 12312-avoiding matchings. The notion of *oscillating tableaux* is introduced by Sundaram [33, 34] in the study of the representations of the symplectic group (see also [9, 27]). These tableaux play an important role in Berele's decomposition formula [2] for powers of defining representations of the complex symplectic groups. In fact, an oscillating tableau is a sequence of Young diagrams (or partitions) starting and ending with the empty diagram $\lambda : \emptyset = \lambda^0, \lambda^1, \ldots \lambda^{k-1}, \lambda^k = \emptyset$ such that the diagram λ^i is obtained from λ^{i-1} by either adding one square or removing one square. An oscillating tableau can be equivalently formulated as a sequence of standard Young tableaux (often abbreviated as SYT). The number k in the above definition is called the length of the oscillating tableau λ.

It has been shown by Stanely [31] that oscillating tableaux of length $2n$ are in one-to-one correspondence with matchings on $[2n]$. In this paper we apply this bijection to 12312-avoiding matchings and obtain the corresponding oscillating tableaux and closed lattice walks. We further provide a one-to-one correspondence between the set of closed lattice walks and the set of lattice paths from $(0,0)$ to $(2n,n)$ using steps $E = (1,0)$ and $N = (0,1)$ without crossing the line $y = x/2$, see [16]. From this perspective, we see that $\mathcal{M}_n(12312)$ is counted by the 3-Catalan numbers.

In addition to the pattern 12312, we find other patterns that are equivalent to 12312 in the sense of Wilf-equivalence. To be more specific, we show that for any pattern $\tau \in \{12312, 12132, 12123, 12321, 12231, 12213\}$, we have $|M_n(\tau)| =$

$C_{n,3}$. We use the technique of generating trees to reach this conclusion. A generating tree is a rooted tree in which each node is associated with a label, and the labels of the children of a node are determined by certain succession rules. The idea of generating trees was introduced by Chung, Graham, Hoggat, JR. and M. Kleiman [7] for the study of Baxter permutations and was further applied to the study of pattern avoidance by Stankova and West [29, 30, 36, 37]. Barcucci et al. [1] developed the ECO method: a methodology for enumeration of combinatorial objects, which is based on the technique of generating trees.

2. Matchings and ternary trees

In this section, we use the sequence representation of a matching as described in the introduction. Our goal is to show that $\mathcal{M}_n(12312)$ is counted by the 3-Catalan number. The first approach is to give a recursive construction of the set $\mathcal{M}_n(12312)$. Intuitively, for a matching $\theta = a_1a_2\cdots a_{2n}$ in $\mathcal{M}_n(12312)$, we may obtain a matching in $\mathcal{M}_{n-1}(12312)$ by removing an edge. Then we need to keep track of all possible ways to recover a matching in $\mathcal{M}_n(12312)$ from a smaller matching. In the recursive generation of matchings with n edges, one is often concerned with the edge whose initial and end points have the label n in the canonical sequential form. However, for the purpose of this paper, we use the edge that is associated with the last node $2n$. We denote by E_θ the edge $(j, 2n)$ that is associated with the last node $2n$. In general, we use the notation E_i to denote the edge with end point i. In this sense, $E_\theta = E_{2n}$.

Let $E_\theta = (j, 2n)$ be the edge of θ associated with the last node $2n$. Clearly, if θ is 12312 avoiding, then the matching θ' obtained from θ by removing the edge E_θ is also 12312-avoiding. Thus, the question becomes how to identify the possibilities of the position j in the matching θ' for which one can add the edge $(j, 2n)$ to form a 12312-avoiding matching.

We need to introduce the notion of the critical crossing of a matching θ. Let F_θ be the edge with the rightmost end point that intersects with the edge E_θ. We call F_θ the critical edge of θ. If θ is 12312-avoiding, then the subgraph induced by the nodes between the end points of F_θ and E_θ is a 12312-avoiding matching. If there does not exist any edge that intersects with E_θ, then the subgraph induced by the nodes between the initial point and the end point of E_θ is a 12312-avoiding matching.

Let us now consider the case when there exists a critical crossing. We have the following lemma on the structure of 12312-avoiding matchings, which is straightforward to verify.

Lemma 2.1 *Let θ be a 12312-avoiding matching that has a critical crossing. Let $E_\theta = (j, 2n)$ and $F_\theta = (x, y)$. Then all the nodes between j and y are end points, and the edges associated with these nodes do not cross each other and have their initial nodes between x and j.*

Suppose that θ has a critical crossing. Let i be initial point of the edge E_{j+1}.

As a consequence of the above lemma, we see that the subgraph induced by the nodes between i and j forms a 12312-avoiding matching. It remains to consider the structure of the edges associated with the nodes before the node i. We need the following observation.

Lemma 2.2 *Let θ be a 12312-avoiding matching that has a critical crossing. Let $E_\theta = (j, 2n)$ and $F_\theta = E_{j+m}$. Then E_θ is the only edge that intersects any two edges E_{j+r} and E_{j+s} for $1 \le r < s \le m$.*

The above lemmas are sufficient to demonstrate the recursive structure of 12312-avoiding matchings. We need to decompose the matching on $\{1, 2, \ldots, j\}$ into segments for the construction of smaller 12312-avoiding matchings. The first step is to find an edge that intersects with F_θ with the rightmost end point v. If such edge does not exist then we get a 12312-avoiding matching induced by the nodes from 1 to the node before the initial point of F_θ. Otherwise, we have an edge with rightmost end point v that intersects with F_θ.

We claim that the nodes from 1 to v, altogether with the node $j+m$ (the end point of F_θ), form a 12312-avoiding matching. This can be seen from the fact that all the nodes between the initial point of F_θ and v are end points since θ is 12312-avoiding.

Now we ready to describe the recursive structure of 12312-avoiding matchings. If a 12312-avoiding matching does not have a critical crossing, then it consists of two smaller 12312-avoiding matchings as illustrated by Figure 2.

Figure 2.

The nontrivial part is the recursive structure of 12312-avoiding matchings that have critical crossings. Let θ be a matching on $[2n]$ having the two edges $E_\theta = (j, 2n)$ and $F_\theta = (i, j+m)$. Let us consider the following two cases: (1) There exists an edge E_y with the rightmost end point crossing the edge F_θ. (2) There does not exist such an edge.

For the above two cases, we see that θ can always be decomposed into smaller 12312-avoiding matchings. In the first case, we obtain three smaller matchings given below.

(1A) The induced subgraph of θ on the nodes $1, 2, \ldots, y, j+m$.

(1B) The induced subgraph of θ on the nodes $y+1, \ldots, j+m-1, 2n$.

(1C) The induced subgraph of θ on the nodes $j+m+1, \ldots, 2n-1$.

In the second case, we obtain three smaller matchings:

(2A) The induced subgraph of θ on the nodes $1, 2, \ldots, i, j+m$.

(2B) The induced subgraph of θ on the nodes $i+1,\ldots,j+m-1,2n$.

(2C) The induced subgraph of θ on the nodes $j+m+1,\ldots,2n-1$.

To find the number of 12312-avoiding matchings on $[2n]$ with a given number of crossings, we need a refinement of the above structure of 12312-avoiding matchings. We see that the matching θ avoids 12312 if and only if the smaller structures θ_1 and α' are 12312-avoiding matchings, where θ_1 is the corresponding induced subgraph in Case (1A) or (2A) (in Case (1) we define θ_1 as the empty matching), and α' is the induced subgraph obtained from θ by deleting the subgraph θ_1. The matching θ_1 has the critical edge of θ, namely E_{j+m}.

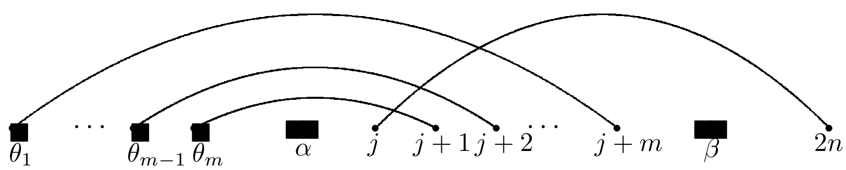


Figure 3.

Now, let us consider α' which is a smaller 12312-avoiding matching. If there exists a critical crossing, say E_{j+m-1}, for the matching α', then the matching α' can be decomposed into two smaller 12312-avoiding matchings θ_2 and α''. Repeating the above procedure m times we see that the matching θ can be decomposed into $m+2$ smaller 12312-avoiding matchings where the first m smaller matchings have critical edges $E_{j+m},\ldots,E_{j+1}$. We call these m edges quasi-critical edges of θ (see Figure 3).

Lemma 2.3 *Let θ be any matching on $[2n]$ with $E_\theta=(j,2n)$. Then, there exist m quasi-critical edges $E_{j+1},E_{j+2},\ldots,E_{j+m}$ of θ such that θ can be decomposed into $m+2$ smaller 12312-avoiding matchings $\theta_1,\ldots,\theta_m,\alpha,\beta$ such that*

(1) *θ_s is the induced subgraph on the nodes $v_{s-1}+1,v_{s-1}+2,\ldots,v_s,j+m+1-s$,*
(2) *α is the induced subgraph on the nodes $v_m+1,v_m+2,\ldots,j-1$,*
(3) *β is the induced subgraph on the nodes $j+m+1,j+m+2,\ldots,2n-1$,*

where $v_0=0$ and v_r, $r=1,2,\ldots,m$, is the rightmost end point of an edge that crosses the edge $E_{j+m+1-r}$. If such an edge does not exist, we define v_r as the initial point of $E_{j+m+1-r}$.

Moreover, as a corollary of Lemma 2.3 we find a formula for the number of 12312-avoiding matchings on $[2n]$ with exactly m crossings.

Theorem 2.4 *The number of 12312-avoiding matchings on $[2n]$ with exactly m crossings is given by*

$$\sum_{i=n}^{2n-1}\frac{(-1)^{n+m+i}}{i}\binom{i}{n}\binom{3n}{i+1+n}\binom{i-n}{m}.$$

Proof. Let
$$G(x,y)=\sum_{n\geq 0}\sum_{\theta\in M_n(12312)} x^n y^{c(\theta)},$$
where $c(\theta)$ is the number of crossings of θ. Let
$$B(x,y)=\sum_{n\geq 1}\sum_{\theta} x^n y^{c(\theta)},$$
where the second summation ranges over matchings θ_s as in the first case of Lemma 2.3. It follows from Lemma 2.3 that the ordinary generating function for the number of 12312-avoiding matchings with exactly m quasi-critical edges $E_{j+1},\ldots,E_{j+m}$ is given by $xy^mG^2(x,y)B^m(x,y)$. Summing over all the possibilities for $m\geq 0$ we arrive at
$$G(x,y)=1+\frac{xG^2(x,y)}{1-yB(x,y)}. \tag{2.1}$$
Applying Lemma 2.3 for matchings of the form θ_j, it follows that that the ordinary generating function for the number of 12312-avoiding matchings θ_j with exactly k quasi-critical edges is given by $xy^kG(x,y)B^k(x,y)$. Therefore, summing over all the possibilities for $k\geq 0$ we get
$$B(x,y)=\frac{xG(x,y)}{1-yB(x,y)}. \tag{2.2}$$
Combining (2.1) and (2.2) we get
$$B(x,y)=\frac{G(x,y)-1}{G(x,y)}. \tag{2.3}$$
It follows from (2.1) and (2.3) that $G(x,y)$ satisfies the following recurrence relation
$$xG(x,y)^3+G(x,y)-G(x,y)^2+y(G(x,y)-1)^2=0. \tag{2.4}$$
Substituting xy by x and $y+1$ by y in above recurrence, we get
$$G(xy,y+1)=1+y(xG^3(xy,y+1)+(G(xy,y+1)-1)^2). \tag{2.5}$$
Using the Lagrange inversion formula we obtain
$$G(xy,y+1)=1+\sum\nolimits_{i\geq 1}\frac{1}{i}\sum\nolimits_{j=0}^{i}\binom{i}{j}\binom{3j}{i+1+j}x^jy^i,$$
which implies that
$$G(x,y)=1+\sum\nolimits_{i\geq 1}\frac{1}{i}\sum\nolimits_{j=0}^{i}\binom{i}{j}\binom{3j}{i+1+j}x^j(y-1)^{i-j}. \tag{2.6}$$
Then $[x^ny^m]G(x,y)$ gives the number of 12312-avoiding matchings on $[2n]$ with exactly m crossings. ∎

Setting $y=1$ in (2.6), we obtain the following conclusion.

Theorem 2.5 *The number of* 12312*-avoiding matchings on* $[2n]$ *equals the* 3*-Catalan number* $C_{n,3}$.

In fact, we may use the above recursive structure of 12312-avoiding matchings to construct a bijection between 12312-avoiding matchings on $[2n]$ and ternary trees with n internal nodes. Instead, we will construct a bijection between 12312-avoiding matchings and oscillating tableaux which are in one-to-one correspondence with lattice paths counted by $C_{n,3}$. Let us consider the case $m = 0$, namely, the number of 12312-avoiding matchings without any crossings. It is clear that any noncrossing matching automatically avoids the pattern 12312. Therefore, the above formula reduces to the Catalan number when $m = 0$.

Corollary 2.6 *For all* $n \geq 1$, *we have the identity*

$$\sum_{i=n}^{2n-1} \frac{(-1)^{n+i}}{i} \binom{i}{n} \binom{3n}{i+1+n} = \frac{1}{n+1} \binom{2n}{n}.$$

3. $\mathcal{M}_n(12312, 121323)$ and Schröder paths

In this section, we are concerned with the matchings avoiding both patterns 12312 and 121323. We need a refinement of Lemma 2.3.

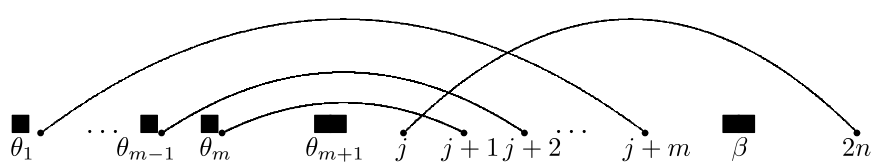


Figure 4.

Lemma 3.1 *Let* θ *be a matching on* $[2n]$ *with* $n \geq 1$. *Assume that* θ *has* m $(m \geq 0)$ *quasi-critical edges* $E_{j+1}, \ldots, E_{j+m}$. *Then* θ *avoids both patterns* 12312 *and* 121323 *if and only if* θ *can be decomposed into smaller matchings* $\theta_1, \ldots, \theta_{m+1}, \beta$ *avoiding both patterns* 12312 *and* 121323 *such that*

(1) θ_s *is the induced subgraph of* θ *on the nodes* $v_{s-1}+1, \ldots, v_s - 1$,
(2) β *is the induced subgraph of* θ *on the nodes* $j+m+1, \ldots, 2n-1$,

where $v_0 = 0$, $v_{m+1} = j$, *and* v_s *is the initial point of the edge* $E_{j+m+1-s}$.

Let

$$F(x) = \sum_{n \geq 0} f_n x^n$$

be the ordinary generating function of the number of matchings on $[2n]$ which avoid both patterns 12312 and 121323. From Lemma 3.1 we have the following recurrence relation

$$F(x) = 1 + \frac{xF^2(x)}{1 - xF(x)}.$$

It follows that

$$F(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4x} = 1 + \sum_{n \geq 1} \frac{1}{n} \sum_{j=1}^{n} 2^{j-1} \binom{n}{j} \binom{n}{j-1} x^n.$$

Now we see that for $n \geq 1$, f_{n-1} turns out to be the n-th super-Catalan number which equals the number of Schröder paths of semilength $n-1$ without peaks at level one.

We proceed to give a bijection ϕ between the set of Schröder paths of semilength n without peaks at level one and the set of matchings on $[2n]$ which avoid both patterns 12312 and 121323. Note that any nonempty Schröder path P has the following unique decompostion:

$$P = HP' \text{ or } P = UP'DP'',$$

where P' and P'' are possibly empty Schröder paths. This is called the *first return decomposition* by Deutsch [10].

Given a Schröder path P of semilength n without peaks at level one, if it is empty, then $\phi(P)$ is the empty matching. Otherwise, we may decompose it by using the first return decomposition. We may use this decomposition recursively to get the matching $\phi(P)$ on $[2n]$ avoiding both patterns 12312 and 121323.

Case 1. If $P = HP'$, we have the structure as shown in Figure 5:

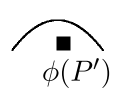
$\phi(P) =$ 

Figure 5. Case 1.

Case 2. If $P = UP'DP''$ and $P' = P_1UDP_2UD \ldots P_kUDP_{k+1}$, where for any $1 \leq i \leq k+1$, P_i is a Schröder path without peaks at level one, then we have the structure as shown in Figure 6:

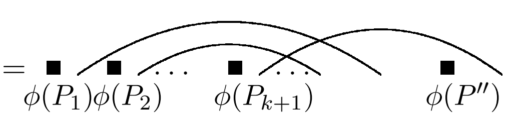
$\phi(P) =$ 

Figure 6. Case 2.

Conversely, given a matching M on $[2n]$ which avoids both patterns 12312 and 121323, we can get a Schröder path P of semilength n without peaks at level one. Suppose that M can be decomposed into smaller matchings $\theta_1, \ldots, \theta_{k+1}, \beta$ avoiding both patterns 12312 and 121323 as described in Lemma 3.1. If $k = 0$ and $\theta_1 = \emptyset$, then we have

$$\phi^{-1}(M) = H\phi^{-1}(\beta).$$

Otherwise, we get

$$\phi^{-1}(M) = U\phi^{-1}(\theta_1)UD\phi^{-1}(\theta_2)UD \ldots \phi^{-1}(\theta_k)UD\phi^{-1}(\theta_{k+1})D\phi^{-1}(\beta),$$

which is clearly a Schröder path of semilength n without peaks at level one. Thus, we have constructed the desired bijection.

Example 3.2 *As illustrated in Figure 7, the Schröder path $UUDDUUUDDHD$ corresponds to the matching* $\{(1,3),(2,12),(4,6),(5,9),(7,8),(10,11)\}$

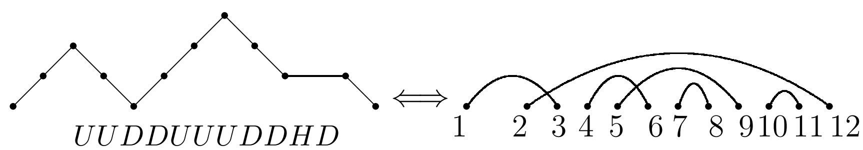


Figure 7. The bijection ϕ.

In view of the bijection ϕ, we see that a peak corresponds to a crossing of the corresponded matching. Denote by $\mathcal{M}_{n,m}(12312, 121323)$ the set of the matchings in $\mathcal{M}_n(12312, 121323)$ with exactly m crossings. We have the following formula.

Theorem 3.3 *For $n, m \geq 0$, we have*

$$|\mathcal{M}_{n,m}(12312, 121323)| = \frac{1}{n}\binom{n}{m}\binom{2n-m}{n+1}.$$

Proof. It is well known that a Schröder path of semilength n can be obtained from a Dyck path of semilength n by turning some peaks of the Dyck path into H steps. A peak is called a *low* peak if it is at level one. Otherwise, it is called a *high* peak. It has been shown by Deutsch [10] that the number of Dyck paths of semilength n with exactly k high peaks is given by the Narayana number

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

Thus the number of Schröder paths of semilength n that contain exactly m high peaks but no peaks at level one equals

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{1}{n}\binom{n}{k}\binom{n}{k+1}\binom{k}{m} \\
&\quad = \sum_{k=0}^{n-1} \frac{1}{n}\binom{n}{m}\binom{n-m}{k-m}\binom{n}{k+1} \\
&\quad = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{m}\binom{n-m}{n-k+1}\binom{n}{k} \\
&\quad = \frac{1}{n}\binom{n}{m}\binom{2n-m}{n+1}.
\end{aligned}$$

This completes the proof. ∎

4. Matchings and Oscillating Tableaux

In this section, we apply Stanley's bijection between matchings and oscillating tableaux to the set of 12312-avoiding matchings [31] to obtain the corresponding restrictions on the oscillating tableaux for 12312-avoiding matchings. From the oscillating tableaux we may construct closed lattice walks and lattice paths that are counted by the 3-Catalan numbers.

It has been shown by Stanley [31] that oscillating tableaux of length $2n$ are in one-to-one correspondence with matchings on $[2n]$. We denote this bijection by ρ. This is stated as follows. Given an oscillating tableau $\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n-1}, \lambda^{2n} = \emptyset$, we may recursively define a sequence $(\pi_0, T_0), (\pi_1, T_1), \ldots,$ (π_{2n}, T_{2n}), where π_i is a matching and T_i is a standard Young tableau (SYT). Let π_0 be the empty matching and T_0 be the empty SYT. The pair (π_i, T_i) can be obtained from (π_{i-1}, T_{i-1}) by the following procedure:

(1) If $\lambda^i \supset \lambda^{i-1}$, then $\pi_i = \pi_{i+1}$ and T_i is obtained from T_{i-1} by adding the entry i in the square $\lambda^i \setminus \lambda^{i-1}$.
(2) If $\lambda^i \subset \lambda^{i-1}$, then let T_i be unique SYT of shape λ^i such that T_{i-1} is obtained from T_i by inserting some number j by the RSK (Robinson-Schensted-Knuth) algorithm. In this case, let $\pi_i = \pi_{i-1} \cup (j, i)$.

If the entry i is added to T_{i-1} to obtain T_i, then we say that i is added at step i. If i is removed from T_{j-1} to obtain T_j, then we say that i leaves at step j. In this bijection, (i, j) is an edge of the corresponding matching if and only if i is added at step i and leaves at step j.

Example 4.1 *For the oscillating tableau*

$$\emptyset, (1), (2), (2,1), (1,1), (1), \emptyset,$$

we get the corresponding sequence of SYTs as follows:

$\emptyset$	1	12	12	1	3	$\emptyset$,
			3	3		

and the corresponding matching with edges $\{(1,5),(2,4),(3,6)\}$.

Theorem 4.2 *There exists a bijection ρ between the set of* 12312*-avoiding matchings on* $[2n]$ *and the set of oscillating tableaux $T_{2n}^{\emptyset}$, in which each partition is of shape (k) or $(k,1)$ such that a partition $(k,1)$ is not followed immediately by the partition $(k+1,1)$.*

Proof. Let M be a 12312-avoiding matching. By definition, there do not exist edges (i_1,j_1), (i_2,j_2) and (i_3,j_3) in M such that $i_1 < i_2 < i_3 < j_1 < j_2$. Suppose that the corresponding sequence of SYTs are $T_0, T_1, \ldots, T_{2n}$ and the corresponding oscillating tableau is $\lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^{2n}$ under the Stanley's bijection.

If λ^{p-1} is of shape (k) for some $1 \le p \le 2n$, then it is possible that $\lambda^{p-1} \subset \lambda^p$. We claim that all the entries in T_{p-1} must leave the tableau in deceasing order. It is clear that the entries in the first row of an SYT are strictly decreasing from right to left. Suppose that i_2 is right to i_1 in T_{p-1} and i_1 leaves the tableau before i_2. Assume that i_1 leaves at step j_1. That is to say there exists an entry $h \ge p$ in the first row of T_{j_1} such that T_{j_1-1} is obtained from T_{j_1} by inserting i_1. According to the RSK algorithm, the insertion of i_1 pushes h up and i_1 takes the place of h. Hence in T_{j_1}, h is left to i_2 in its first row, which contradicts with the fact that T_{j_1} is an SYT. So there do not exist two crossing edges (i_1,j_1) and (i_2,j_2) such that $i_1 < i_2 < p < j_1 < j_2$. So T_p can be obtained from T_{p-1} by adding an entry p.

Now let us consider the case that λ^{i-1} is of shape (k) and λ^i is of shape $(k,1)$. By the bijection ρ, i is added in the square $\lambda^i \setminus \lambda^{i-1}$ to obtain T_i. Suppose that i is moved to the first row in T_{j_1}. Then there exists a unique entry j such that T_{j_1-1} is obtained from T_{j_1} by row-inserting the entry j by the RSK algorithm. Hence $j < i$ and j leaves before i. Suppose that i leaves the tableau at step i_1 with $j < i < j_1 < i_1$, which implies that (j,j_1) and (i,i_1) are two crossing edges of the matching M. For any $i+1 \le p \le j_1$, we have $\lambda^p \subset \lambda^{p-1}$. Otherwise, T_p is obtained from T_{p-1} by inserting the entry p in the square $\lambda^p \setminus \lambda^{p-1}$. It is obvious that p is the initial point of an edge of M. Let (p,p_1) be an edge of M. Then (j,j_1), (i,i_1) and (p,p_1) are three edges of M such that $j < i < p < j_1 < i_1$, which contradicts with the fact that M is a 12312-avoiding matching. Furthermore, λ^{j_1} is of shape (h) for some integer h. Thus we come to the assertion that for the case when λ^{p-1} is of shape $(k,1)$, no square is added to obtain λ^p for any $1 \le p \le 2n$. This completes the proof. ∎

Given a matching $\pi \in \mathcal{M}_n(12312)$, we may define a closed lattice walk $\{\overrightarrow{v_i} = (x_i, y_i)\}_{i=0}^{2n}$ with $x_i \ge y_i$ in the (x,y)-plane from the origin to itself by letting x_i (resp. y_i) be the number of squares in the first (resp. second) row of the partition λ^i of the corresponding oscillating tableau. If $(x_{i+1}, y_{i+1}) - (x_i, y_i) = (0, 1)$, then by Theorem 4.2 we see that the size of the next partition does not increase. Thus we have the following corollary.

Corollary 4.3 *There is a one-to-one correspondence between* 12312*-avoiding matchings on* $[2n]$ *and closed lattice walks of length* $2n$ *in the* (x, y)*-plane from the origin to itself consisting of the steps* $E = (1, 0)$*,* $W = (-1, 0)$*,* $N = (0, 1)$ *and* $S = (0, -1)$ *such that a step* N *is followed immediately by some consecutive* W *steps and one step* S *and no step crosses the line* $y = x$*.*

Example 4.4 *The closed lattice walk corresponding to the matching* $\{(1, 5), (2, 4), (3, 6)\}$ *is* $EENWSW$*.*

Denote by L_n the set of such closed lattice walks in the (x, y)-plane as specified in the above corollary, and denote by P_n the set of lattice paths from $(0, 0)$ to $(2n, n)$ consisting of steps $E = (1, 0)$ and $N = (0, 1)$ and never crossing the line $y = \frac{x}{2}$. Hilton and Pedersen [19] have shown that the cardinality of P_n is the 3-Catalan number. Now, let us describe a one-to-one correspondence between L_n and P_n. Given a closed lattice walk $p \in L_n$, we define a map τ by traversing the steps of p along the path and changing the steps of p by the following rule:

$$\begin{array}{lcl} E & \rightarrow & EE, \\ W & \rightarrow & N, \\ N & \rightarrow & EN, \\ S & \rightarrow & E. \end{array}$$

Denote by $|x|_p$ the number of x steps in the path p. Clearly, we have that $|E|_p = |W|_p$, $|N|_p = |S|_p$ and $|E|_p + |N|_p = n$ since p is a lattice path going from the origin to itself with $2n$ steps. From the map τ, we see that $|E|_{\tau(p)} = 2|E|_p + |N|_p + |S|_p = 2n$ and $|N|_{\tau(p)} = |N|_p + |W|_p = n$. Hence $\tau(p)$ is a path from $(0, 0)$ to $(2n, n)$. We claim that $\tau(p)$ never crosses the line $y = x/2$. Otherwise, let $\tau(p) = p_1 p_2 \ldots p_{3n}$ such that $p_k = N$ is the first step going above the line $y = x/2$. Let $p'' = p_1 p_2 \ldots p_k$ and $\tau(p') = p''$. We get

$$|E|_{p''} = 2|E|_{p'} + |N|_{p'} + |S|_{p'} < 2|N|_{p''} = 2|N|_{p'} + 2|W|_{p'},$$

which implies that either $|E|_{p'} - |W|_{p'} < |N|_{p'} - |S|_{p'}$ or $|E|_{p'} < |W|_{p'}$. This contradicts with the fact that p never goes above the line $y = x$, implying that $\tau(p) \in P_n$.

Conversely, given a lattice path $p \in P_n$, let E_i be its i-th E step from left to right. If E_{2k-1} and E_{2k} are consecutive steps in p, then E_{2k-1} altogether with E_{2k} corresponds to a E step. Otherwise, E_{2k} corresponds to one S step and E_{2k-1} altogether with next N step corresponds to a N step. For the remaining N steps, each N step corresponds to a W step.

Denote by p' the resulted path. From the map we see that each N step is followed by some W steps and one S step in p' and $|N|_{p'} = |S|_{p'}$. Moreover we have the relations $|E|_p = 2|E|_{p'} + |N|_{p'} + |S|_{p'} = 2n$ and $|N|_p = |N|_{p'} + |W|_{p'} = n$. It follows that $|E|_{p'} + |N|_{p'} = n$ and $|E|_{p'} = |W|_{p'}$, which implies that p' is a path going from the origin to itself with $2n$ steps. We claim that p' is a path never crossing the line $y = x$. Otherwise, let $p' = p_1 p_2 \ldots p_{2n}$, $p'' = p_1 p_2 \ldots p_k$ and $\tau^{-1}(p''') = p''$. From the map we have

$$|E|_{p'''} = 2|E|_{p''} + |N|_{p''} + |S|_{p''} \geq 2|N|_{p'''} = 2|N|_{p''} + 2|W|_{p''},$$

and either $|N|_{p''} = |S|_{p''}$ or $|N|_{p''} - |S|_{p''} = 1$. Therefore, we obtain $|E|_{p''} - |W|_{p''} \geq |N|_{p''} - |S|_{p''}$, which implies that p' is a path never crossing the line $y = x$. That is to say $p' \in L_n$. Up to now we have proved that the map τ is a bijection between L_n and P_n, and we have the following conclusion.

Theorem 4.5 *The map τ is a bijection between L_n and P_n. Moreover, we have*

$$|L_n| = |P_n| = |\mathcal{M}_n(12312)| = \frac{1}{2n+1}\binom{3n}{n}.$$

Example 4.6 *For $n = 2$, we have*

$$\begin{array}{cccc} L_2: & EEWW & ENSW & EWEW \\ & \downarrow & \downarrow & \downarrow \\ P_2: & EEEENN & EEENEN & EENEEN \end{array}$$

5. Matchings and generating trees

In this section we use the methodology of generating trees to deal with other partial patterns 12132, 12123, 12321, 12231, 12213. In fact, they are all Wilf-equivalent to 12312. Given a matching π on $[2n]$, a *position* s of π is meant to be the position between the nodes s and $s+1$ in the canonical sequential form if $1 \leq s \leq 2n-1$, and the *position* $2n$ is meant to be the position to the right of the node $2n$. In the terminology of generating trees, a position is called a *site*.

Definition 5.1 *Let τ be a pattern on $[k]$. The position s of π is an active site if there exists a position t, $1 \leq s \leq t \leq 2n$, such that inserting an edge starting at position s and ending at position t gives a τ-avoiding matching on $[2n+2]$. Otherwise, the position s is said to be an inactive site.*

Given a partial pattern τ, we use $T(\tau)$ to denote the generating tree for the set of τ-avoiding matchings on $[2n]$.

Lemma 5.2 *For any $\tau \in \{12312, 12132, 12123, 12321, 12231, 12213\}$, the generating tree $T(\tau)$ is given by*

$$\begin{cases} \textit{Root}: (0) \\ \\ \textit{Rule}: (k) \rightsquigarrow (k+1)^1\,(k)^2\,(k-1)^3 \dots (0)^{k+2}, \end{cases} \tag{5.1}$$

where the matching $\pi \in \mathcal{M}_n(\tau)$ is labeled by (k) such that $k+2$ is the number of its active sites.

Proof. We only consider the cases for $\tau = 12312,\ 12123,\ 12321$. The other cases can be dealt with in the same manner.

The case $\tau = 12312$: The matching π on [2] has two active sites. This is consistent with the root label (0). Let π be a 12312-avoiding matching on $[2n]$ labeled by (k) with active sites $i_1, i_2, \dots, i_{k+2}$. Let π' be a matching in $\mathcal{M}_{n+1}(12312)$ obtained from π by inserting an edge from position i_s to position i_t with $i_s \leq i_t$. Hence the active sites of π' are $i_t+1, i_t+2, i_{t+1}+2, \dots, i_{k+2}+2$. There are $k+4-t$ such active sites. So the children of the node (k) are exactly the nodes (k') where $k' = k+2-t$. If s ranges over $1, 2, \dots, k+2$ and t ranges over $s, s+1, \dots, k+2$, we get the desired rule 5.1.

The case $\tau = 12123$: The proof is analogous to the previous proof. The only difference lies in the following counting argument. The active sites of π' are $i_s+1, i_{s+1}+1, \dots, i_t+1$ when $i_s \neq i_t$ and $i_s+1, i_s+2, i_{s+1}+2, \dots, i_{k+2}+2$ when $i_s = i_t$. Thus, the label of π' is (k') where $k' = t-s-1$ if $i_s \neq i_t$ and $k' = k-s+2$, otherwise. The rest of the argument is similar to that in the previous case.

The case $\tau = 12321$: We only need to mention that the active sites of π' are $i_t+2, i_{t+1}+2, \dots, i_{k+2}+2$ when $t < k+2$ and $i_s+1, i_s+2, i_{s+1}+2 \dots, i_{k+2}+2$ when $t = k+2$. It follows that the label of π' is (k') where $k' = k-t+1$ when $t < k+2$ and $k' = k+2-s$, otherwise. ∎

Applying Theorem 2.5 and Lemma 5.2, we reach the following assertion.

Theorem 5.3 *For any $\tau \in \{12312, 12132, 12123, 12321, 12231, 12213\}$, we have $|M_n(\tau)| = C_{n,3}$.*

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