Extending Fine sequences: a link with forbidden patterns

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Abstract

We propose a natural, bivariate, generalization of the nonsingular similarity relations considered by T. Fine. We also provide an enumeration formulae and a generating tree for those relations. The latter allow us to give a new bijection between 321-avoiding derangements and Fine sequences. Moreover, we establish that two special cases are in a one-to-one correspondence with subsets of permutations characterized by forbidden subsequences on the symmetrical group. All our results are established using the technique of generating tree, thus giving entirely bijective proofs.

Keywords: Fine sequences, permutations with forbidden patterns or subsequences, similarity relations, generating trees, bijection, enumeration.

1 Introduction and motivation

1.1 Similarity relations and permutations with forbidden patterns

Similarity relations are *symmetrical* and *reflexive* binary relations R operating on $[n] = \{1, 2, \ldots, n\}$ such that $x < y < z$ and xRz implies xRy and yRz. These relations can be coded by an *integer sequence* $\alpha_1\alpha_2 \ldots \alpha_n$ such that for every y: $\alpha_y = y - x$ where x is the minimal integer verifying xRy. From the above definitions, one can see that $\alpha_1 = 0$ and $0 \leq \alpha_{x+1} \leq \alpha_x + 1$. Such relations are enumerated by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$. The figure below illustrates the similarity relation on $n = 8$ coded by the integer sequence 01100121 (an edge connects two vertices x, y iif xRy).

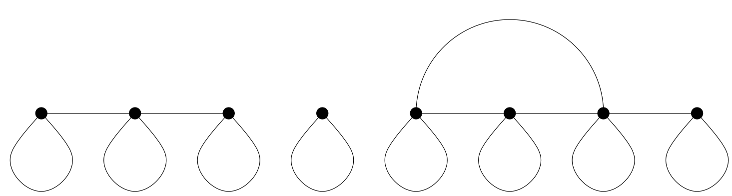

Figure 1: a similarity relation on $n = 8$ coded by the integer sequence 01100121.

In the present work, we shall consider *nonsingular similarity relations* on $[n]$, that is similarity relations with the restriction that every element must be in relation with another one at least, namely: $\forall x \in [n], \exists y \neq x$ such that xRy. We can code these relations on $[n]$ with the same integer sequence as above by adding the condition that every 0 is followed by 1. From now on, the cardinality of these relations on $[n]$ will be denoted F_n, with $n \geq 1$, the first values of which are: $1, 2, 6, 18, 57, 186, 622, 2120, \ldots$ These are the Fine sequences, enumerated by $F_n = \frac{1}{2}\sum_{i=0}^{n-1}(-\frac{1}{2})^i C_{n+1-i}$, a formulae due to D.G. Roger [Ro]. The figure hereafter illustrates a nonsingular similarity relation on $n = 6$ coded by the integer sequence 010122.

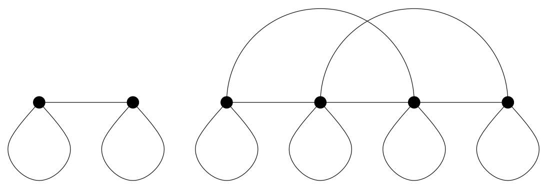

Figure 2: a nonsingular similarity relation on $n = 6$ coded by the integer sequence 010122.

This Fine sequence was considered notably by L.W. Shapiro [Sh] who stated that $2F_n + F_{n-1} = C_{n+1}$. It was also discussed by V. Strehl [Str] who established that $F_n = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} C_{n+1,2k}$ for $n \geq 1$ where $C_{n,j} = \binom{2n-j-1}{n-1} - \binom{2n-j-1}{n}$ enumerates the nonsingular similarity relation on $[n+1]$ for which the transitive closure consists of k blocks. We recall that $C_{n,j}$ is known as ballot numbers, Delannoy numbers [E] or distribution α of the Catalan numbers [Krew]. More recently, Deutsch and Shapiro wrote a survey [DS] and Callan give also some identities for the Fine numbers [C].

The study of *permutations with forbidden patterns* can be traced back to Simion and Schmidt [Si]. It is, nowadays, a growing domain of combinatorics with its own annual conference (for instance see [EV]).

Permutations with forbidden patterns (or subsequences) constitute a subset of the symmetrical group characterized by the exclusion of permutations containing at least one subsequence orderisomorphic to the forbidden one. More specifically, a permutation π of length n contains the subsequence (pattern) of type τ of length k if and only if one can find $1 \leq i_{\tau(1)} < i_{\tau(2)} < \cdots < i_{\tau(k)} \leq n$ such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. We denote by $\mathcal{S}_n(\tau)$ the set of permutations of length n which does not contain any subsequences of type τ. Moreover, let $\mathcal{S}_n(\tau_1, \tau_2, \ldots, \tau_l)$ be $\mathcal{S}_n(\tau_1) \cap \mathcal{S}_n(\tau_2) \cap \ldots \cap \mathcal{S}_n(\tau_l)$. As an example, the following permutation $\pi = 364125$ contains two occurences of the pattern 123 (namely the subsequences 345 and 125) but none of the pattern 321, so it belongs, for example, to $\mathcal{S}_6(321)$.

The aim of our present work is to give a natural generalization of nonsingular similarity relations. In section 2, we define these generalized Fine sequences as both Catalan paths with constraints or words on the natural numbers. We also give an enumeration formulae for those objects. Thereafter, in section 3, we state four theorems. The first one relates the generalized Fine sequences with generating trees. The second one establish a link between those generating trees and 321-avoiding derangements, which ones are notably in bijection with non singular similarity relation [DS]. The others relate two cases of generalized Fine sequences with permutations with forbidden patterns. Next, in section 4, we introduce some needed background on those topics. We then proceed to demonstrate these theorems, the former in section 5 and the latter ones in sections 6, 7 and 8.

2 Generalized Fine sequences

A natural way of generalizing nonsingular similarity relations is to consider a Fine sequence as a Catalan path and operate a congruous/modulo fonction on the α-distribution of those paths. Moreover, as we will see later, this generalization relate to well-known sequences. So, a generalized Fine sequence on $[n]$ congruous q modulo p will be a Catalan path of length $2n$ such that every primitive path —subpaths that doesn't touch the x-axis— starts with a rise at least equal to q except the first one, who starts with p. Equivalently: Catalan paths such that the first rise is equal to $k\,q+p$, with $k < \lfloor \frac{n-p}{q} \rfloor$. We shall denote the set of generalized Fine sequences on $[n]$ congruous q modulo p by $F_n^{p,q}$.

As presented earlier in subsection 1.1, we can also code the generalized Fine sequences with words ω on $\mathbb{N}^n$.

Definition 2.1 *Generalized Fine sequences, $F_n^{p,q}$, are words $\omega = \omega_1\omega_2\ldots\omega_n$ on $\mathbb{N}^n$, with $0 \le p < q$, such that:*

(i) $\omega = \beta\gamma$ *with* $\beta = 01\ldots(p-1)$,

(ii) $\forall i \in [n-1], 0 \le \omega_{i+1} \le \omega_i + 1$,

(iii) $\forall i \in [p+1, n-q], \omega_i = 0$ *implies* $\omega_{i+1}\omega_{i+2}\ldots\omega_{i+q} = 12\ldots(q-1)$.

Remark The following mappings give two bijections on generalized Fine sequences: the first one between words and Catalan paths with constraint on the primitive paths and the second one between the latter and Catalan paths with constraint on the first rise. Let x be a $(+1,+1)$ step and $\overline{x}$ a $(+1,-1)$ step.

- consider a word $\omega = \omega_1\omega_2\ldots\omega_n$ on $\mathbb{N}^n$, satisfying Definition 2.1. First, start the path with a x step, then, for each ω_i, if $\omega_i > \omega_{i-1}$ append a x step, else, append $(\omega_{i-1} - \omega_i + 1)\,\overline{x}$ steps and one x step. Finally, append $(\omega_n + 1)\,\overline{x}$ steps. The reverse of this fonction is direct, thus giving us a bijection,
- now, given a Catalan path with $k+1$ primitive paths: $x^p(\nu_p)\overline{x}\,x^q_{q_1}(\nu_{q_1})\ldots x^q_{q_i}(\nu_{q_i})\ldots x^q_{q_k}(\nu_{q_k})$, where $x^q_{q_i}(\nu_{q_i})$ is the $(i+1)^{th}$ primitive path and ν_{q_i} a postfix of this path. The corresponding path with constraint on the first rise will be: $x^{kq}\,x^p\,\overline{x}(\nu_p)\,(\nu_{q_1})\ldots(\nu_{q_i})\ldots(\nu_{q_k})$.

As an example, Figure 3 illustrates the generalized Fine sequence, with $q = 3$ and $p = 1$, coded by the integer sequence 011201220123345.

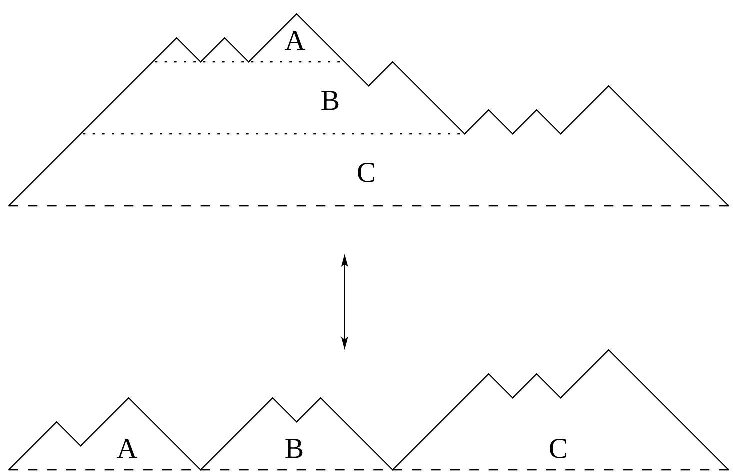

Figure 3: Catalan paths corresponding to the generalized Fine sequence coded by 011201220123345.

The result which ensues from the above definitions is a direct enumerative formulae:

$$Card\{F_n^{p,q}\} = \sum_{k=0}^{\lfloor \frac{n-p}{q} \rfloor} \left[\binom{2n-(kq+p)-1}{n-1} - \binom{2n-(kq+p)-1}{n} \right]$$

where $\binom{2n-k-1}{n-1} - \binom{2n-k-1}{n}$ is the α-distribution of the Catalan numbers. That is a Catalan path with a first rise of height k.

3 Main theorems

We first state a general theorem about generalized Fine sequences. We then proceed with permutations with forbidden patterns.

Definition 3.1 *We consider the following succession system:*

$$\left\{ \begin{array}{lll} root = [P] & & \\ [T] & \longrightarrow & [T], [3] \\ & \underset{q-1}{\longrightarrow} & [q] \\ [t] & \longrightarrow & [T], [3], \ldots, [t+1] \end{array} \right.$$

with $P = \left\{ \begin{array}{l} p\ if\ p \geq 1 \\ q\ else \end{array} \right.$

Take note that if $t < 2$, then the last succession rule only generates $[T]$.

Theorem 3.2 *Generalized Fine sequences, congruous p modulo q: $F_n^{p,q}$, can be charaterized by the succession system 3.1, given just below.*

Theorem 3.3 *Derangements —permutations such that $\forall i, \pi(i) \neq i$— avoiding pattern* 321*, which are in a one-to-one correspondence with nonsingular similarity relations [DS], can also be characterized by the succession system 3.1, with $p = 0$ and $q = 2$.*

Among the generalized Fine sequences, two particular cases can be related to permutations with forbidden patterns. Those are:

Theorem 3.4 *Nonsingular similarity relations, namely $F_n^{0,2}$, are in a one-to-one correspondence with three sets of permutations with forbidden patterns: $S_n(\mathcal{F}_1)$, $S_n(\mathcal{F}_2)$ and $S_n(\mathcal{F}_3)$. With, $\mathcal{F}_1 = \{1234, 1243, 1324, 2134, 2314, 3124\}$, $\mathcal{F}_2 = \{1324, 2134, 2143, 2314, 3124, 3214\}$ and $\mathcal{F}_3 = \{1342, 2341, 2413, 2431, 3142, 3241\}$.*

Theorem 3.5 *Generalized Fine sequences congruous to* one *modulo* three*, namely $F_n^{1,3}$, are in a one-to-one correspondence with five sets of permutations with forbidden patterns: $S_{n-1}(\mathcal{H}_1)$, $S_{n-1}(\mathcal{H}_2)$, $S_{n-1}(\mathcal{H}_3)$, $S_{n-1}(\mathcal{H}_4)$ and $S_{n-1}(\mathcal{H}_5)$. With, $\mathcal{H}_1 = \{1324, 2314, 2413, 3124, 3142, 3214\}$, $\mathcal{H}_2 = \{1234, 1243, 1324, 1423, 2314, 3124\}$, $\mathcal{H}_3 = \{2341, 2413, 2431, 3412, 3421, 4231\}$, $\mathcal{H}_4 = \{2134, 2143, 2314, 3124, 3214, 4213\}$ and $\mathcal{H}_5 = \{1234, 1243, 1324, 1423, 2134, 3124\}$.*

4 Some background

In order to follow the demonstration of the theorems, we give some background and outline the sketch of the proves. But first of all, let us define a usefull notation: say ${}_f\tau$ as "the fordibben pattern τ". This notation will come in handy thereafter. The reader already familiarized with the technique of generating trees may now skip this section.

4.1 Background on generating tree

Originaly, the technique of *generating tree* was introduced by F.R.K. Chung, R.L. Graham, V.E. Hoggat and M. Kleiman [CGHK] in order to enumerate Baxter permutations who avoided two patterns. Later, it was also applied to the study of various permutations with forbidden subsequences by different authors (see for example [BdLPP, Gi, Gu, P, W1, W2, W3]).

A generating tree is a rooted, labeled tree such that the label of any vertex exclusively determined the number and the labels of his children. Thus, any particular generating tree can be recursively defined by a *succession system*, which is a set of succession rules consisting of a basis (the label of the root) and an inductive step (a set of labels), which are the children generated by any label. Moreover, each vertex sharing the same depth is associated with a combinatorial object of the same cardinality.

Consequently, if a succession system is shared by different generating trees, the combinatorial objects related to those trees are in a direct bijection. We say that they are *charaterized* by the same succession system.

Moreover, any succession system can be used to obtain recurrence relations from which one may compute a closed form counting the objects themselves.

The relationship between structural properties of rules and the rationality, algebraicity or transcendence of the corresponding generating function has been investigated [BB-MDFGG-B]. See also [B-M]. This technique also permits the random generation of the objects considered [BdLP] (see also [Gu, section 2.4]).

4.2 Sketch of the proofs

In order to prove the main theorems we will have to show, many times, that a given set of permutations with forbidden subsequences can be characterized by a given succession system.

Given a set $\mathcal{E}$ of forbidden patterns and a permutation π in $S(\mathcal{E}) = \bigcup_{n\geq 0} S_n(\mathcal{E})$. To prove that $S(\mathcal{E})$ can be characterized by a succession system we have to:

- give a generating tree of $S(\mathcal{E})$ characterized by the succession system,
- associate each permutation belonging to $S(\mathcal{E})$ with a label of the generating tree. Thus, the set of labels must form a partition of $S(\mathcal{E})$. Furthermore, each labelized permutation belonging to $S(\mathcal{E})$ must have a unique father in the generating tree,
- prove that given a labeled permutation, all its children will have the same labels as given by the generating tree. Moreover, they must be unique and belongs to $S(\mathcal{E})$. We also need to prove that the root has the right label.

Alltogether, these points define a bijection between a path in the generating tree and a permutation in $S(\mathcal{E})$.

Remark In the following, we will not explicitly give the generating trees associated with the succession systems: they can be obtained thoroughly with the definition of the labels and by inserting $n+1$ in the active sites from left to right. The resulting permutations will have, orderwise, the labels given by the succession rules. Thereafter, we will not explicitly prove that the set of labels form a correct partition, that all childrens are unique nor that the root has the right label, as this can be done at first glance. Finally, since new permutations are obtained by insertion, determining the unique father is trivialy done by suppressing the greatest integer. Consequently this point will not be developed either.

Definition 4.1 *Given a set $\mathcal{E}$ of forbidden patterns and π a permutation in $S_n(\mathcal{E})$. We call a* site *i* active *if the permutation resulting from the insertion of $n+1$ between $\pi(i-1)$ and $\pi(i)$ is in $S_{n+1}(\mathcal{E})$. A* site *is called* inactive *otherwise. Moreover, a site is* always active *if it is active for any π in $S_n(\mathcal{E})$. In the subsequent figures, a dot will denote a inactive site and a blank an active one.*

Remark Consequently, to prove the inactivity of a site i, we have to find a subsequence of π, containing $n+1$ in position i orderisomorph to some forbidden pattern τ in $\mathcal{E}$. Note that $n+1$ corresponds to the term of greatest ordinality in τ.

Example Set $\tau = 4312$ a forbidden pattern and $\pi = 326415$. The first three sites are inactive as the subsequence 7615 is orderisomorph to 4312. Sites from four to seven are active as no subsequences in the resulting permutations will be forbidden. Note that the last three sites are always active as the value of greatest ordinality in τ is in the first position.

5 Generalized Fine sequences can be characterized by the succession system 3.1

Definition 5.1 *We consider the following generating tree:*

$$\begin{cases} root = 01\ldots(p-1)[P] & \\ w = w'1[T] & \longrightarrow \quad w1[T], w2[3] \\ & \underset{q-1}{\longrightarrow} \quad w'01\ldots(q-1)[q] \\ w \neq w'1[t] & \longrightarrow \quad w1[T], w2[3], \ldots, wt[t+1] \end{cases}$$

with w labelized:

- $[T]$ *if* $w_n = 1$,
- $[t]$ *otherwise.*

In order to prove Theorem 3.2, we have to show that the following points hold true:

(i) exclusivity: considering an object generated by the tree in $F_n^{p,q}$, all of its children belong to $F_m^{p,q}$ with $m > n$. Moreover, the root is in $F_p^{p,q}$,

(ii) completeness: every object in $F_n^{p,q}$ is generated by the generating tree given in Definition 5.1,

(iii) unicity: no object in $F_n^{p,q}$ appears more than once in the generating tree.

Proof of Theorem 3.2 To prove the first point, we have to consider definitions 2.1 and 5.1. Clearly, all children of an object labeled $[t]$ fulfill condition (ii) of Definition 2.1. The same holds for the objects labeled $[T]$, note that the second rule corresponds to condition (iii) of Definition 2.1. Finally, condition (i), the root belongs to $F_p^{p,q}$. Now, given any generalized Fine sequences ω in $F_n^{p,q}$, we can assign to it an unique labelized father with the following mapping:

- $\omega_1 \ldots \omega_{n-1}$ $[T]$ if $\omega_{n-1} = 1$,
- $\omega_1 \ldots \omega_{n-q}\, 1$ $[T]$ if $\omega_{n-q+1} \ldots \omega_n = 01\ldots(q-1)$,
- $\omega_1 \ldots \omega_{n-1}$ $[\omega_n]$ otherwise.

This give a one-to-one correspondence between paths in the generating tree and generalized Fine sequences and thus proves the second and third points. Q.E.D.

6 Derangements avoiding 321 are characterized by the succession system 3.1 with $p = 1$ and $q = 3$

Let's say $D_n(321)$ as 321-avoiding derangements.

Definition 6.1 *We consider the following generating tree:*

$$\begin{cases} root = 21[2] & \\ \pi(1)\ldots\pi(n-2)(n)\pi(n)[T] & \rightarrow \pi(1)\ldots\pi(n-2)(n)(n+1)\pi(n)[T] \\ & \rightarrow \pi(1)\ldots\pi(n-2)\pi(n)(n+1)(n)[2] \\ & \rightarrow \pi(1)\ldots\pi(n-2)(n+1)\pi(n)(n)[3] \\ \pi(1)\ldots(n)\ldots\pi(n)[t] & \rightarrow \pi(1)\ldots(n)\ldots(n+1)\pi(n)[T] \\ & \rightarrow \pi(1)\ldots(n)\ldots(n+1)\ldots\pi(n-1)\pi(n)[n+2-\pi^{-1}(n+1)] \\ & \quad \text{with } \pi^{-1}(n)+1 < \pi^{-1}(n+1) \leq n-1 \\ & \rightarrow \pi(1)\ldots(n+1)\ldots\pi(n)(n)[t+1] \end{cases}$$

with π labelized:

- $[T]$ *if* $\pi(n-1) = n$, *and* $\pi(n) \neq n-1$,
- $[t]$ *else, with* $t = n + 1 - \pi^{-1}(n)$.

Lemma 6.2 *The generating tree 6.1, just given, generate 321-avoiding derangements.*

Proof Set π in $D_n(321)$:

- if π is labeled $[T]$ the succession system generate three new derangements:
  - $n+1$ is inserted in position n, we clearly obtain a derangement and, by Definition 6.1, it has label $[T]$ and still avoid ${}_f321$,
  - $\pi(n)$ substitute n in π, $n+1$ substitute $\pi(n)$ and n is placed at the end of the derangement. As $\pi(n) \neq n-1$, no fixed point can appear. Now, the new derangement has label [2] as $\pi^{-1}(n+1) = n$ and $\pi(n+1) = n$. Moreover it clearly avoid ${}_f321$,
  - $n+1$ substitute n in π and n is placed at the end of the derangement. Again, no fixed point can appear. So, from Definition 6.1 the new derangement has label [3] and it clearly avoid ${}_f321$.
- if π is labeled $[t]$, with $\pi(i) = n$, the succession system generate t new derangements:
  - $n+1$ is inserted in position n, since $\pi(n) \neq n$, no fixed point can appear and from Definition 6.1 the new derangement will have label $[T]$,
  - $n+1$ is inserted in position k in $[i+1\ldots n-1]$. First, take note that $\pi(i+1)\ldots\pi(n)$ is a strictly increasing subsequence. As a consequence, the derangement obtained by the insertion of $n+1$ in position k is still ${}_f321$-avoiding. Now, if a fixed point, $k+1 \leq \pi(j) \leq n$ appear, then the subsequence $\pi(j)\ldots\pi(n)$ should not have been strictly increasing in π since $\pi(n) \neq n$. It follows a contradiction and consequently the new permutation is still a derangement. Definition 6.1 account for the label of those new derangements,
  - $n+1$ substitute n in π and n is placed at the end of the derangement. As above, no fixed point can appear and the new derangement is still ${}_f321$-avoiding. Since $n+1$ is in position i in the new derangement, it has label $[t+1]$.

This point implies the exclusivity. Now, set $i = \pi^{-1}(n)$, we can assign a unique father to a given π in $D_n(321)$ with the following mapping:

If π has label:

- $[T]$ and $\begin{cases} \pi(n-2) = n-1 \mapsto \pi(1)\ldots\pi(n-2)\pi(n)[T] \\ \pi(n-2) \neq n-1 \mapsto \pi(1)\ldots\pi(n-2)\pi(n)[n-\pi^{-1}(n-1)] \end{cases}$

- $[2] \mapsto \pi(1)\ldots\pi(n-3)(n-1)\pi(n-2)[T]$

- $[3]$ and $\begin{cases} \pi(n-1) \neq n-2,\ \pi(n) = n-1 & \mapsto \pi(1)\ldots\pi(n-3)(n-1)\pi(n-1)[T] \\ \pi(n-1)\pi(n) = (n-2)(n-1) & \mapsto \pi(1)\ldots\pi(i-1)(n-1)\pi(i+1)\ldots\pi(n-1)[2] \end{cases}$

- $[t \geq 3]$ and $\pi(n) \neq n-1 \mapsto \pi(1)\ldots\pi(i-1)\pi(i+1)\ldots\pi(n)[n-\pi(n-1)]$

- $[t \geq 4]$ and $\pi(1)\ldots(n)\ldots(n-1) \mapsto \pi(1)\ldots\pi(i-1)(n-1)\pi(i+1)\ldots\pi(n-1)[t-1]$

This mapping, along with the generating tree give us a one-to-one correspondence between a path in the generating tree and a derangement in $D_n(321)$. Since the root is trivialy in $D_1(321)$ the unicity and completness are achevied. Q.E.D.

Proof of Theorem 3.3 First, recall that the succession system 3.1, with $p = 0$ and $q = 2$, is the following:

$$\begin{cases} root = [2] & & \\ [T] & \rightarrow & [T],[2],[3] \\ [t] & \rightarrow & [T],[3],\ldots,[t+1] \end{cases}$$

As one can see, this succession system is isomorph to the succession system charaterizing the generating tree 6.1. This remark and Lemma 6.2 finish the proof and give us a bijection between non singular similarity relations and 321-avoiding derangements. Q.E.D.

7 Nonsingular similarity relations are in a one-to-one correspondence with three sets of forbidden patterns

Lemma 7.1 *The generating trees of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ can all be characterized by the succession system given by Definition 3.1, with $p = 0$ and $q = 2$.*

Proof of Theorem 3.4 The proof follows directly from the above lemma. In order to prove the latter, we have to define generating trees associated with $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ whose succession systems are isomorph with the one given in Definition 3.1, with $p = 0$ and $q = 2$. This is done hereafter. Q.E.D.

7.1 Generating tree of $S_n(\mathcal{F}_1)$

Definition 7.2 *We consider the following succession system and labeling:*

$$\begin{cases} root = [2] & & \\ [T] & \rightarrow & [3],[T],[2] \\ [t] & \rightarrow & [t+1],[T],[3],\ldots,[t] \end{cases}$$

with t an integer.
Given π in $S_n(\mathcal{F}_1)$, π as label:

- $[T]$ *if $\pi(2) = n$,*

- $[t]$ *else, with t the number of active sites of π.*

Property 7.3 *Given π in $S_n(\mathcal{F}_1)$:*

(i) take an active site, say k, if $k+1$ is inactive, then all sites from the first to k are active and all others are inactive,

(ii) if π has label $[T]$, then only the first three sites are active,

(iii) if π has label $[t]$ and $n+1$ is inserted in an active site k greater than two, then, in the resulting permutation, site k is active and $k+1$ inactive.

Proof

(i) results directly from the structure of the forbidden subsequences in $\mathcal{F}_1$,

(ii) set π with label $[T]$ and consider the following permutation, $\pi(1)(n)\pi(3)(n+1)\ldots\pi(n)$, obtained by the insertion $n+1$ in its fourth site. Clearly, this permutation is forbidden as $\pi(1)(n)\pi(3)(n+1)$ will either be orderisomorph to ${}_f2314$ or ${}_f1324$. Now, the third site is active as neither 1342 nor 2341 are in $\mathcal{F}_1$. Applying Property 7.3 (i) finished the proof,

(iii) set π with label $[t]$. The insertion of $n+2$ in position $k+1$ will give the subsequence $\pi(1)\pi(2)(n+1)(n+2)$ orderisomorph to either ${}_f2134$ or ${}_f1234$. Now, site k remains active as if not, then some subsequence $a\,b\,(n+2)(n+1)$, with $a<b$, must be orderisomorph to ${}_f1243$. Hence a contradiction as subsequence $a\,b\,(n+1)$ along with n will have been orderisomorph either to ${}_f3124$, ${}_f1324$, ${}_f1234$ or ${}_f1243$. And consequently k shouldn't have been an active site in π.

Q.E.D.

Now, we prove that given a labelized permutation, its children will have the labels provided by the generating tree.

Proof Set π in $S_n(\mathcal{F}_1)$:

- if π is labeled $[T]$. As proven in Property 7.3 (ii), π has three active sites. Now, if $n+1$ is inserted in:
  - the first site: the new permutation has label [3]. Indeed, insertion of $n+2$ in the fourth site will result in the subsequence $(n+1)\pi(1)(n)(n+2)$ being orderisomorph to ${}_f3124$. However, site three is active as neither 3142 nor 3241 are forbidden patterns,
  - the second site: from Definition 7.2, the new permutation has label $[T]$,
  - the third site: the new permutation has label [2]. Indeed, the first and second sites are active and the third is inactive as $\pi(1)(n)(n+2)(n+1)$ is orderisomorph to ${}_f1243$.
- if π is labeled $[t]$ it has t active sites. Now, if $n+1$ is inserted in:
  - the first site: the new permutation has label $[t+1]$ since all sites of π will remain active at which we add the new first site,
  - the second site: the new permutation has label $[T]$,
  - site k in $[3,\ldots,t]$: using Property 7.3 (iii) the new permutation will have label $[k]$.

Q.E.D.

7.2 Generating tree of $S_n(\mathcal{F}_2)$

Definition 7.4 *We consider the following succession system and labeling:*

$$\begin{cases} root = [2] & \\ [T] & \rightarrow \quad [T],[3],[2] \\ [t] & \rightarrow \quad [T],[3],\ldots,[t+1] \end{cases}$$

with t an integer.
Given π in $S_n(\mathcal{F}_2)$, π has label:

- *$[T]$ if $\pi(1) = n$,*
- *$[t]$ else, with t the number of active sites of π.*

Property 7.5 *Given π in $S_n(\mathcal{F}_2)$:*

(i) *take an active site, say k ; if $k+1$ is inactive, then all sites from the first to k are active and all others are inactive,*

(ii) *if π has label $[T]$; then only the first three sites are active,*

(iii) *if π has label $[t]$ and $n+1$ is inserted in an active site k greater than two ; then, in the resulting permutation, site $k+1$ is active and $k+2$ is inactive.*

Proof

(i) results directly from the structure of the forbidden subsequences in $\mathcal{F}_2$,

(ii) set π with label $[T]$ and consider the following permutation: $(n)\pi(2)\pi(3)(n+1)\ldots\pi(n)$, obtained by the insertion $n+1$ in its fourth site. Clearly, this permutation is forbidden as $(n)\pi(2)\pi(3)(n+1)$ will either be orderisomorph to ${}_f3124$ or ${}_f3214$. Now, the third site is active as no patterns in $\mathcal{F}_2$ are orderisomorph to 1342 or 2341. Property 7.5 (i), finishes the proof,

(iii) set π with label $[t]$. The insertion of $n+2$ in site $k+2$ will give the subsequence $\pi(1)(n+1)\pi(k)(n+2)$ orderisomorph to either ${}_f1324$ or ${}_f2314$. Now, site $k+1$ is active as if not, then some subsequence $a\,b\,(n+1)(n+2)$, with $a > b$, must be orderisomorph to ${}_f2134$. Hence a contradiction appears, as subsequence $a\,b\,(n+1)$ along with n will have been orderisomorph either to ${}_f2143$, ${}_f2134$, ${}_f2314$ or ${}_f3214$ and consequently k should not have been an active site in π.

Q.E.D.

Now, we prove that given a labelized permutation, its children will have the labels provided by the generating tree.

Proof Set π in $S_n(\mathcal{F}_2)$:

- if π is labeled $[T]$. As proven in Property 7.5 (ii), π has three active sites. Now, if $n+1$ is inserted in:
  - the first site: the new permutation has label $[T]$,
  - the second site: then the new permutation has label [3]. Indeed, the fourth site is inactive as the subsequence $(n)(n+1)\pi(2)(n+2)$ will be orderisomorph to ${}_f2314$. Moreover, the third site is active as pattern 2341 is allowed,

– the third site: the new permutation has label [2]. Indeed, the first and second sites are active and the third is inactive as $(n)\pi(2)(n+2)(n+1)$ is orderisomorph to ${}_f2143$.

- if π is labeled $[t]$ it as t active sites. Now, if $n+1$ is inserted in:
  – the first site: the new permutation as label $[T]$,
  – site k in $[2,\dots,t]$. Using Property 7.5 (*iii*) we see that the new permutation will have label $[k+1]$.

Q.E.D.

7.3 Generating tree of $S_n(\mathcal{F}_3)$

Definition 7.6 *We consider the following succession system and labeling:*

$$\begin{cases} root = [2] & \\ [T] & \rightarrow \quad [T],[2],[3] \\ [t] & \rightarrow \quad [T],[3],\dots,[t+1] \end{cases}$$

with t an integer.
Given π in $S_n(\mathcal{F}_3)$, π has label:

- $[T]$ *if* $\pi(1)=n$,
- $[t]$ *else, with t the number of active sites of π.*

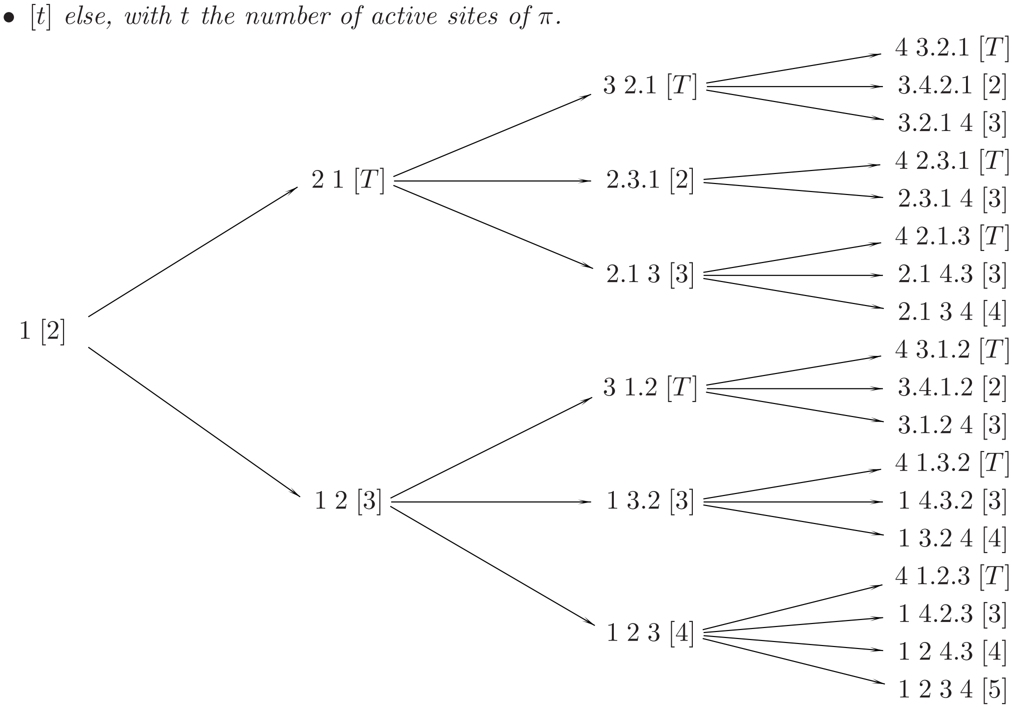

Figure 4: generating tree of $S_n(\mathcal{F}_3)$.

Property 7.7 *Given π in $S_n(\mathcal{F}_3)$:*

(i) The first and last sites are always active,

(ii) if π has label $[T]$, then only the first, the second and last sites are active,

(iii) if π has label $[t]$ and $n+1$ is inserted in site $1 < k < n+1$, then all sites from 2 to k remain active and all sites from $k+1$ to n are inactive.

Proof

(i) results directly from the structure of the forbidden subsequences in $\mathcal{F}_3$,

(ii) the second site is active as, with $\pi(1) = n$, patterns $_f2413$ and $_f2431$ cannot appear. Moreover, all sites from three to n are inactive as subsequence $(n)\pi(2)(n+1)\pi(n)$ will be either orderisomorph to $_f3142$ or $_f3241$,

(iii) consider the following permutation, $\pi(1)\ldots(n+1)\ldots\pi(n)$, obtained after insertion of $n+1$ in π in position k. All sites from $k+1$ and n are inactive as the subsequence $\pi(1)(n+1)(n+2)\pi(n)$ is orderisomorph either to patterns $_f1342$ or $_f2341$. Now, if a site l before k becomes inactive, it must contain a subsequence orderisomorph to patterns $_f2413$ or $_f2431$, with $n+1$ and $n+2$ respectively in position k and l. But then, the subsequence 241 along with n will have been orderisomorph to one of the following patterns: $_f3241$, $_f2341$, $_f2431$ or $_f2413$. Hence a contradiction. So l remains active.

Q.E.D.

Now, we prove that given a labelized permutation, its children will have the labels provided by the generating tree.

Proof Set π in $S_n(\mathcal{F}_3)$:

- if π is labeled $[T]$. As proven in Property 7.7 (ii), π has three active sites. Now, if $n+1$ is inserted in:
  - the first site: the new permutation has label $[T]$,
  - the second site: the new permutation has label $[2]$. Indeed, the first and last sites are always active, as stated in Property 7.7 (i), moreover, site two is inactive as $(n)(n+2)(n+1)\pi(n)$ is orderisomorph to $_f2431$. Finally, all sites from three to n are inactive as $(n)(n+1)(n+2)\pi(n)$ is orderisomorph to $_f2341$.
- if π is labeled $[t]$, it has at least two active sites. Now, if $n+1$ is inserted in:
  - the first site: the new permutation has label $[T]$,
  - the k^{th} active site, with $k > 1$, then Properties 7.7 (ii) and (iii) imply that the new permutation has label $[k+1]$.

Q.E.D.

8 Generalized Fine sequences congruous three modulo one are in one-to-one correspondence with five sets of forbidden patterns

We begin by stating three lemmas which are proven in the next sections.

Lemma 8.1 *Generalized Fine sequences congruous three modulo one, namely $F_n^{1,3}$, are in a one-to-one correspondence with permutations with forbidden patterns $S_{n-1}(\mathcal{H}_1)$.*

Lemma 8.2 *Permutations with forbidden patterns $S_n(\mathcal{E})$, with $\mathcal{E}$ taking value in $\{\mathcal{H}_1^\star$, $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_4\}$ can be characterized by the same succession system, thus giving a bijective correspondence among them.*

Lemma 8.3 *Permutations with forbidden patterns $S_n(\mathcal{H}_3^{-1\,c})$ and $S_n(\mathcal{H}_5)$ can be characterized by the same succession system, thus giving a bijective correspondence between them.*

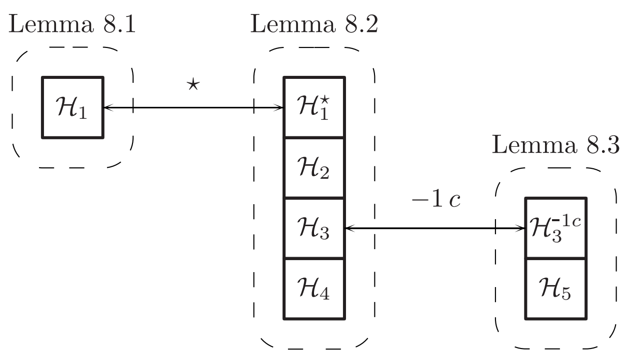

Figure 5: an overview of the correspondences between the five sets.

Proof of Theorem 3.5 Lemma 8.1 establish a bijection between $F_n^{1,3}$ and $S_{n-1}(\mathcal{H}_1)$. Now, the operation of *mirror* —denoted by $\star : \forall i \in n,\ \pi^\star(i) = \pi(n-i+1)$— on $\mathcal{H}_1$ gives also a one-to-one correspondence between $S_n(\mathcal{H}_1)$ and $S_n(\mathcal{H}_1^\star)$. It follows that all permutations with forbidden patterns given in Lemma 8.2 are, through $S_{n-1}(\mathcal{H}_1^\star)$ and $S_{n-1}(\mathcal{H}_1)$, in a one-to-one correspondence with $F_n^{1,3}$. Finally, the last set, $\mathcal{H}_5$, is obtained with the operations of *inverse* —denoted by -1— and *complementary* —denoted by $c : \forall i \in n,\ \pi^c(i) - \pi(i) = n+1$— on $\mathcal{H}_3$ along with Lemma 8.3. Q.E.D.

8.1 A bijection between $S_{n-1}(\mathcal{H}_1)$ and $F_n^{1,3}$

8.1.1 Generating tree of $S_n(\mathcal{H}_1)$

Definition 8.4 *We consider the following generating tree:*

$$\begin{cases} root = 1[T] & & \\ (n)\pi(2)\dots\pi(n)[T] & \rightarrow & (n+1)(n)\pi(2)\dots\pi(n)[T], (n)(n+1)\pi(2)\dots\pi(n)[3] \\ & \underset{2}{\rightarrow} & (n+1)(n)(n+2)\pi(2)\dots\pi(n)[3] \\ \pi(1)\dots\pi(n)[t] & \rightarrow & (n+1)\pi(1)\dots\pi(n)[T] \\ & \rightarrow & \pi(1)(n+1)\dots\pi(n)[3], \dots, \pi(1)\dots\pi(\pi^{-1}(n))(n+1)\dots\pi(n)[t+1] \end{cases}$$

with π labelized:

- $[T]$ *if* $\pi(1) = n$,
- $[t]$ *else, with* t *the number of active sites of* π.

Property 8.5 *Given* π *in* $S_n(\mathcal{H}_1)$:

(i) *if* π *has label* $[T]$ *and* $\pi(2) \neq n-1$ *then only the first two sites are active. Moreover, if* $\pi(2) = n-1$ *then only the first three sites are active,*

(ii) *if* π *has label* $[t]$. *Set* i *such that* $\pi(i) = n$. *Then all sites from* 1 *to* $i+1$ *are active except the second one if* $\pi(1)\pi(2)\pi(3)$ *is orderisomorph to* 213. *Moreover,* $t > 2$.

Proof

(i) results directly from the structure of the forbidden subsequences in $\mathcal{H}_1$,

(ii) the first site is trivialy active as no pattern in $\mathcal{H}_1$ begins with 4. Now, if a site j in $[3 \dots i]$ is inactive, then some subsequence $\pi(k)(n+1)\pi(l)(n)$, with $1 < k < l < i+1$ must be orderisomorph to the forbidden pattern ${}_f2413$. Then, either $\pi(1) > \pi(k)$ and $\pi(1)\pi(k)\pi(l)\pi(i)$ is orderisomorph to ${}_f3214$, either $\pi(1) < \pi(k)$ and the same subsequence is now orderisomorph to ${}_f2314$. Hence a contradiction since those two patterns are forbidden. Site $i+1$ is active as no patterns in $\mathcal{H}_1$ end with 34. All sites j in $[i+2 \dots n+1]$ are inactive as patterns ${}_f1324$ and ${}_f2314$ are forbidden. It follows, since i is greater than one, that the first, third and fourth sites are always active and, consequently, $t > 2$. Finally, as ${}_f2413$ is a forbidden pattern, the second site is inactive if $\pi(1)\pi(2)\pi(3)$ is orderisomorph to 213.

Q.E.D.

Lemma 8.6 *The generating tree 8.4, just given, generate* $S_n(\mathcal{H}_1)$.

Proof First, we prove that given a labelized permutation, its children will have the labels given by the generating tree, this point implies exclusivity.

Set π in $S_n(\mathcal{H}_1)$:

- if π is labeled $[T]$. As proven in Property 8.5 (i), π has always two active sites. Three succession rules apply:
  - $n+1$ is inserted in the first site: from Definition 8.4, the resulting permutation has label $[T]$,
  - $n+1$ is inserted in the second site: the resulting permutation corresponds to label [3] as all sites i in $[4 \dots n+2]$ are inactive since $(n)(n+1)\pi(2)(n+2)$ is orderisomorph to ${}_f2314$ and $(n)(n+1)\pi(2)$ isn't orderisomorph to 213 (Property 8.5 (ii)).
  - $n+1$ is inserted in the first site and $n+2$ in the second one: from Property 8.5 (ii), the resulting permutation, $(n+1)(n)(n+2)\pi(2)\dots\pi(n)$, has label [3].
- if π is labeled $[t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the first site: from Definition 8.4, the resulting permutation has label $[T]$,
  - site k in $[3 \dots t+1]$: using Property 8.5 (ii), we see that the resulting permutations will have labels ranging from [3] to $[t+1]$.

Now, we can assign a unique father to a given π in $S_n(\mathcal{H}_1)$ with the following mapping:

$$\begin{cases} \pi(2)\pi(4)\dots\pi(n)[T] \text{ if } \pi(1) = n-1 \text{ and } \pi(3) = n \\ \pi(1)\dots\pi(\pi^{-1}(n)-1)\pi(\pi^{-1}(n)+1)\dots\pi(n) \text{ else, with the label given by Definition 8.4.} \end{cases}$$

This point give us both unicity and compleness. Q.E.D.

Proof of Lemma 8.1 First, remark that the succession system associated to the generating tree 8.4 is isomorph to the succession system 3.1 associated with $F_n^{1,3}$ with one application of the succession rules on the root. The last point account for the cardinality's shift between the objects. Finally, Lemma 8.6 finish the proof. Q.E.D.

8.2 Permutations avoiding $\mathcal{H}_1^\star$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ are all characterized by the same succession system

Proof of Lemma 8.2 All the succession systems associated with the generating trees defined hereafter are equivalent up to an isomorphism. As those generating trees produces $S_n(\mathcal{H}_1^\star)$, $S_n(\mathcal{H}_2)$, $S_n(\mathcal{H}_3)$ and $S_n(\mathcal{H}_4)$, they are all characterized by the same succession system. Q.E.D.

8.2.1 Generating tree of $S_n(\mathcal{H}_1^\star)$

Remember that $\mathcal{H}_1^\star$ is the *mirror* of $\mathcal{H}_1$. So we have $\mathcal{H}_1^\star = \{2341, 2413, 2431, 4231, 3142, 3241\}$.

Definition 8.7 *We consider the following succession system and labeling:*

$$\begin{cases} root = [A,2] & \\ [A,t] & \rightarrow \quad [B,3],\ldots,[B,t+1],[A,t+1] \\ [B,t] & \rightarrow \quad [B,3],\ldots,[B,t],[A,1],[A,t] \end{cases}$$

Given π in $S_n(\mathcal{H}_1^\star)$ and i,j such that $\pi(i)=n-1$ and $\pi(j)=n$, π has label:

- *$[A,t]$ if $i<j$ and π has t active sites,*
- *$[B,t]$ if $j<i$ and π has t active sites.*

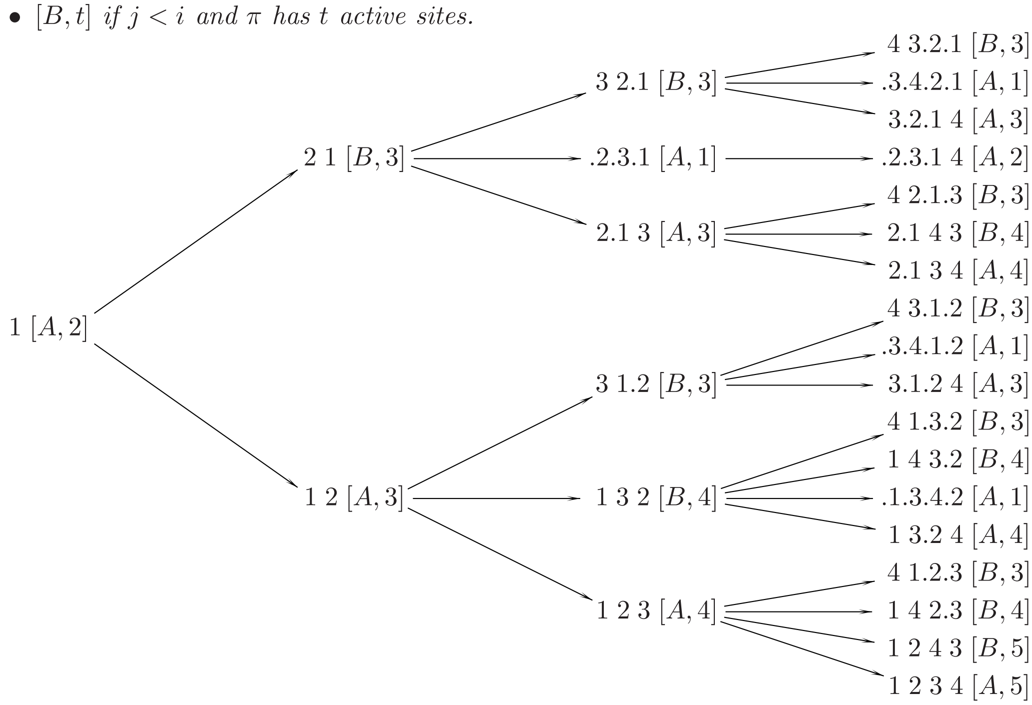

Figure 6: generating tree of $S_n(\mathcal{H}_1^\star)$.

Property 8.8 *Set π in $S_n(\mathcal{H}_1^\star)$:*

(i) if π has label $[A,t]$ and $j < n$, then the only active site is $n+1$,

(ii) if π has label $[A,t]$ and $j = n$, then sites n, $n+1$ are always active, all sites from $j+1$ to $n-1$ are inactive. Moreover if $n+1$ is inserted in position $n+1$, all active sites in π remain active in the resulting permutation,

(iii) if π has label $[B,t]$. Then all sites from $j+2$ to n are inactive. Sites j, $j+1$ and $n+1$ are always active. Moreover if $n+1$ is inserted in a position k less than $j+1$, all active sites belonging to $[1\ldots k]$ in π remain active in the resulting permutation.

Proof

(i) By hypothesis, $\pi = \pi(1)\ldots\pi(i)\ldots\pi(j)\ldots\pi(n)$. Now, insertion of $n+1$ in sites $[1\ldots i]$, $[i+1\ldots j]$ and $[j+1\ldots n]$ will result in subsequences being respectively orderisomorph to the forbidden patterns ${}_f4231$, ${}_f2431$ and ${}_f2341$. Therefore, as no forbidden patterns in $\mathcal{H}_1^\star$ end with 4, site $n+1$ is active,

(ii) sites n, $n+1$ are always active as forbidden patterns end neither with 4 nor 43. Moreover the resulting permutations will keep their active sites as none of patterns 1423, 4123, 1432 or 4132 are forbidden. Now, consider site k in $[j+1\ldots n-1]$, the permutations resulting from the insertion of $n+1$ in site k will be forbidden as the subsequence $(n-1)(n+1)\pi(n-1)(n)$ is orderisomorph to ${}_f2413$, a forbidden pattern,

(iii) consider $\pi = \pi(1)\ldots(n)\ldots(n-1)\ldots\pi(n)$. As patterns ${}_f2413$ and ${}_f4231$ must be avoided, it follows that $\pi(1)\ldots\pi(j-1) < \pi(j+1)\ldots\pi(i-1) < \pi(i+1)\ldots\pi(n)$. Consequently, all sites k in $[j+2\ldots n]$ are inactive as $(n)\pi(k-1)(n+1)\pi(n)$ is orderisomorph either to ${}_f3142$ or ${}_f3241$. Site $n+1$ is active as no forbidden patterns end with 4. Now, sites j and $j+1$ are active as neither pattern ${}_f2341$ nor ${}_f2431$ can appear. Next, if an active site becomes inactive after insertion of $n+1$, then some subsequence containing n and $n+1$ must be orderisomorph to the forbidden patterns 4231, 2431 or 2413. The first two cases are dismissed as no value before n is greater than after (see above). Next consider subsequence $\pi(1)(n+2)\pi(2)(n+1)$ orderisomorph to ${}_f2413$. As $\pi(1)(n+1)\pi(2)(n)$ is orderisomorph to the forbidden pattern ${}_f2413$, this site would not have been active in π, hence a contradiction.

Q.E.D.

Now, we prove that given a labelized permutation, its children will have the labels provided by the generating tree.

Proof Set π in $S_n(\mathcal{H}_1^\star)$:

- if π is labeled $[A,t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the last active site: if $j < n$, as stated in Property 8.8 (i), $t = 1$. The resulting permutation, $\pi(1)\ldots(n-1)\ldots(n)\ldots\pi(n)(n+1)$, has label $[A,2]$. Indeed, the insertion of $n+2$ in sites, $[1\ldots i]$, $[i+1\ldots j]$ and $[j+1\ldots n]$, will, respectively, create subsequences orderisomorph to ${}_f4231$, ${}_f2431$ and ${}_f2413$. Now, if $j = n$, the resulting permutation, as a direct consequence of the structure of the forbidden patterns, has label $[A,t+1]$,
  - active sites from the first to the last but one, which imply $j = n$. From Property 8.8 (ii), and Definition 8.7, all permutations resulting from insertion of $n+1$ in all but the last active sites will be labelized $[B,k]$, with k taking values in $[3\ldots t+1]$. Indeed, all sites before the insertion are conserved and both sites, $j+1$ and $n+1$, are always active (Property 8.8 (iii)).

- if π is labeled $[B, t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the first $(t-2)^{th}$ sites: again, we apply Property 8.8 (*iii*), and, consequently, those permutations will be labelized $[B, k]$, with k taking values in $[3 \ldots t+1]$,
  - site $j+1$: from Property 8.8 (*i*) the resulting permutation will have label $[A, 1]$,
  - site $n+1$: the resulting permutation will have label $[A, t]$, since, from Property 8.8 (*iii*), the $t-2$ active sites before n will be conserved and the two last sites are always active (Property 8.8 (*ii*)).

Q.E.D.

8.2.2 Generating tree of $S_n(\mathcal{H}_2)$

Definition 8.9 *We consider the following succession system and labeling:*

$$\begin{cases} root = [A,2] \\ [A,t] & \rightarrow \quad [A,t+1],[B,3],\ldots,[B,t+1] \\ [B,t] & \rightarrow \quad [A,t],[B,3],\ldots,[B,t],[A,1] \end{cases}$$

Given π in $S_n(\mathcal{H}_2)$ and i, j such that $\pi(i) = n-1$ and $\pi(j) = n$, π has label:

- $[A, t]$ *if either $j = 1$ or $1 < i < j$ and π has t active sites,*
- $[B, t]$ *if $1 < j < i$ and π has t active sites.*

Property 8.10 *Set π in $S_n(\mathcal{H}_2)$:*

(i) if π has t active sites then, all sites from $t+1$ to $n+1$ are inactive and consequently, all sites in $[1 \ldots t]$ are active. Moreover, if π has label $[A, t]$ and $1 < i < j$ then only the first site is active,

(ii) π has label $[B, t]$ if and only if $\pi(t-1) = n$.

Proof

(i) if a site k is inactive because some subsequence $a\,(n+1)\,b\,c$ is orderisomorph to the forbidden pattern ${}_f1423$ then all sites from $k+1$ to $n+1$ are inactive as ${}_f1243$ and ${}_f1234$ are also forbidden. The same holds true if some subsequence $a\,b\,(n+1)\,c$ is orderisomorph to the forbidden pattern ${}_f1243$,

(ii) site $j+2$ is inactive as permutation $\pi(1)\pi(j)\pi(j+1)(n+1)$ is orderisomorph to the forbidden pattern 1324. Now, site $j+1$ must be active. If not, some subsequence, $a\,b\,c\,(n+1)$, $a\,b\,(n+1)\,c$ or $a\,(n+1)\,b\,c$ of the permutation obtained by insertion of $n+1$ in position $j+1$, shall be orderisomorph to, at least, one of the forbidden patterns. Then, remembering that $\pi(i) = n-1$ and $\pi(j) = n$, $a\,b\,c\,\pi(j)$, $a\,b\,\pi(j)\pi(i)$, $a\,b\,\pi(j)\,c$ or $a\,\pi(j)\,b\,c$ will have been, also, orderisomorph to the same patterns. As a consequence π was not in $S_n(\mathcal{H}_2)$, hence a contradiction. Next, using Property 8.10 (*i*), all sites from 1 to $j+1$ are active. Finally, Definition 8.9 implies $\pi(t-1) = n$.

Q.E.D.

Now, we prove that given a labelized permutation, its children will have the labels provided by the generating tree.

Proof Set π in $S_n(\mathcal{H}_2)$:

- if π is labeled $[A, t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the first active site $j \neq 1$: then, only the first site is active as any other site will result in subsequences, $\pi(1)(n+1)\pi(i)\pi(j)$, $\pi(1)\pi(i)(n+1)\pi(j)$ and $\pi(1)\pi(i)\pi(j)(n+1)$ being all, respectively orderisomorph to ${}_f1423$, ${}_f1243$ and ${}_f1234$. The resulting permutation has label $[A, 2]$. Indeed, the second site is active as forbidden patterns begin with neither 342 nor 341 and the third site is inactive as $\pi(1)(n+2)\pi(i)\pi(j)$ is orderisomorph to ${}_f1423$. Now, if $j = 1$, the resulting permutation has label $[A, t+1]$. Indeed, if site t of π becomes inactive then some subsequence $(n+1)\,a\,b\,(n+2)$, with $n+2$ in position $t+1$, must be orderisomorph to ${}_f3124$. But then $\pi(j)\,a\,b\,(n+1)$, with $n+1$ in position t, will also, have been orderisomorph to the same pattern. Hence a contradiction. It follows that all active sites of π remain active, to which, we add the new first site,
  - the second to last active sites: the resulting permutations have labels in $[B, 3] \dots [B, t]$ as a direct consequence of Definition 8.9 and Property 8.10 (*iii*).
- if π is labeled $[B, t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the first active site: the resulting permutation has label $[A, t]$. Indeed, site t becomes inactive as $(n+1)\pi(1)(n)(n+2)$ is orderisomorph to ${}_f3124$. Site $t-1$ remains active as, in contradiction, if some subsequence $(n+1)\,a\,b\,(n+2)$, with $n+2$ in the former position $t-1$, is orderisomorph to ${}_f3124$ then $a\,b\,\pi(j)\pi(i)$ would have been orderisomorph to ${}_f1243$,
  - the second to the $t-1^th$ active sites: this will result with permutations labelized from $[B, 3]$ to $[B, t]$ as a direct consequence of Definition 8.9 and Property 8.10 (*iii*),
  - the last active site: the resulting permutation, $\pi(1)\dots(n)(n+1)\dots\pi(n)$, has label $[A, 1]$ as seen previously.

Q.E.D.

8.2.3 Generating tree of $S_n(\mathcal{H}_3)$

Definition 8.11 *We consider the following succession system and labeling:*

$$\begin{cases} root = [A, 2] & \\ [A, t] & \rightarrow \quad [B, 3], \dots, [B, t+1], [A, t+1] \\ [B, t] & \rightarrow \quad [B, 3], \dots, [B, t], [A, 1], [A, t] \end{cases}$$

Given π in $S_n(\mathcal{H}_3)$ and i, j such that $\pi(i) = n-1$ and $\pi(j) = n$, π has label:

- $[A, t]$ *if $i > j$ and π has t active sites,*
- $[B, t]$ *if $i < j$ and π has t active sites.*

Property 8.12 *Set π in $S_n(\mathcal{H}_3)$:*

(i) if π has label $[A, t]$ and $j < n$, then the only active site is $n+1$,

(ii) if π has label $[A, t]$ and $j = n$, then sites n, $n+1$ are always active, all sites from $j+1$ to $n-1$ are inactive. Moreover if $n+1$ is inserted in position $n+1$, all active sites in π remain active in the resulting permutation,

(iii) if π has label $[B, t]$. Then all sites from $j+1$ to $n-1$ are inactive. Sites j, n and $n+1$ are always active. Moreover if $n+1$ is inserted in a position k less than $j+1$, all active sites belonging to $[1 \dots k]$ in π remain active in the resulting permutations.

Proof

(i) by hypothesis, $\pi = \pi(1)\dots\pi(i)\dots\pi(j)\dots\pi(n)$. Now, insertion of $n+1$ in sites $[1\dots i]$, $[i+1\dots j]$ and $[j+1\dots n]$ will result in subsequences being respectively orderisomorph to ${}_f4231$, ${}_f2431$ and ${}_f2341$. Therefore, as no forbidden patterns in $\mathcal{H}_3$ end with 4, site $n+1$ is active,

(ii) sites n, $n+1$ are always active as forbidden patterns end neither with 4 nor 43. Moreover the resulting permutations will keep its active sites as patterns 1423, 4123, 1432, 4132 are not forbidden. Now, consider site k in $[j+1\dots n-1]$, the permutation resulting from the insertion of $n+1$ in site k will be forbidden as the subsequence $(n-1)(n+1)\pi(n-1)(n)$ is orderisomorph to ${}_f2413$,

(iii) consider $\pi = \pi(1)\dots(n)\dots(n-1)\dots\pi(n)$. As patterns ${}_f2413$ and ${}_f4231$ must be avoided it follows that $\pi(1)\dots\pi(j-1) < \pi(j+1)\dots\pi(i-1) < \pi(i+1)\dots\pi(n)$. Consequently, all sites k in $[j+1\dots n-1]$ are inactive as $(n)(n+1)\pi(n-1)\pi(n)$ is orderisomorph either to ${}_f3412$ or ${}_f3421$. Site $n+1$ is active as no forbidden patterns end with 4. Site $n-1$ is active, as supposing it generates a subsequence $b\,c\,(n+1)\,a$ orderisomorph to ${}_f2341$; then the same subsequence with $\pi(j)$ would have been orderisomorph to ${}_f4231$, ${}_f2431$ or ${}_f2341$. Site j is active as pattern ${}_f2431$ cannot appear. Now, if $n+1$ is inserted in position $k < j+1$, then all active sites before k remain active. In contrary, if such a site becomes inactive, then some subsequence, involving $n+1$ and $n+2$ must be orderisomorph to ${}_f2413$, ${}_f2431$ or ${}_f4231$. For the two latter cases we have a direct contradiction as all values, except $n+2$, before $n+1$ are lesser than those after $n+1$. For the former, the site should not have been active as the same subsequence, along with $n+1$ in position k and n would have been orderisomorph to ${}_f2413$.

Q.E.D.

Now, we prove that given a labelized permutation, its children will have the labels provided by the generating tree.

Proof Set π in $S_n(\mathcal{H}_3)$:

- if π is labeled $[A,t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the first $(t-1)^th$ active sites: if $j < n$, as stated in Property 8.12 (ii), $t = 1$. The resulting permutation, $\pi(1)\dots(n-1)\dots(n)\dots\pi(n)(n+1)$, has label $[A,2]$. Indeed, the insertion of $n+2$ in sites, $[1\dots i]$,$[i+1\dots j]$ and $[j+1\dots n]$ will, respectively, create subsequences orderisomorph to forbidden patterns ${}_f4231$, ${}_f2431$ and ${}_f2413$. Now, from Property 8.12 (ii), and Definition 8.11 if $j = n$, all permutations resulting from insertion of $n+1$ in all but the last active sites will be labelized $[B,k]$. With k taking value in $[3\dots t+1]$. Indeed, all sites before the insertion are preserved and sites, $j+1$, $n+1$ are always active (Property 8.12 (iii)),
  - site $n+1$: from Property 8.12 (ii), the resulting permutation has label $[A,t+1]$.
- if π is labeled $[B,t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the first $(t-2)^{th}$ sites: again, we apply Property 8.12 (iii) and consequently, these permutations have label $[B,k]$. With k taking value in $[3\dots t+1]$,
  - site $j+1$: by Property 8.12 (i), its label will be $[A,1]$,
  - site $n+1$: the resulting permutation has label $[A,t]$. Indeed, from Property 8.12 (ii), the $t-2$ active sites before n will be preserved and the last two sites are always active.

Q.E.D.

8.2.4 Generating tree of $S_n(\mathcal{H}_4)$

Definition 8.13 *We consider the following succession system and labeling:*

$$\begin{cases} root = [A,2] \\ [A,t] \quad \rightarrow \quad [B,3],\ldots,[B,t+1],[A,t+1] \\ [B,t] \quad \rightarrow \quad [B,3],\ldots,[B,t],[A,t],[A,1] \end{cases}$$

Given π in $S_n(\mathcal{H}_4)$ and i, j such that $\pi(i) = n-1$ and $\pi(j) = n$, π has label:

- *$[A,t]$ if $i < j$ and π has t active sites,*
- *$[B,t]$ if $i > j$ and π has t active sites.*

Property 8.14 *Set π in $S_n(\mathcal{H}_4)$:*

(i) if π has label $[A,t]$ then $i+1 \leq j \leq i+2$,

(ii) if π has label $[A,t]$ and $j = i+1$. Then all sites from the first to the $j+1^{th}$ are active, all others being inactive,

(iii) if π has label $[A,t]$ and $j = i+2$. Then the only active site is $i+1$,

(iv) if π has label $[B,t]$. All sites from 1 to $j+2$ are active. Moreover, all sites from $j+3$ to n are inactive.

Proof

(i) set $j > i+2$, the following permutation: $(n)\pi(i+1)\pi(i+2)(n+1)$ will be either orderisomorph to ${}_f3124$ or ${}_f3214$,

(ii) consider the following permutation, satisfying the hypothesis: $\pi(1)\ldots(n-1)(n)\ldots\pi(n)$. All sites from $j+2$ to $n+1$ are inactive as the subsequence $\pi(i)\pi(j)\pi(j+1)(n+1)$ is orderisomorph to the forbidden pattern ${}_f2314$. Now, all sites from the first to the $j+1^{th}$ are active. As before, we prove this by contradiction. If some subsequence $a\,b\,c\,(n+1)$ is orderisomorph to forbidden patterns who end with 4, then, $a\,b\,c\,\pi(j)$ would have been orderisomorph to the same patterns. Now, if the subsequence $a\,b\,(n+1)\,c$ is orderisomorph to ${}_f2143$. Then, either $a\,b\,\pi(i)\pi(j)$ or $a\,b\,\pi(i)c$ would have been too. The remaining case is if a subsequence $(n+1)\,a\,b\,c$ is orderisomorph to ${}_f4213$, then either, $a\,b\,c\,\pi(i)$ or $\pi(i)\,a\,b\,c$ would have been orderisomorph to ${}_f2134$ or ${}_f4213$,

(iii) consider the following permutation, satisfying the hypothesis: $\pi(1)\ldots(n-1)\,\pi(i+1)(n)\ldots\pi(n)$. Insertion of $n+1$ in sites $[1\ldots i]$, $i+2$ and $[j+1\ldots n+1]$ will result with subsequences, respectively orderisomorph to ${}_f4213$, ${}_f2143$ and ${}_f2134$. Consequently all those sites are inactive. Now, insertion of $n+1$ in position $i+1$ cannot result in a subsequence $a\,b\,c\,(n+1)$ orderisomorph to ${}_f2134$, ${}_f2314$, ${}_f3124$ nor ${}_f3214$. Indeed, if such a subsequence exists then $a\,b\,c\,\pi(j)$ would also have been orderisomorph to the same patterns which contradicts the hypothesis. Now, if $a\,b\,(n+1)\,c$ is orderisomorph to ${}_f2143$ then $a\,b\,\pi(i)\pi(j)$ would have been orderisomorph to ${}_f2134$. Finally, if $(n+1)\,a\,b\,c$ is orderisomorph to ${}_f4213$, then $\pi(i)\,a\,b\,c$ would have been too, as c cannot correspond to $\pi(j)$ in the subsequence,

(iv) if $n+1$ is inserted in position $k > j+2$, then the following subsequence: $(n)\pi(j+1)\pi(j+2)(n+1)$ will be orderisomorph to either ${}_f3124$ or ${}_f3214$ since all sites greater than $j+2$ are inactive. If site $j+1$ is inactive, then some subsequence $b\,a\,(n)(n+1)$, with $a < b$, must be orderisomorph to ${}_f2134$. This is contradiction as $b\,a\,(n)(n-1)$ would have been orderisomorph to ${}_f2143$. Now, if a site $1 \leq k \leq j$ is inactive, then some subsequence involving n and $n+1$

must be oderisomorph to either ${}_f2143$ or ${}_f4213$. In both case, we get a contradiction. Finally, if site $i+2$ is inactive, then some subsequence $a\,(n)\,\pi(j+1)(n+1)$ must be orderisomorph to ${}_f2314$. Hence a contradiction since subsequence $a\,\pi(j+1)\pi(j+2)(n-1)$ would have been orderisomorph in π to either ${}_f2134$, ${}_f3124$ or ${}_f3214$.

Q.E.D.

Now, we prove that given a labelized permutation, its children will have the labels provided by the generating tree.

Proof Set π in $S_n(\mathcal{H}_4)$:

- if π is labelized $[A,t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the first to the last but one active site k: from Definition 8.13 along with Property 8.14 (iv),the new permutation has label $[B,k+2]$,
  - the last site, which is the only active site if $t=1$: from Property 8.14 (iv), the new permutation has label $[A,t+1]$.
- if π is labelized $[B,t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the first to $(t-2)^{th}$ active site k: from Property 8.14 (iv), the new permutation has label $[B,k+2]$,
  - the last but one active site: from Property 8.14 (ii), the new permutation has label $[A,t+1]$,
  - the last active site: from Property 8.14 (iii), the new permutation has label $[A,1]$.

Q.E.D.

8.3 Permutations avoiding $\mathcal{H}_3^{-1\,c}$ and $\mathcal{H}_5$ are characterized by the same succession system

Proof of Lemma 8.3 Both succession systems associated with the generating trees defined hereafter are equivalent up to an isomorphism. As those generating trees produces $S_n(\mathcal{H}_3^{-1\,c})$ and $S_n(\mathcal{H}_5)$, both are characterized by the same succession system. Q.E.D.

8.3.1 Generating tree of $S_n(\mathcal{H}_3^{-1\,c})$

Remember that $\mathcal{H}_3^{-1\,c}$ is the *inverse* and *complement* of $\mathcal{H}_3$. So we have
$\mathcal{H}_3^{-1\,c} = \{1324, 2134, 2143, 2314, 2413, 3214\}$.

Definition 8.15 *We consider the following succession system and labeling:*

$$\begin{cases} root = [A,2] & \\ [P] & \rightarrow \quad [A,2] \\ [A,t] & \rightarrow \quad [B,t+1],[A,3],\ldots,[A,t+1] \\ [B,t] & \rightarrow \quad [B,3],[A,3],[P]^{t-2} \end{cases}$$

Given π in $S_n(\mathcal{H}_3^{-1\,c})$ and integers i,j such that $\pi(i)=n-1$ and $\pi(j)=n$, π is labeled:

- $[P]$ *if* $i=1$ *and* $2<j$,
- $[A,t]$ *if either* $2<j<i$ *or* $j=i+1$,
- $[B,t]$ *if* $j=1$.

Property 8.16 *Set π in $S_n(\mathcal{H}_3^{-1\,c})$:*

(i) if a site k is inactive, all sites to the right of k are inactive. Therefore, if π has t active sites, all sites from the first to t are inactive. All others being inactive,

(ii) if $j = 1$ and $i > j + 1$, then all, but the first, sites are inactive. As a consequence, if π has label $[P]$ only its first site is active,

(iii) if π has label $[A, t]$ then site $j + 1$ is active and site $j + 2$ is inactive. Thus, with property (i) above, all sites from the first to the $(j + 1)^{th}$ are active, all others being inactive.

Proof

(i) results directly from the structure of the forbidden subsequences in $\mathcal{H}_3^{-1\,c}$,

(ii) set π with label $[P]$: $(n-1)\pi(2)\ldots(n)\ldots\pi(n)$. Clearly, all sites from the second to the last will yield a subsequence orderisomorph to either: ${}_f2413$, ${}_f2143$ or ${}_f2134$,

(iii) site $j + 2$ is inactive since either $2 < j < i$ and then, subsequence $\pi(1)(n)\pi(j + 1)(n + 1)$ is orderisomorph to ${}_f1324$ or ${}_f2314$, or $j = i+1$ and then subsequence $(n-1)(n)\pi(j+1)(n+1)$ is orderisomorph to ${}_f2314$. Finally, site $j + 1$ is active, as if not then some subsequence $b\,a\,(n)(n + 1)$ must be orderisomorph to ${}_f2134$, from which follows a contradiction as then either subsequences, $b\,a\,(n)(n-1)$ or $b\,a\,(n-1)(n)$ would have been orderisomorph to ${}_f2143$ or ${}_f2134$ respectively.

Q.E.D.

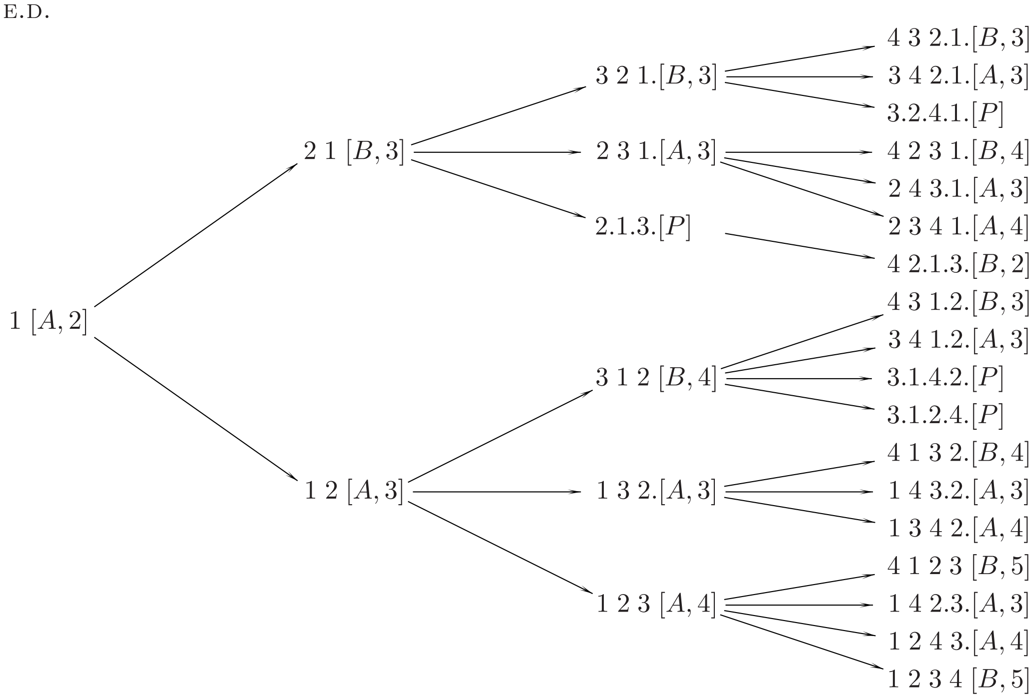

Figure 7: generating tree of $S_n(\mathcal{H}_3^{-1\,c})$.

Now, we prove that given a labelized permutation, its children will have the labels provided by the generating tree.

Proof Set π in $S_n(\mathcal{H}_3^{-1\,c})$:

- if π is labelized $[P]$. We conclude from Definition 8.15 and Properties 8.16 (ii) and (iii), that the solechild has label $[A,2]$,
- if π is labelized $[A,t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the first site: the resulting permutation has label $[B,t+1]$, since all active sites of π remain active. Indeed, all sites before j necessary remain active. Site $j+1$ too, as no subsequence involving $(n+1)(n)(n+2)$ can be orderisomorph to ${}_f1324$ or ${}_f3214$,
  - site k in $[2\dots t]$: from Property 8.16 (iii), the resulting permutation has label $[A,k+1]$.
- if π is labelized $[B,t]$, it has t active sites. Now, if $n+1$ is inserted in:
  - the first site: then the resulting permutation has label $[B,3]$. Indeed, the permutation will be: $(n+1)(n)\pi(2)\dots\pi(n)$. The first three sites are active and the fourth will result with a subsequence, namely, $(n+1)(n)\pi(2)(n+2)$, orderisomorph to ${}_f3214$,
  - the second site: then the resulting permutation has label $[A,3]$,
  - site k in $[3\dots t]$: then, from Property 8.16 (i), the resulting permutation has label $[P]$.

Q.E.D.

8.3.2 Generating tree of $S_n(\mathcal{H}_5)$

Definition 8.17 *We consider the following succession system and labeling:*

$$\begin{cases} root=[A,2] & \\ [P] & \rightarrow \quad [A,2] \\ [A,t] & \rightarrow \quad [A,t+1],[B,t+1],[A,3],\dots,[A,t] \\ [B,t] & \rightarrow \quad [3,A],[3,B],[P]^{t-2} \end{cases}$$

Given π in $S_n(\mathcal{H}_5)$ and integers i,j such that $\pi(i)=n-1$ $\pi(j)=n$, π is labeled:

- $[P]$ *if* $1<i<j$,
- $[A,t]$ *if either,* $j\neq 2$ *and* $1<i<j$ *or,* $i=1$ *and* $j=3$,
- $[B,t]$ *if* $j=2$.

Property 8.18 *Consider π in $S_n(\mathcal{H}_5)$, if site k is inactive, then all sites from k to $n+1$ are inactive. Consequently, if π has t active sites, then all sites from the first to the k^{th} are active.*

Proof results directly from the structure of the forbidden subsequences in $\mathcal{H}_5$, Q.E.D.

Now, we proove that given a labelized permutation, its children will have the labels provided by the generating tree.

Proof Set π in $S_n(\mathcal{H}_5)$:

- if, π has label $[P]$, the first site is active as no forbidden patterns begin with 4. Moreover, the second site is inactive as $\pi(1)(n+1)\dots(n-1)\dots(n)$ is orderisomorph to ${}_f1423$. Next, we consider the only resulting permutation: $(n+1)\pi(1)\dots(n-1)\dots(n)$. The third site is inactive as it corresponds to the former second site. The second site is active as no forbidden patterns begin with 34. Finally, since $(n+1)$ is in first position, the new label is $[A,2]$.
- if π has label $[A,t]$, it has t active sites. Now, if $n+1$ is inserted in:

- the first site: the resulting permutation as label [A,t+1]. Indeed, if site $t+1$ becomes inactive then some subsequence $(n+1)\,a\,b\,(n+2)$ with $(n+2)$ in position $t+1$ and $a<b$ must be orderisomorph to ${}_f3124$. Hence a contradiction as the subsequence formed with $a\,b\,(n+1)$ (with $n+1$ in position t) and n would have been orderisomorph with either ${}_f3124$, ${}_f1324$, ${}_f1234$ or ${}_f1243$. Site $t+2$ is inactive as it corresponds to the former site $t+1$ in π,
- the second site: the resulting permutation has label $[B, t+1]$. As above, if site $t+1$ becomes inactive then some subsequence $a\,(n+1)\,b\,(n+2)$ with $(n+2)$ in position $t+1$ and $a<b$ must be orderisomorph to ${}_f1324$ and from this follows a contradiction,
- site k in $[3\dots t]$: the resulting permutation has label $[A,k]$, as $n+1$ isn't in the first or third place. Moreover site k remains active and site $k+1$ is inactive as patterns ${}_f1234$ and ${}_f2134$ are forbidden.

• if π has label $[B,t]$, it has t active sites. Now, if $n+1$ is inserted in:

- the first site: the resulting permutation, $(n+1)\pi(1)(n)\dots\pi(n)$, has label $[A,3]$, as the third site is active and the fourth inactive ($(n+1)\pi(1)(n)(n+2)$ is orderisomorph to pattern ${}_f3124$),
- the second site: the resulting permutation, $\pi(1)(n+1)(n)\dots\pi(n)$, has label $[A,3]$ since the third site is active and the fourth inactive ($\pi(1)(n+1)(n)(n+2)$ is orderisomorph to pattern ${}_f1324$),
- all other active sites: the resulting permutations have label $[P]$ as one can find $1<i<j$ such that $\pi(i)=n$ and $\pi(j)=n+1$.

Q.E.D.

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