

# On Conjugates for Set Partitions and Integer Compositions

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## Abstract

There is a familiar conjugate for integer partitions: transpose the Ferrers diagram, and a conjugate for integer compositions: transpose a Ferrers-like diagram. Here we propose a conjugate for set partitions and show that it interchanges  $\#$  singletons and  $\#$  adjacencies. Its restriction to noncrossing partitions cropped up in a 1972 paper of Kreweras. We also exhibit an analogous pair of statistics interchanged by the composition conjugate.

**0 The Conjugate of an Integer Partition** A partition of  $n$  is a weakly decreasing list of positive integers, called its parts, whose sum is  $n$ . The Ferrers diagram of a partition  $a_1 \geq a_2 \geq \dots \geq a_k \geq 1$  is the  $k$ -row left-justified array of dots with  $a_i$  dots in the  $i$ -th row. The conjugate, obtained by transposing the Ferrers diagram, is a well known involution on partitions of  $n$  that interchanges the largest part and the number of parts.

**1 A Conjugate for Set Partitions** The partitions of an  $n$ -element set, say  $[n] = \{1, 2, \dots, n\}$ , into nonempty blocks are counted by the Bell numbers, [A000110](#) in OEIS. A *singleton* is a block containing just 1 element and an *adjacency* is an occurrence of two consecutive elements of  $[n]$  in the same block. Consecutive is used here in the cyclic sense so that  $n$  and 1 are also considered to be consecutive (the ordinary sense is considered in [2]). We say  $i$  *initiates* an adjacency if  $i$  and  $i + 1 \bmod n$  are in the same

block and analogously for terminating an adjacency. The number of  $k$ -block partitions of  $[n]$  containing *no* adjacencies is considered in a recent Monthly problem proposed by Donald Knuth [3]. Suppressing the number of blocks, it turns out that #singletons and #adjacencies have the same distribution.

**Theorem 1.** *There is a bijection  $\phi$  on partitions of  $[n]$  that interchanges number of singletons and number of adjacencies.*

To prove this, we need to consider partitions on arbitrary subsets rather than just initial segments of the positive integers. (I thank Robin Chapman [1] for pointing out that my original proof was incorrect.) The notions of adjacency, adjacency initiator and adjacency terminator generalize in the obvious way. We always write a partition with elements increasing within each block and blocks arranged in increasing order of their first (smallest) elements. Thus, for example, with a dash separating blocks, the partition  $\pi = 3\ 5\ 12 - 4\ 8\ 10 - 7$  has support  $\text{supp}(\pi) = \{3, 4, 5, 7, 8, 10, 12\}$  and adjacencies  $(8, 10), (12, 3)$ . In a partition with one-element support, this element is considered to be both an adjacency initiator and an adjacency terminator.

Consider the operation `SeparateIS` on positive-integer-support partitions defined by  $\text{SeparateIS}(\pi) = (\rho, (I, S))$  where  $I$  is the set of adjacency initiators of  $\pi$ ,  $S$  is the set of singleton elements of  $\pi$ , and  $\rho$  is the partition obtained by suppressing the elements of  $I \cup S$  in  $\pi$ . Thus, for  $\pi = 3\ 5\ 12 - 4\ 8\ 10 - 7$ , we have  $I = \{8, 12\}$ ,  $S = \{7\}$  and  $\rho = 3\ 5 - 4\ 10$ . Also, for a one-block partition  $\pi = a_1 a_2 \dots a_k$ ,  $\text{SeparateIS}(\pi) = (\epsilon, (\{a_1, \dots, a_k\}, \emptyset))$  where  $\epsilon$  denotes the empty partition, unless  $k = 1$  in which case it is  $(\epsilon, (\{a_1\}, \{a_1\}))$ . Clearly, for  $\rho$  a partition and  $A, B$  finite sets of positive integers, the pair  $(\rho, (A, B))$  lies in the range of `SeparateIS` iff (i)  $\rho = \epsilon$ ,  $A = B$  and both are singletons, or (ii)  $\text{supp}(\rho)$ ,  $A$ ,  $B$  are disjoint and for no successive pair  $(a, b)$  in  $\text{supp}(\rho) \cup A \cup B$  is  $a \in A$  and  $b \in B$  (the successor of an adjacency initiator cannot be a singleton).

Analogously, define `SeparateST` with  $S$  the set of singleton elements and  $T$  the set of adjacency terminators. Note that the condition for  $(\rho, (A, B))$  to lie in  $\text{range}(\text{SeparateST})$  is precisely the same as for it to lie in  $\text{range}(\text{SeparateIS})$ . Both are injective and so we may define their respective inverses `CombineIS` and `CombineST` and we will make use of the crucial property that these inverses have identical domains. For example, `CombineST` is defined on  $(3\ 10 - 4\ 7 - 12, (\{11\}, \{1, 2\}))$  and yields the partition  $1\ 2\ 12 - 3\ 10 - 4\ 7 - 11$ .

Now we can define the desired bijection. Given a partition  $\pi$  on  $[n]$ , form a sequence  $(\rho_1, (I_1, S_1)), (\rho_2, (I_2, S_2)), \dots, (\rho_k, (I_k, S_k))$  by setting  $(\rho_1, (I_1, S_1)) = \text{SeparateIS}(\pi)$ ,

$(\rho_2, (I_2, S_2)) = \text{SeparateIS}(\rho_1), \dots, (\rho_k, (I_k, S_k)) = \text{SeparateIS}(\rho_{k-1})$  stopping when  $\rho_k$  has no adjacency initiators and no singletons (as must eventually occur). For example, with  $n = 12$  and  $\pi = 1 - 2 - 3 11 12 - 4 7 10 - 5 9 6 8$ , the results are laid out in the following table

$j$	$\rho_j$	$I_j$	$S_j$
1	3 12 - 4 7 10 - 5 9 - 6 8	{11}	{1, 2}
2	3 - 4 7 10 - 5 9 - 6 8	{12}	$\emptyset$
3	4 7 10 - 5 9 - 6 8	$\emptyset$	{3}
4	4 7 - 5 9 - 6 8	{10}	$\emptyset$

and  $\rho_4$  has no adjacency initiators (hence, no adjacencies) and no singletons. Next, form a sequence of partitions  $\tau_k, \tau_{k-1}, \dots, \tau_1, \tau_0$  by reversing the procedure but using `CombineST` rather than `CombineIS`. More precisely, set  $\tau_k = \rho_k$ ,  $\tau_{k-1} = \text{CombineST}$  on  $(\tau_k, (I_k, S_k))$ ,  $\tau_{k-2} = \text{CombineST}$  on  $(\tau_{k-1}, (I_{k-1}, S_{k-1}))$ ,  $\dots$ ,  $\tau_0 = \text{CombineST}$  on  $(\tau_1, (I_1, S_1))$ . Note that for  $j = k, k-1, \dots, 1$  in turn, `CombineST` is defined on  $(\tau_j, (I_j, S_j))$  because (i)  $\text{supp}(\tau_j) = \text{supp}(\rho_j)$ , (ii) `CombineIS` is certainly defined on  $(\rho_j, (I_j, S_j))$ , and (iii) `CombineST`, `CombineIS` have the same domain. The example yields

$j$	$\tau_j$	$I_j$	$S_j$
4	4 7 - 5 9 - 6 8	{10}	$\emptyset$
3	4 7 - 5 9 - 6 8 - 10	$\emptyset$	{3}
2	3 10 - 4 7 - 5 9 - 6 8	{12}	$\emptyset$
1	3 10 - 4 7 - 5 9 - 6 8 - 12	{11}	{1, 2}
0	1 2 12 - 3 10 - 4 7 - 5 9 - 6 8 - 11		

Now  $\phi : \pi \rightarrow \tau_0$  is the desired bijection:  $\phi$  is clearly reversible and sends adjacency initiators to singletons and singletons to adjacency terminators, and hence interchanges  $\#$  adjacencies and  $\#$  singletons.  $\square$

The *complement* of a partition  $\pi$  on  $[n]$  is  $n + 1 - \pi$  (elementwise). Our proposed conjugate is as follows.

**Definition** *The conjugate of a partition  $\pi$  on  $[n]$  is obtained by applying the map  $\phi$  followed by complementation.*

Since complementation is an involution and sends adjacency initiators to adjacency terminators and vice versa, it is not hard to deduce the following result.

**Theorem 2.** *Conjugation is an involution on partitions of  $[n]$  that interchanges  $\#$  singletons and  $\#$  adjacencies.*

## 2 Restriction to Noncrossing partitions

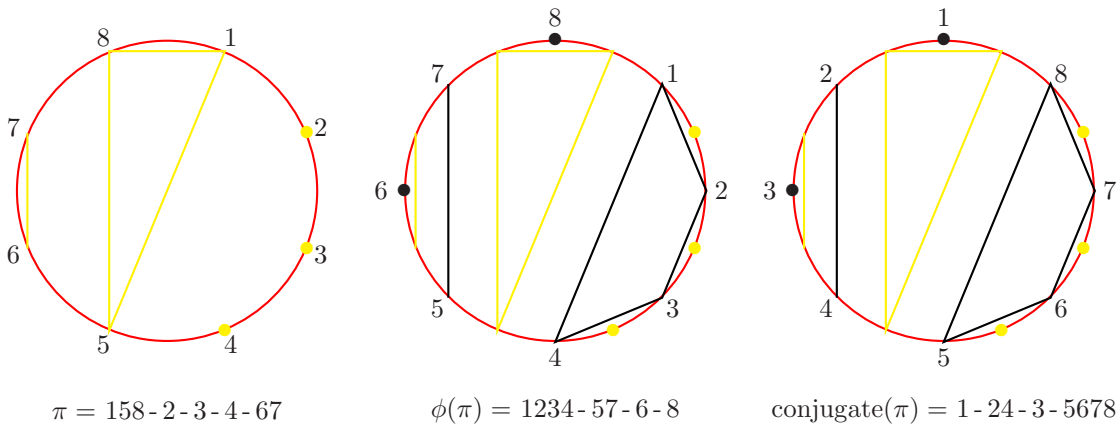
A *noncrossing* partition of  $[n]$  is one for which no quadruple  $a < b < c < d$  in  $[n]$  has  $a, c$  in one block and  $b, d$  in another. This implies that if the elements of  $[n]$  are arranged in a circle and neighboring elements within each block are joined by line segments, then no line segments cross one another in the resulting polygon diagram (see figure below). The bijection  $\phi$  of the previous section associates with each partition  $\pi$  on  $[n]$  a partition  $\rho_k$  with no adjacency initiators and no singletons. Since a noncrossing partition cannot avoid both singletons and adjacencies, induction yields the following characterization.

**Proposition.**  $\pi$  is noncrossing iff  $\rho_k$  is the empty partition.

Both  $\phi$  and conjugate preserve the noncrossing property and so we have

**Proposition.** Conjugation is an involution on noncrossing partitions of  $[n]$  that interchanges  $\#$  singletons and  $\#$  adjacencies.

On noncrossing partitions, the conjugate coincides with a graphically defined bijection first considered by Kreweras [4] (see also [5]). Given a partition  $\pi$  on  $[n]$ , draw its polygon diagram. Insert new vertices interleaving the old ones and form the partition with largest possible blocks subject to its polygon diagram being disjoint from that of  $\pi$ . Label the new vertices clockwise as shown to get  $\phi(\pi)$  and counterclockwise as shown to get the conjugate of  $\pi$ .



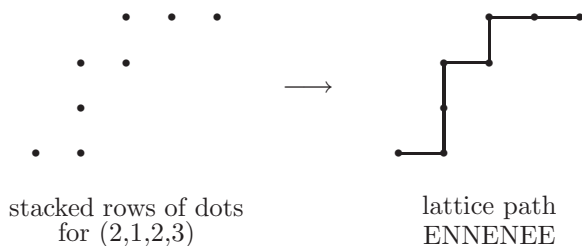
polygon diagrams of noncrossing partitions

### 3 The Conjugate of an Integer Composition

A *composition* of  $n$  is a list of positive integers—its parts—whose sum is  $n$ . There is a bijection from compositions of  $n$  to subsets of  $[n - 1]$  via partial sums:  $(c_i)_{i=1}^k \mapsto \{\sum_{j=1}^k c_j\}_{i=1}^{k-1}$ , and a further bijection from subsets  $S$  of  $[n - 1]$  to lattice paths of  $n - 1$  unit steps North ( $N$ ) or East ( $E$ ): the  $i$ th step is  $N$  if  $i \in S$  and  $E$  otherwise. The conjugate of a composition is defined by: pass to lattice path, flip the path in the  $45^\circ$  line, and pass back. For example, with  $n = 8$  and  $k = 4$ ,

$$\begin{array}{ccccccc} \text{composition} & \text{subset} & \text{path} & \text{flip} & \text{path} & \text{subset} & \text{conjugate} \\ (2, 1, 2, 3) & \rightarrow \{2, 3, 5\} & \xrightarrow{1\ 2\ 3\ 4\ 5\ 6\ 7} \text{ENNENEE} & \rightarrow & \xrightarrow{1\ 2\ 3\ 4\ 5\ 6\ 7} \text{NEENENN} & \rightarrow \{1, 4, 6, 7\} & \rightarrow (1, 3, 2, 1, 1). \end{array}$$

There is a neat graphical construction for the lattice path of a composition using a kind of shifted Ferrers diagram [6]. Represent a part  $a_i$  as a row of  $a_i$  dots. Stack the rows so each starts where its predecessor ends. Then join up the dots with  $E$  and  $N$  steps.



If the compositions of  $n$  of a given length are listed in lex (dictionary) order, then so are the corresponding subsets, and the length of the conjugate is  $n + 1 -$  length of the original. It follows that if the compositions of  $n$  are sorted, primarily by length and secondarily by lex order ( $n = 4$  is shown),

$$(4), (1\ 3), (2\ 2), (3\ 1), (1\ 1\ 2), (1\ 2\ 1), (2\ 1\ 1), (1\ 1\ 1\ 1)$$

then the conjugate of the  $i$ th composition from the left is the  $i$ th composition from the right.

There are two statistics on compositions of  $n$  that are interchanged by conjugation, and these statistics again involve “singletons” (parts = 1) and “adjacent” parts. (For partitions, since they are unordered, adjacency necessarily means “in value” but here it means “in position”). To define them, observe that each part has two neighbors except the end parts which have only one or, in the case of a one-part composition, none (“wraparound” neighbors are not allowed here). Say a part  $\geq 2$  is big; a part = 1 is small.

Now define

$$\mu = \text{sum of the big parts,}$$

$$\nu = \text{sum of the small parts + total number of neighbors of the big parts.}$$

For example, in the composition  $(3, 1, 1, 4, 2)$ , the big parts  $3, 4, 2$  have  $1, 2, 1$  neighbors respectively; so  $\mu = 3 + 4 + 2 = 9$  and  $\nu = (1 + 1) + (1 + 2 + 1) = 6$ .

**Theorem 3.** *Conjugation interchanges the statistics  $\mu$  and  $\nu$  on compositions of  $n$  except when  $n = 1$ .*

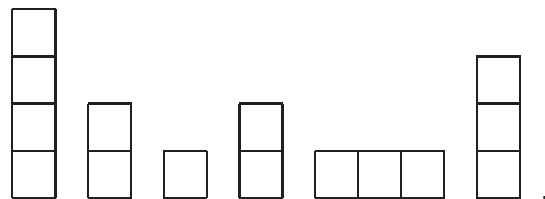
**Proof** Carefully translate  $\mu$  and  $\nu$  to the corresponding lattice paths. Using the Iverson notation that  $[statement] = 1$  if *statement* is true,  $= 0$  if it is false, we find

$$\mu = \# Es + \# ENs + [\text{path ends } E], \text{ and}$$

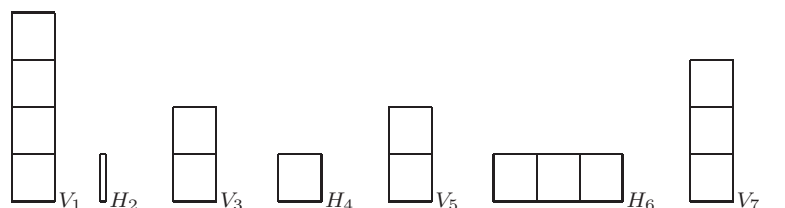
$$\nu = \# Ns + \# ENs + [\text{path starts } N]$$

It is then routine to check that  $\mu$  on the flipped path agrees with  $\nu$  on the original.  $\square$

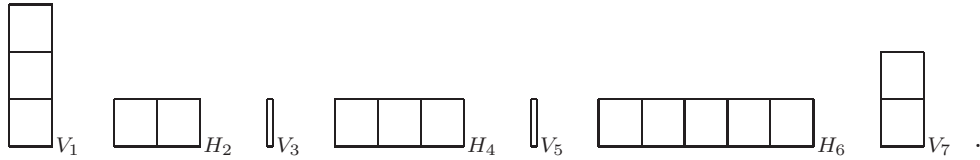
Finally, we remark that the genesis of the statistics  $\mu$  and  $\nu$  was the following graphical construction of the conjugate that involves “local” rather than “global” flipping. Suppose given a composition, say  $(4, 2, 1, 2, 1^3, 3)$ , where consecutive 1s have been collected so that  $1^3$  is short for  $1, 1, 1$ . Represent it as a list of vertical strips (for parts  $> 1$ ) and horizontal strips (for the 1s):



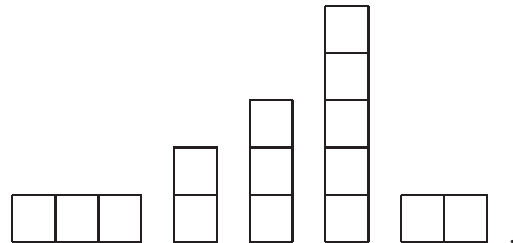
The vertical strips thus consist of 2 or more squares and may occur consecutively, but no two horizontal strips are consecutive. Insert an (initially) empty horizontal strip between each pair of consecutive vertical strips so that horizontal strips  $H$  and vertical strips  $V$  alternate (the first strip may be either an  $H$  or a  $V$ ):



For each horizontal strip  $H$ , transfer one square from each of its neighboring vertical strips to  $H$  (there will be two such neighbors in general, but possibly just one or even none in the case of the all-1s composition):



Since each vertical strip originally contained  $\geq 2$  squares this will always be possible, though some vertical strips may afterward be empty, in which case just erase them. Finally, rotate all strips  $90^\circ$  so that  $V$ s become  $H$ s and vice versa:



The result is the conjugate composition:  $(1^3, 2, 3, 5, 1^2)$ .

## References

- [1] Robin Chapman, personal communication.
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- [6] P. A. MacMahon, *Combinatory Analysis*, Vol. 1 *Cambridge University Press*, 1915; reprinted 2 vols. in 1, *Chelsea Publishing Company*, New York, 1984, p. 153.