Digit Reversal Without Apology

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In *A Mathematician's Apology* [1] G. H. Hardy states, "8712 and 9801 are the only four-figure numbers which are integral multiples of their reversals"; and, he further comments that "this is not a serious theorem, as it is not capable of any significant generalization."

However, Hardy's comment may have been short-sighted. In 1966, A. Sutcliffe [2] expanded this obscure fact about reversals. Instead of restricting his study to base 10 integers and their reversals, Sutcliffe generalized the problem to study all integer solutions of

$$k(a_h n^h + a_{h-1} n^{h-1} + \cdots + a_1 n + a_0) = a_0 n^h + a_1 n^{h-1} + \cdots + a_{h-1} n + a_h$$

with $n \geq 2$, $1 < k < n$, $0 \leq a_i \leq n-1$ for all i, $a_0 \neq 0$, $a_h \neq 0$. We shall refer to such an integer $a_0 ... a_h$ as an $(h+1)$-*digit solution for* n and write $k(a_h, a_{h-1}, ..., a_1, a_0)_n = (a_0, a_1, ..., a_{h-1}, a_h)_n$. For example, 8712 and 9801 are 4-digit solutions in base $n = 10$ for $k = 4$ and $k = 9$ respectively. After characterizing all 2-digit solutions for fixed n and generating parametric solutions for higher digit solutions, Sutcliffe left the following open question: Is there any base n for which there is a 3-digit solution but no 2-digit solution?

Two years later T. J. Kaczynski [1] [3] answered Sutcliffe's question in the negative. His elegant proof showed that if there exists a 3-digit solution for n, then deleting the middle digit gives a 2-digit solution for n. Together with Sutcliffe's work, this proved that there exists a 2-digit solution for n if and only if there exists a 3-digit solution for n.

[1] Better known for other work.

Given the nice correspondence between 2- and 3-digit solutions described by Sutcliffe and Kaczynski, it is natural to ask if there exists such a correspondence for higher digit solutions. In this paper, we will explore the relationship between 4- and 5-digit solutions. Unfortunately, there is not a bijection between these solutions, but there is a nice family of 4- and 5- digit solutions which have a natural one-to-one correspondence.

A second extension of Sutcliffe and Kaczynski's results is to ask, "Is there any value of n for which there is a 5-digit solution but no 4-digit solution?" We will answer this question in the negative; and, furthermore, we will show that there exist 4- and 5-digit solutions for every $n \geq 3$.

An attempt at generalization

In the case of 3-digit solutions, Kaczynski proved that if $n+1$ is prime and $k(a,b,c)_n = (c,b,a)_n$ is a 3-digit solution for n, then $k(a,b)_n = (b,a)_n$ is a 2-digit solution. Thus, we consider the following:

Question 1. *Let $k(a,b,c,d,e)_n = (e,d,c,b,a)_n$ be a 5-digit solution for n. If $n+1$ is prime, then is $k(a,b,d,e)_n = (e,d,b,a)_n$ a 4-digit solution for n?*

First, following Kaczynski, let $p = n+1$. We have

$$k(an^4 + bn^3 + cn^2 + dn + e) = en^4 + dn^3 + cn^2 + bn + a. \tag{1}$$

Reducing this equation modulo p, we obtain

$$k(a - b + c - d + e) \equiv e - d + c - b + a = a - b + c - d + e \bmod p.$$

Thus, $(k-1)(a-b+c-d+e) \equiv 0 \bmod p$, and

$$p \mid (k-1)(a-b+c-d+e). \tag{2}$$

If $p \mid (k-1)$, then $k-1 \geq p$, which is impossible because $k < n$. Therefore, $p \mid (a-b+c-d+e)$. But $-2p < -2n < a-b+c-d+e < 3n < 3p$, so there are four possibilities:

(i) $a - b + c - d + e = -p$,
(ii) $a - b + c - d + e = 0$,
(iii) $a - b + c - d + e = p$,
(iv) $a - b + c - d + e = 2p$.

Write $a - b + c - d + e = fp$, where $f \in \{-1, 0, 1, 2\}$. Substituting $c = -a + b + d - e + fp$ into equation 1 gives:

$$k[n^2(n^2-1)a + n^2(n+1)b + fpn^2 + n(n+1)d - (n^2-1)e]$$

$$= n^2(n^2-1)e + n^2(n+1)d + fpn^2 + n(n+1)b - (n^2-1)a.$$

After substituting for p, dividing by $n+1$, and rearranging, one sees that $k[an^3 + (b-a+f)n^2 + (d-e)n + e] = en^3 + (d-e+f)n^2 + (b-a)n + a$. Indeed, this is a 4-digit solution for n if $f = 0$, $b - a \geq 0$, and $d - e \geq 0$, but not necessarily a 4-digit solution of the form conjectured in Question 1.

As in Kaczynski's proof for 2- and 3-digit solutions, it would be ideal if three of the four possible values for f lead to contradictions and the fourth leads to a "nice" pairing of 4- and 5-digit solutions. Unlike Kaczynski, we now have the added advantage of exploring these cases with computer programs such as Maple. Experimental evidence suggests that the cases $f = -1$ and $f = 2$ are impossible. The cases $f = 0$ and $f = 1$ are discussed below.

A counterexample

Unfortunately, Kaczynski's proof does not completely generalize to higher digit solutions. Most 5-digit solutions do, in fact, yield 4-digit solutions in the manner described in Question 1, but for sufficiently large n there are examples where $(a, b, c, d, e)_n$ is a 5-digit solution but $(a, b, d, e)_n$ is not a 4-digit solution.

A computer search shows that the smallest such counterexamples appear when $n = 22$:

$$7(2, 8, 3, 13, 16)_{22} = (16, 13, 3, 8, 2)_{22}, 3(2, 16, 11, 5, 8)_{22} = (8, 5, 11, 16, 2)_{22}.$$

However, there is no integer k for which $k(2,8,13,16)_{22} = (16,13,8,2)_{22}$ or $k(2,16,5,8)_{22} = (8,5,16,2)_{22}$. Note that $-2+8+13-16 = 3$ and $-2+16+5-8 = 11$; that is, both of these counterexamples to Question 1 occur when $f = 0$. The next smallest counterexamples are

$$3(3,22,15,7,11)_{30} = (11,7,15,22,3)_{30}, 8(2,13,8,16,9)_{30} = (9,16,8,13,2),$$

which occur when $f = 0$ and $n = 30$.

A family of 4- and 5-digit solutions

Although Kaczynski's proof does not generalize entirely, there exists a family of 5-digit solutions when $f = 1$ that has a nice structure.

Theorem 1. *Fix $n \geq 2$ and $a > 0$. Then*

$$k(a, a-1, n-1, n-a-1, n-a)_n = (n-a, n-a-1, n-1, a-1, a)_n$$

is a 5-digit solution for n if and only if $a \mid (n-a)$.

Proof. We have

$$\frac{(n-a)n^4 + (n-a-1)n^3 + (n-1)n^2 + (a-1)n + a}{an^4 + (a-1)n^3 + (n-1)n^2 + (n-a-1)n + (n-a)}$$
$$= \frac{(n-a)(n^4+n^3-n-1)}{a(n^4+n^3-n-1)} = \frac{n-a}{a},$$

and the result is clear. $\square$

Notice that

$$(-a + (a-1)) + ((n-a-1) - (n-a)) + p = -1 + -1 + (n+1) = n-1.$$

That is, this family of solutions occurs when $f = 1$. Moreover, this family follows the pattern described in Question 1; that is, for each 5-digit solution described in Theorem 1, deleting its middle digit gives a 4-digit solution.

Theorem 2. *If*

$$k(a, a-1, n-1, n-a-1, n-a)_n = (n-a, n-a-1, n-1, a-1, a)_n$$

is a 5*-digit solution for* n*, then*

$$k(a, a-1, n-a-1, n-a)_n = (n-a, n-a-1, a-1, a)_n$$

is a 4*-digit solution for* n*.*

Proof. By Theorem 1, $\frac{n-a}{a} \in \mathbb{N}$. Now

$$\frac{(n-a)n^3 + (n-a-1)n^2 + (a-1)n + a}{an^3 + (a-1)n^2 + (n-a-1)n + (n-a)}$$

$$= \frac{(n-a)(n^3 + n^2 - n - 1)}{a(n^3 + n^2 - n - 1)} = \frac{n-a}{a}.$$

□

These 4-digit solutions were first described by Klosinski and Smolarski [4] in 1969, but their relationship to 5-digit solutions was not made explicit before now.

It is also interesting to note that 9801 and 8712, the two integers in Hardy's discussion of reversals, are included in this family of solutions.

We conclude with the following corollary.

Corollary 1. *There is a* 4*-digit solution and a* 5*-digit solution for every* $n \geq 3$*.*

Proof. Let $a = 1$ in the statements of Theorem 1 and Theorem 2 above. □

Some open questions

We have shown that there is no n for which there is a 5-digit solution but no 4-digit solution. More specifically, we know that there are 4- and 5-digit solutions for every $n \geq 3$.

Although Kaczynski's proof does not generalize directly to 4- and 5-digit solutions, it does bring to light several questions about the structure of solutions to the digit reversal problem.

First, it would be interesting to completely characterize 4- and 5-digit solutions for n. Namely,

1. All known counterexamples to Question 1 occur when $f = 0$. Are there counterexamples for which $f \neq 0$? Is there a parameterization for all such counterexamples?

2. Theorems 1 and 2 exhibit a family of 4- and 5-digit solutions for $f = 1$ with a particularly nice structure. To date, no other 4- or 5-digit solutions are known for $f = 1$. Do such solutions exist?

More generally,

3. Solutions to the digit reversal problem have not been explicitly characterized for more than 5 digits. Do there exist analogous results to Theorems 1 and 2 for higher digit solutions?

A Maple package for exploring these questions is available from the author's web page at `http://www.math.rutgers.edu/~lpudwell/maple.html`.

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References

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[5] N. J. A. Sloane, Sequence A031877 in “The On-Line Encyclopedia of Integer Sequences”,
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