The Universal Amplitude Ratio Γ^-/Γ^+ for Two-Dimensional Percolation

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The amplitude ratio of the susceptibility (or second size-moment) for two-dimensional percolation is calculated by two series methods and also by Monte-Carlo simulation. The first series method is a new approach based upon integrating approximations to the scaling function. The second series method directly uses low- and high-density series expansions of the susceptibility, going to unprecedented orders for both bond and site percolation on the square lattice. Putting all methods together we find a consistent value $\Gamma^-/\Gamma^+ = 162.5 \pm 2$, a significant improvement over previous results that placed the value of this ratio variously in the range of 14 to 220.

Percolation is one of the fundamental problems in statistical mechanics [1, 2] and is perhaps the simplest system exhibiting critical behavior. Through its mapping to the q-state Potts model (for $q \to 1$) many theoretical predictions follow, such as exact critical exponents in two dimensions. Yet many unanswered questions remain. One of these is the value of amplitude ratios, which represent universal ratios of quantities related to integrals of the scaling function. A great amount of work has been done investigating amplitude ratios of various systems, both to demonstrate that systems expected to be in a given universality class have the same ratios, and to determine their values accurately [3]. The study of amplitude ratios remains an active area of research (e.g., [4, 5, 6, 7, 8, 9, 10, 11]).

In this paper we study specifically the universal amplitude ratio Γ^-/Γ^+ for percolation, where Γ^- and Γ^+ are the amplitudes of the second size-moment (also called the susceptibility) in the low- and high-density phases, respectively. This ratio has been especially difficult to estimate accurately and the status of the estimates is very controversial. In [3] values are quoted in a wide range from 14 to 220 based on numerical estimates from Monte-Carlo simulations and series expansions for various percolation models. Furthermore, there are no reliable field-theoretical estimate for this quantity. Several years ago Delfino and Cardy [12] studied the q-state Potts model using methods from quantum field theory and predicted a value of 74.2 for percolation, by extrapolating the results for $q = 2$, 3, and 4 to $q = 1$. This was consistent with the numerical work of Corsten et al. [13], who gave the value 75 $(+40, -26)$, but inconsistent with many other measurements [3]. In this work, we study bond and site percolation on the square lattice and use extensive exact enumerations to obtain estimates for Γ^-/Γ^+, using two different approaches: one a novel approach based upon directly integrating approximations to the scaling function, and the second a more conventional analysis of the high- and low-density series for the susceptibility. The estimates for the amplitude ratio are consistent with the value $\Gamma^-/\Gamma^+ = 162 \pm 3$. We also carried out a Monte-Carlo calculation, which gave an almost identical value of 163 ± 2. These results are a significant improvement on previous published numerical estimates.

Percolation models are commonly formulated in a lattice setting with the edges and/or vertices occupied (or vacant) with probability p (or $1-p$). In this paper we limit our study to bond and site percolation on the square lattice $\mathbb{Z}^2$. We shall refer to *occupied* edges and vertices as bonds and sites, respectively. Nearest neighbor bonds (sites) are said to be connected and clusters are sets of connected bonds (sites). The behavior of the model is controlled by the occupation probability (or density of bonds/sites) p. When p is smaller than a critical value p_c all clusters remain finite. Above p_c there is a non-zero probability of finding an infinite cluster. The critical occupation probability is know exactly for bond percolation, $p_c = 1/2$ [14], and to a high degree of numerical accuracy for site percolation, $p_c = 0.59274621(13)$ [15]. The average cluster size $S(p) \sim \Gamma^-(p_c - p)^{-\gamma}$, which diverges as $p \to p_c^-$, plays a role similar to a susceptibility, with $\gamma = 43/18$.

Percolation problems are closely related to the combinatorial problem of the enumeration of lattice animals, which are connected subgraphs of a lattice. The size of a lattice animal is the number of connected sites (or bonds). A vertex (or edge) is said to be a *perimeter site* if it is a nearest neighbor of a site in the lattice animal. Series expansions for various percolation properties, such as the average cluster size, can be obtained as weighted sums over the number of lattice animals, $g_{s,t}$, enumerated according to the number of sites (bonds) s and perimeter t. *Perimeter polynomials* are defined as

$$D_s(q) = \sum_t g_{s,t} q^t. \qquad (1)$$

We have calculated the perimeter polynomials up to size 35 (bond) and 40 (site). The central property describing the cluster statistics in percolation is n_s, defined as the number of clusters (per site) containing s occupied sites or bonds, as a function of the occupation probability p,

$$n_s(p) = p^s D_s(1-p) = \sum_t g_{s,t} p^s (1-p)^t. \qquad (2)$$

The scaling hypothesis [2] states that n_s behaves as

$$n_s(p) = c_0 s^{-\tau} f(c_1(p-p_c)s^\sigma) \qquad (p \to p_c,\ s \to \infty), \quad (3)$$

where the critical exponents τ and σ and the scaling function f are universal, while c_0 and c_1 are non-universal metric factors. In two dimensions $\tau = 187/91$ and $\sigma = 36/91$. We shall often use the scaling variable $z = c_1(p-p_c)s^\sigma$. It then follows that [2]

$$\begin{aligned} S(p) &= \sum_s s^2 n_s(p) \propto \int c_0 s^{2-\tau} f(c_1(p-p_c)s^\sigma) \mathrm{d}s \\ &= \frac{c_0 c_1^{-\gamma}}{\sigma} |p-p_c|^{-\gamma} \int |z|^{-1+\gamma} f(z) \mathrm{d}z \\ &= \Gamma^\pm |p-p_c|^{-\gamma}, \end{aligned} \qquad (4)$$

where $\gamma = (3-\tau)/\sigma$. The integration in the high- or low-density case extends from 0 to ∞ or from $-\infty$ to 0, respectively. The amplitudes Γ^+ and Γ^- are given by the non-universal constant $c_0 c_1^{-\gamma}/\sigma$ multiplied by the corresponding integrals of the universal scaling function $f(z)$. It thus follows that the ratio Γ^-/Γ^+ is universal. Now, from the knowledge of $n_s(p)$ for finite s, where $n_s(p)$ is a polynomial in p, we can estimate the integrals in (4) by approximating $f(z)$ by $\bar{n}_s(z) = s^\tau n_s(z)$, where $z = (p-p_c)s^\sigma$, and taking the ratio of the relevant integrals involving $\bar{n}_s(z)$. Obviously, $\bar{n}_s(z)$ is just a polynomial approximation to $f(z)$ so we can't extend the integration to infinity (the integral would just diverge). However, there is a natural cut-off provided by the scaling variable z and the fact that the physical low-density region is $0 \le p < p_c$, so the integral involving z over the low-density region runs over the interval $[-z_- = -s^\sigma p_c, 0]$. Likewise, integrals over the high-density region include the interval $[0, s^\sigma(1-p_c) = z_+]$. The integrals are easily evaluated since $\bar{n}_s(z) = \sum_k a_k z^k$ are polynomials, which we can determine by exact enumeration. We get in the high-density case:

$$\begin{aligned} I_s^+ &= \int_0^{z_+} |z|^{\gamma-1} \bar{n}_s(z) \mathrm{d}z = \int_0^{z_+} z^{\gamma-1} \sum_{k=0} a_k z^k \mathrm{d}z \\ &= \sum_{k=0} a_k (z_+)^{k+\gamma}/(k+\gamma), \end{aligned}$$

and in the low-density region

$$\begin{aligned} I_s^- &= \int_{-z_-}^0 (-z)^{\gamma-1} \bar{n}_s(z) \mathrm{d}z = \int_0^{z_-} z^{\gamma-1} \sum_{k=0} a_k (-z)^k \mathrm{d}z \\ &= \sum_{k=0} (-1)^k a_k (z_-)^{k+\gamma}/(k+\gamma), \end{aligned}$$

from which we obtain the estimate $\Gamma_s^-/\Gamma_s^+ = I_s^-/I_s^+$. We shall call this ratio r_s for short. The amplitude ratio Γ^-/Γ^+ is obtained from the limit of r_s as $s \to \infty$.

In Fig. 1 we show (in the left panel) the estimates of the ratio $r_s = I_s^-/I_s^+$ for bond percolation. The estimates display some curvature when plotted against $1/s$, but seem to become close to a straight line and extrapolate to a value around 160. In order to extrapolate to the limit $s \to \infty$ with a bit more confidence we need to try and work out the asymptotic form of r_s. In the middle panel we show a log-log plot of the difference between consecutive ratios, $d_s = r_s - r_{s-1}$, against $1/s$. Clearly, d_s has a power-law decay with $1/s$. This means that $r_s \approx \Gamma^-/\Gamma^+ + a/s^\alpha$, and is is straightforward to show that then $d_s \propto 1/s^{\alpha+1}$. We now try to estimate the decay exponent α. In the right panel of Fig. 1 we plot estimates for the *local exponent* α_s against $1/s$, where α_s is obtained from a linear regression of $\log d_s$ vs. $\log s$ using the 5 values from s to $s-4$. From this figure we estimate that $\alpha + 1 = 1.85(5)$.

Next we extrapolate the data for r_s by fitting directly to an assumed asymptotic form

$$r_s = \frac{\Gamma^-}{\Gamma^+} + \sum_{i \ge 0} \frac{a_i}{s^{\alpha_i}}, \qquad (5)$$

where the exponents α_i form a strictly increasing sequence. By way of justification we can mention that similar forms are commonly found in the study of the asymptotic behavior of series coefficients and arise from corrections to scaling. We don't know the values of α_i, except for the leading exponent $\alpha_0 = \alpha$, so we assume that $\alpha_i = \alpha + i$ (this is akin to including only one non-analytic correction to scaling), and fit to the form

$$r_s = \frac{\Gamma^-}{\Gamma^+} + \sum_{i=0}^{k-1} \frac{a_i}{s^{\alpha+i}}. \qquad (6)$$

That is, we take a sub-sequence of terms $\{r_s, r_{s-1}, \ldots, r_{s-k}\}$, plug into the formula above and solve the $k+1$ linear equations to obtain estimates for the amplitudes. It is then advantageous to plot estimates for the leading amplitude Γ^-/Γ^+ against $1/s$ for several values of k. The results using the value $\alpha = 0.85$ are plotted in Fig. 2. We notice that the estimates obtained with $k = 2$, 3, and 4 are quite stable and show little dependence on s for large values. This would indicate that r_s is well approximated by the assumed asymptotic form. We estimate from this that $\Gamma^-/\Gamma^+ = 159.2 \pm 0.2$, where the error bars represent fluctuations but not systematic error associated with this method. We also tested the sensitivity of the extrapolation procedure to the value of α by fitting to the form (6) using the values $\alpha = 0.8$ and 0.9, respectively. The resulting estimates for the amplitude ratio were only marginally lower or higher and the procedure is

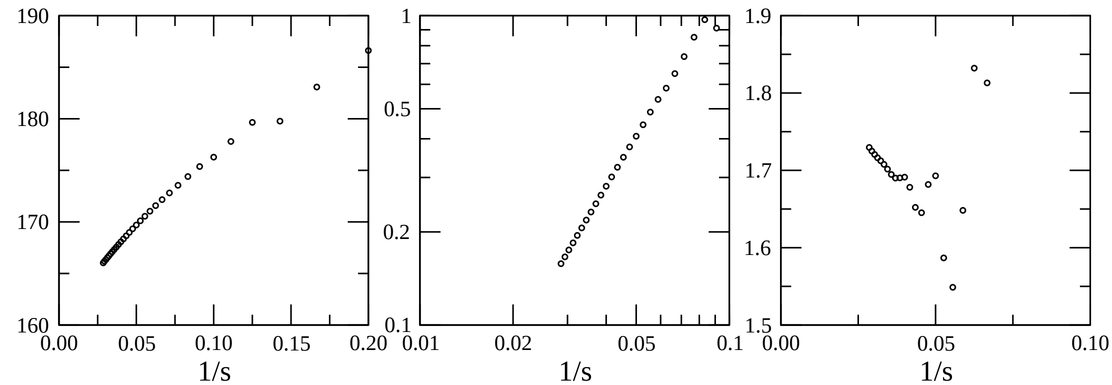

FIG. 1: The amplitude ratio r_s for bond percolation (left panel), a log-log plot of the difference between consecutive ratios (middle panel), and the local exponent of the difference plot (right panel).

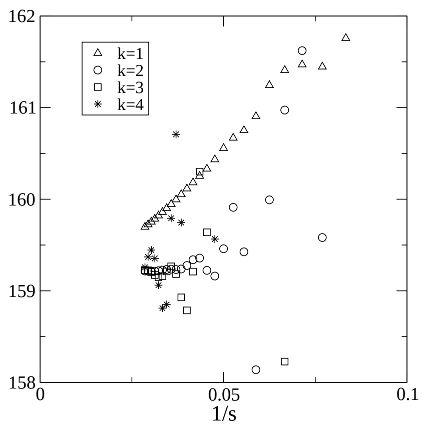

FIG. 2: Estimates of the amplitude ratio Γ^-/Γ^+ for bond percolation from fits to the form (6) with $\alpha = 0.85$.

not very sensitive to changes in α, at least within a reasonable range.

A similar analysis was carried out for site percolation. In this case the estimates for r_s displayed a pronounced curvature when plotted against $1/s$ and the extrapolation to the limit $s \to \infty$ was more challenging than in the bond case. We estimate that $\Gamma^-/\Gamma^+ = 164.5 \pm 1.5$.

For our second approach, we have obtained estimates for the amplitudes Γ^- and Γ^+ directly from the low- and high-density series for the second size-moment $M_2(p)$. In the low-density region the series was simply obtained from the perimeter polynomials

$$M_2(p) = \sum_s s^2 p^s \sum_t (1-p)^t g_{s,t} \sim \Gamma^-(p_c - p)^{-\gamma},$$

while the high-density series was calculated separately to order 51 (bond) and 55 (site). Estimates for the amplitudes were obtained by transforming the original series, using our knowledge of the critical point and exponents, into series which have a simple pole at p_c, with a residue from which we can calculate the amplitude. The following two methods are quite standard [16]. We can raise the series to the power $1/\gamma$ to get a series with the behavior $(\Gamma^-)^{1/\gamma}/(p_c - p)$ or we can look at the series $M_2(p)(p_c - p)^{\gamma-1}$ which should have the behavior $\Gamma^-/(p_c - p)$. The amplitudes can then be estimated by forming Padé approximants to the transformed series and calculating the residues. Another approach is to get completely rid of the physical singularity at p_c by studying the transformed series $M_2(p)(p_c-p)^\gamma$ or $M_2(p)^{1/\gamma}(p_c-p)$ and evaluating Padé approximants to this series at p_c. As pointed out by Daboul *et al.* in their recent study of 2D percolation [17], such biased methods for calculating the amplitudes generally lead to more accurate estimates when judged purely on the spread among the estimates as obtained from individual approximants. However, this higher apparent accuracy can be misleading in that biased approximations can display systematic corrections on a scale exceeding the fluctuations. Thus great care must be taken and one should be careful not to rely too heavily on the biased estimates, particularly when it comes to determining the error bars on the estimates.

In Fig. 3 we have plotted estimates for the high-density amplitude $\tilde{\Gamma}$ for site percolation. The estimates are obtained from Padé approximants to the series $\frac{1}{q^4}M_2(q)(q_c - q)^{\gamma-1}$ (upper panel) and $\frac{1}{q^4}M_2(q)(q_c - q)^{\gamma}$ (lower panel), where the factor $1/q^4$ is included in order that the transformed series has a non-zero constant term. As expected the biased estimates (lower panel) are quite well-converged and appear to settle down at a value around 0.02208(2). This is obviously slightly at odds with the unbiased estimates (upper panel) which favor a higher value, but these estimates have a greater spread and a pronounced downwards drift. Estimates from the other methods display similar trends and in particular have a greater spread than those in the lower panel. To be cautious we adopt the rather conservative estimate $\tilde{\Gamma} = 0.02215(15)$ and from this we get $\Gamma^+ = q_c^4\tilde{\Gamma} = 0.000609(4)$. Similarly we estimate that in the low-density case $\Gamma^- = 0.09819(6)$ and thus $\Gamma^-/\Gamma^+ = 161.2 \pm 1.2$. From a similar analysis we estimate that

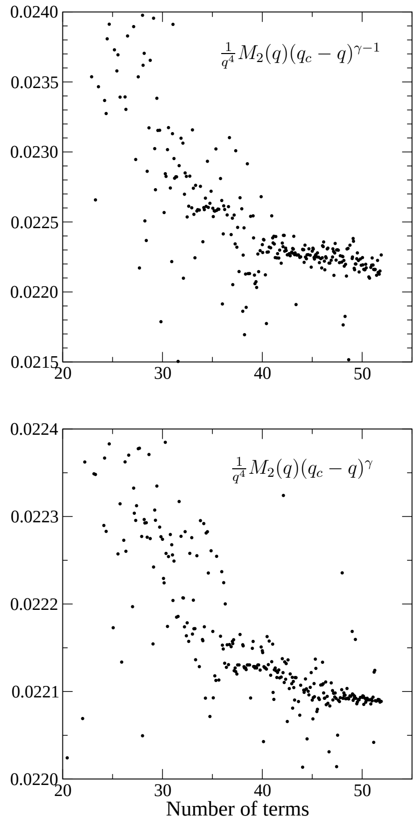

FIG. 3: The high-density amplitude $\tilde{\Gamma}$ for site percolation.

the amplitudes of the second size-moments of bond percolation in the low- and high-density phases are $\Gamma^- = 0.15001(8)$ and $\Gamma^+ = 0.0009290(15)$, which yields the estimate $\Gamma^-/\Gamma^+ = 161.5 \pm 0.4$ for the amplitude ratio.

The estimates from the series studies are consistent with the value $\Gamma^-/\Gamma^+ = 162 \pm 3$, which is a significant improvement over previous results. We have also carried out a Monte-Carlo test of this amplitude ratio, studying bond percolation on the square lattice. The method used was to generate individual clusters by a growth algorithm on a large lattice at $p = 0.47$, 0.48, 0.49, 0.495, 0.505, 0.51, 0.53, and 0.53. For $p < p_c$, all the clusters terminate, while for $p > p_c$, the clusters that keep growing beyond a clear cutoff can be identified as being part of the infinite cluster and discarded. This method proved superior to previous MC works which generally count clusters on fully populated lattices and have significant finite-size effects; here, there were essentially no finite-size effects as long as p is kept sufficiently away from p_c and the lattice is made large enough. This work yielded the result $\Gamma^-/\Gamma^+ = 163 \pm 2$, which was quoted previously in [18].

The two determinations of the amplitude ratio (by series and Monte-Carlo) were done separately and the analysis of each was done by the two authors independently. Putting these two unbiased results together, we propose a final value of 162.5 ± 2, where the estimated error is at the 68% confidence interval. We have thus found consistent precise series-expansion and Monte-Carlo results, laying to rest the controversy on the value of this amplitude ratio. Most likely, the wide spread in previous measurements were due to the large finite-size effects in Monte-Carlo simulations, and the relatively short series used in exact enumerations. We have eliminated both of those shortcomings in the present work.

As part of a larger on-going project we have also calculated perimeter polynomials on the honeycomb lattice, and a preliminary analysis yields the estimates $\Gamma^-/\Gamma^+ = 166 \pm 5$ (bond) and 170 ± 7 (site). Finally, we mention in passing that we have studied the *area* moments of bond percolation on the square lattice (where the area is the volume enclosed by the external hull walk) (see [19] for details) and find that the amplitude ratio of the second area-moment is 175 ± 10. This value is close to the one for the second size-moments, but it is expected to have a somewhat different value because it represents a different moment of the scaling function $f(z)$.

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