On uniquely k-determined permutations

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Abstract

There are several approaches to study occurrences of consecutive patterns in permutations such as the inclusion-exclusion method, the tree representations of permutations, the spectral approach and others. We propose yet another approach to study occurrences of consecutive patterns in permutations. The approach is based on considering the graph of patterns overlaps, which is a certain subgraph of the de Bruijn graph.

While applying our approach, the notion of a uniquely k-determined permutation appears. We give two criteria for a permutation to be uniquely k-determined: one in terms of the distance between two consecutive elements in a permutation, and the other one in terms of directed hamiltonian paths in the certain graphs called path-schemes. Moreover, we describe a finite set of prohibitions that gives the set of uniquely k-determined permutations. Those prohibitions make applying the transfer matrix method possible for determining the number of uniquely k-determined permutations.

1 Introduction

A *pattern* τ is a permutation on $\{1, 2, \ldots, k\}$. An occurrence of a *consecutive pattern* τ in a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ is a word $\pi_i\pi_{i+1}\ldots\pi_{i+k-1}$ that

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is *order-isomorphic* to τ. For example, the permutation 253164 contains two occurrences of the pattern 132, namely 253 and 164. In this paper we deal only with consecutive patterns, which courses omitting the word "consecutive" in defining a pattern to shorten the notation.

There are several approaches in the literature to study the *distribution* and, in particular, *avoidance*, of consecutive patterns in permutations. For example, direct combinatorial considerations are used in [8]; the *method of inclusion-exclusion* is used in [6, 9]; the *tree representations of permutations* are used in [4]; the *spectral theory of integral operators* on $L^2([0,1]^k)$ is used in [3]. In this paper we introduce yet another approach to study occurrences of consecutive patterns in permutations. The approach is based on considering the *graph of patterns overlaps* defined below, which is a subgraph of the *de Bruijn graph* studied broadly in the literature mainly in connection with combinatorics on words and graph theory.

Suppose we are interested in the number of occurrences of a pattern τ of length k in a permutation π of length n. To find this number, we scan π from left to right with a "window" of length k, that is, we consider $P_i = \pi_i \pi_{i+1} \dots \pi_{i+k-1}$ for $i = 1, 2, \dots, n-k+1$: if we meet an occurrence of τ, we register it. Each P_i forms a pattern of length k, and the procedure of scanning π gives us a path in the *graph* $\mathcal{P}_k$ *of patterns overlaps of order* k defined as follows (graphs of patterns/permutations overlaps appear in [1, 2, 7]). The nodes of $\mathcal{P}_k$ are all $k!$ k-permutations, and there is an arc from a node $a_1a_2\dots a_k$ to a node $b_1b_2\dots b_k$ if and only if $a_2a_3\dots a_k$ and $b_1b_2\dots b_{k-1}$ form the same pattern. Thus, for any n-permutation there is a path in $\mathcal{P}_k$ of length $n-k+1$ corresponding to it. For example, if $k=3$ then to the permutation 13542 there corresponds the path $123 \to 132 \to 321$ in $\mathcal{P}_3$.

Our approach to study the distribution of a consecutive pattern τ of length k among n-permutations is to take $\mathcal{P}_k$ and to consider all paths of length $n-k+1$ passing through the node τ exactly ℓ times, where $\ell = 0, 1, \dots, n-k+1$. Then we could count the permutations corresponding to the paths. Similarly, for the "avoidance problems" that attracted much attention in the literature, we proceed as follows: given a set of patterns of length k to avoid, we remove the corresponding nodes with the corresponding arcs from $\mathcal{P}_k$, consider all the paths of certain length in the graph obtained, and then count the permutations of interest.

However, a complication with the approach is that a permutation does not need to be reconstructible uniquely from the path corresponding to it. For example, the permutation 13542 above has the same path in $\mathcal{P}_3$ corresponding to it as the permutations 23541 and 12543. Thus, different paths in $\mathcal{P}_k$ may have different contributions to the number of permutations with required properties; in particular, some of the paths in $\mathcal{P}_k$ give exactly one permutation corresponding to them. We call such permutations *uniquely k-determined*. Study of such permutations is the main concern of the paper, and it should be considered as the first step in understanding how to use our approach to the problems described. Also, in our considerations we assume that all the nodes in $\mathcal{P}_k$ are allowed while dealing with uniquely k-determined permutations, that is, we do not prohibit any pattern.

The paper is organized as follows. In Section 2 we study the set of uniquely k-determined permutations. In particular, we give two criteria for a permutation to be uniquely k-determined: one in terms of the distance between two *consecutive elements* in a permutation, and the other one in terms of directed hamiltonian paths in the certain graphs called *path-schemes*. We use the second criteria to establish (rough) upper and lower bounds for the number of uniquely k-determined permutations. Moreover, given an integer k, we describe a finite set of prohibitions that determines the set of uniquely k-determined permutations. Those prohibitions make applying the *transfer matrix method* [13, Thm. 4.7.2] possible for determining the number of uniquely k-determined permutations and we discuss this in Subsection 2.3. As a corollary of using the method, we get that the generating function for the number of uniquely k-determined permutations is rational. Besides, we show that there are no *crucial permutations* in the set of uniquely k-determined permutations. (Crucial objects, in the sense defined below, are natural to study in infinite sets of objects defined by prohibitions; for instance, see [5] for some results in this direction related to words.) We consider in more details the case $k = 3$ in Subsection 2.4. Finally, in Section 3, we state several open problems for further research.

2 Uniquely k-determined permutations

2.1 Distance between consecutive elements; a criterion on unique k-determinability

Suppose $\pi = \pi_1\pi_2\dots\pi_n$ is a permutation and $i < j$. The *distance* $d_\pi(\pi_i, \pi_j) = d_\pi(\pi_j, \pi_i)$ between the elements π_i and π_j is $j-i$. For example, $d_{253164}(3, 6) = d_{253164}(6, 3) = 2$.

Theorem 1. [*First criterion on unique k-determinability*] *An n-permutation π is uniquely k-determined if and only if for each $1 \leq x < n$, the distance $d_\pi(x, x+1) \leq k-1$.*

Proof. Suppose for an n-permutation π, $d(x, x+1) \geq k$ for some $1 \leq x < n$. This means that x and $x+1$ will never be inside a "window" of length k while scanning consecutive elements of π. Thus, these elements are incomparable in π in the sense that switching x and $x+1$ in π will lead to another permutation π' having the same path in $\mathcal{P}_k$ as π has. So, π is not uniquely k-determined.

On the other hand, if for each $1 \leq x < n$, the distance $d_\pi(x, x+1) \leq k-1$, then the positions of the elements $1, 2, \ldots, n$ are uniquely determined (first we note that the position of 1 is uniquely determined, then we determine the position of 2 which is a 1's neighbor in a "window" of length k, then the position of 3, etc.) leading to the fact that π is uniquely k-determined. $\square$

The following corollary to Theorem 1 is straightforward.

Corollary 2. *An n-permutation π is not uniquely k-determined if and only if there exists x, $1 \leq x < n$, such that $d_\pi(x, x+1) \geq k$.*

So, to determine if a given n-permutation is uniquely k-determined, all we need to do is to check the distance for $n-1$ pairs of numbers: $(1,2)$, $(2,3)$,..., $(n-1, n)$. Also, the language of uniquely determined k-permutations is *factorial* in the sense that if $\pi_1\pi_2 \ldots \pi_n$ is uniquely k-determined, then so is the pattern of $\pi_i\pi_{i+1} \ldots \pi_j$ for any $i \leq j$ (this is a simple corollary to Theorem 1).

Coming back to the permutation 13542 above and using Corollary 2, we see why this permutation is not uniquely 3-determined ($k = 3$): the distance $d_{13542}(2, 3) = 3 = k$.

2.2 Directed hamiltonian paths in path-schemes; another criterion on unique k-determinability

Let $V = \{1, 2, \ldots, n\}$ and M be a subset of V. A *path-scheme* $P(n, M)$ is a graph $G = (V, E)$, where the edge set E is $\{(x, y) \mid |x - y| \in M\}$. See Figure 1 for an example of a path-scheme.

Path-schemes appeared in the literature, for example, in connection with counting independent sets (see [10]). However, we will be interested in path-schemes having $M = \{1, 2, \ldots, k-1\}$ for some k (the number of independent

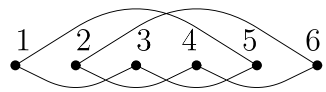

Figure 1: The path-scheme $P(6, \{2, 4\})$.

sets for such M in case of n nodes is given by the $(n+k)$-th k-*generalized Fibonacci number*). Let $\mathcal{G}_{k,n} = P(n, \{1, 2, \ldots, k-1\})$, where $k \leq n$. Clearly, $\mathcal{G}_{k,n}$ is a subgraph of $\mathcal{G}_{n,n}$.

Any permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ determines uniquely a directed hamiltonian path in $\mathcal{G}_{n,n}$ starting with π_1, then going to π_2, then to π_3 and so on. The reverse is also true: given a directed hamiltonian path in $\mathcal{G}_{n,n}$ we can easily construct the permutation corresponding to it.

Theorem 3. [*Second criterion on unique k-determinability*] *Let Φ be a map that sends a uniquely k-determined n-permutation π to the directed hamiltonian path in $\mathcal{G}_{n,n}$ corresponding to π^{-1}. Φ is a bijection between the set of all uniquely k-determined n-permutations and the set of all directed hamiltonian paths in $\mathcal{G}_{k,n}$.*

Proof. Let π be a uniquely k-determined n-permutation. We claim that the directed hamiltonian path in $\mathcal{G}_{n,n}$ corresponding to π^{-1} is actually a directed hamiltonian path in $\mathcal{G}_{k,n}$. Indeed, suppose the elements x and $x+1$, $1 \leq x < n$, are located in π in positions i and j respectively. According to Theorem 1, $|j - i| \leq k - 1$. Now, ij is a factor in π^{-1}, and the directed hamiltonian path corresponding to π^{-1} contains the arc from i to j, which is an arc in $\mathcal{G}_{k,n}$. Obviously, Φ is injective. Also, it is easy to see how to find the inverse to Φ mapping a directed hamiltonian path in $\mathcal{G}_{k,n}$ to a permutation that, due to Theorem 1, is uniquely k-determined. $\square$

Theorem 3 suggests a quick checking of whether an n-permutation π is uniquely k-determined or not. One simply needs to consider $n-1$ differences of the adjacent elements in π^{-1} and check whether at least one of those differences exceeds $k-1$ or not. Moreover, one can find the number of uniquely k-determined n-permutations by listing them and checking for each of them the differences of consecutive elements in the manner described above. Using this approach, one can run a computer program to get the number of uniquely k-determined n-permutations for initial values of k and n, which we record in Table 1.

$k=2$	$1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ldots$
$k=3$	$1,\ 2,\ 6,\ 12,\ 20,\ 34,\ 56,\ 88,\ 136,\ldots$
$k=4$	$1,\ 2,\ 6,\ 24,\ 72,\ 180,\ 428,\ 1042,\ 2512,\ldots$
$k=5$	$1,\ 2,\ 6,\ 24,\ 120,\ 480,\ 1632,\ 5124,\ 15860,\ldots$
$k=6$	$1,\ 2,\ 6,\ 24,\ 120,\ 720,\ 3600,\ 15600,\ 61872,\ldots$
$k=7$	$1,\ 2,\ 6,\ 24,\ 120,\ 720,\ 5040,\ 30240,\ 159840,\ldots$
$k=8$	$1,\ 2,\ 6,\ 24,\ 120,\ 720,\ 5040,\ 40320,\ 282240,\ldots$

Table 1: The initial values for the number of uniquely k-determined n-permutations.

It is remarkable that the sequence corresponding to the case $k = 3$ in Table 1 appears in [12, A003274], where we learn that the inverses to the uniquely 3-determined permutations are called the *key permutations* and they appear in [11]. Another sequence appearing in Table 1 is [12, A003274]: 0, 2, 12, 72, 480, 3600, ... In our case, this is the number of uniquely n-determined $(n+1)$-permutations, $n \geq 1$; in [12], this is the number of $(n+1)$-permutations that have 2 predetermined elements non-adjacent (e.g., for $n = 2$, the permutations with say 1 and 2 non-adjacent are 132 and 231). It is clear that both of the last objects are counted by $n!(n-1)$. Indeed, to create a uniquely n-determined $(n+1)$-permutation, we take any permutation (there are $n!$ choices) and extend it to the right by one element making sure that the extension is not adjacent to the leftmost element of the permutation (there are $n-1$ possibilities; here we use Theorem 1). On the other hand, to create a "good" permutation appearing in [12], we take any of $n!$ permutations, and insert one of the predetermined elements into any position not adjacent to the other predetermined element (there are $(n-1)$ choices). A bijection between the sets of permutations above is given by the following: Suppose a and b are the predetermined elements in $\pi = \pi_1 \ldots \pi_n$, and $\pi_i = a$ and $\pi_j = b$. We build the permutation π' corresponding to π by setting $\pi'_1 = i$, $\pi'_n = j$, and $\pi'_2 \ldots \pi'_{n-1}$ is obtained from π by first removing a and b, and then, in what is left, by replacing i by a and j by b. For example, assuming that 2 and 4 are the determined elements, to $13\underline{4}5\underline{2}6$ there corresponds $\underline{5}1426\underline{3}$ which is a uniquely 5-determined 6-permutation.

Another application of Theorem 3 is finding lower and upper bounds for the number $A_{k,n}$ of uniquely k-determined n-permutations.

Theorem 4. *We have* $2((k-1)!)^{\lfloor n/k \rfloor} < A_{k,n} < 2(2(k-1))^n$.

Proof. According to Theorem 3, we can estimate the number of directed hamiltonian paths in $\mathcal{G}_{k,n}$ to get the desired. This number is two times the number of (non-directed) hamiltonian paths in $\mathcal{G}_{k,n}$, which is bounded from above by $(2(k-1))^n$, since $2(k-1)$ is the maximum degree of $\mathcal{G}_{k,n}$ (for $n \geq 2k-1$). So, $A_{k,n} < 2(2(k-1))^n$.

To see that $A_{k,n} > 2((k-1)!)^{\lfloor n/k \rfloor}$, consider hamiltonian paths starting at node 1 and *not* going to any of the nodes i, $i \geq k+1$ unless a path goes through *all* the nodes $1, 2, \ldots, k$. Going through all the first k nodes can be arranged in $(k-1)!$ different ways. After covering the first k nodes we send the path under consideration to node $k+1$, which can be done since we deal with $\mathcal{G}_{k,n}$. Then the path covers *all*, but not any other, of the $k-1$ nodes $k+2, k+3, \ldots, 2k$ (this can be done in $(k-1)!$ ways) and comes to node $2k+1$, etc. That is, we subdivide the nodes of $\mathcal{G}_{k,n}$ into groups of k nodes and go through all the nodes of a group before proceeding with the nodes of the group to the right of it. The number of such paths can be estimated from below by $((k-1)!)^{\lceil n/k \rceil}$. Clearly, we get the desired result after multiplying the last formula by 2 (any hamiltonian path can be oriented in two ways). $\square$

2.3 Prohibitions giving unique k-determinability

The set of uniquely k-determined n-permutations can be described by the language of prohibited patterns $\mathcal{L}_{k,n}$ as follows. Using Theorem 1, we can describe the set of uniquely k-determined n-permutations by prohibiting patterns of the forms $xX(x+1)$ and $(x+1)Xx$, where X is a permutation on $\{1, 2, \ldots, |X|+2\} - \{x, x+1\}$ ($|X|$ is the number of elements in X), the length of X is at least $k-1$, and $1 \leq x < n$. We collect all such patterns in the set $\mathcal{L}_{k,n}$; also, let $\mathcal{L}_k = \cup_{n \geq 0} \mathcal{L}_{k,n}$.

A prohibited pattern $X = aYb$ from $\mathcal{L}_k$, where a and b are some consecutive elements and Y is a (possibly empty) word, is called *irreducible* if the patterns of Yb and aY are not prohibited, in other words, if the patterns of Yb and aY are uniquely k-determined permutations. Without loss the generality, we can assume that $\mathcal{L}_k$ consists only of irreducible prohibited patterns.

Theorem 5. *Suppose k is fixed. The number of (irreducible) prohibitions in $\mathcal{L}_k$ is finite. Moreover, the longest prohibited patterns in $\mathcal{L}_k$ are of length $2k-1$.*

Proof. Suppose that a pattern $P = xX(x+1)$ of length $2k$ or larger belongs to $\mathcal{L}_k$ (the case $P = (x+1)Xx$ can be considered in the same way). Then obviously X contains either $x-1$ or $x+2$ on the distance at least $k-1$ from either x or $x+1$. In any case, clearly we get either a prohibited pattern $P' = yY(y+1)$ or $P' = (y+1)Yy$, which is a proper factor of P. Contradiction with P being irreducible. $\square$

Theorem 5 allows us to use the transfer matrix method to find the number of uniquely k-determined permutations. Indeed, we can consider the graph $\mathcal{P}_{2k-1}(\mathcal{L}_k)$, which is the graph $\mathcal{P}_{2k-1}$ of patterns overlaps without nodes containing prohibited patterns as factors. Then the number $A_{k,n}$ of uniquely k-determined n-permutation is equal to the number of paths of length $n-2k+1$ in the graph, which can be found using the transfer matrix method [13, Thm. 4.7.2][1]. In particular, the method makes the following statement true.

Theorem 6. *The generating function $A_k(x) = \sum_{n\geq 0} A_{k,n}x^n$ for the number of uniquely k-determined permutations is rational.*

A permutation is called *crucial* with respect to a given set of prohibitions, if it does not contain any prohibitions, but adjoining any element to the right of it leads to a permutation containing a prohibition. In our case, an n-permutation is crucial if it is uniquely k-determined, but adjoining any element to the right of it, and thus creating an $(n+1)$-permutation, leads to a *non*-uniquely k-determined permutation[2]. If such a π exists, then the path in $\mathcal{P}_{2k-1}(\mathcal{L}_k)$ corresponding to π ends up in a sink. However, the following theorem shows that there are no crucial permutations with respect to the set of prohibitions $\mathcal{L}_k$, thus any path in $\mathcal{P}_{2k-1}(\mathcal{L}_k)$ can always be continued.

Theorem 7. *There do not exist crucial permutations with respect to $\mathcal{L}_k$.*

Proof. If $k=2$ then only the monotone permutations are uniquely k-determined, and we always can extend to the right a decreasing permutation by the least element, and the increasing permutation by the largest element.

[1]In fact, one can use a smaller graph, namely $\mathcal{P}_{2k-2}(\mathcal{L}_k)$, in which we mark arcs by corresponding permutations of length $2k-1$; then we remove arcs containing prohibitions and use the transfer matrix method. In this case, to an n-permutation there corresponds a path of length $n-2k+2$. See Figure 2 for such a graph in the case $k=3$.

[2]As it is mentioned in the introduction, crucial *words* are studied, for example, in [5]. We define crucial permutations with respect to a set of prohibited patterns in a similar way. However, as Theorem 7 shows, there are no crucial permutations with respect to $\mathcal{L}_k$.

Suppose $k \geq 3$ and let Xx be an n-permutation avoiding $\mathcal{L}_k$, that is, Xx is uniquely k-determined. If $x = 1$ then Xx can be extended to the right by 1 without creating a prohibition; if $x = n$ then Xx can be extended to the right by $n+1$ without creating a prohibition. Otherwise, due to Theorem 1, both $x-1$ and $x+1$ must be among the k leftmost elements of Xx. In particular, at least one of them, say y, is among the $k-1$ leftmost elements of Xx. If $y = x-1$, we extend Xx by x (the "old" x becomes $(x+1)$); if $y = x+1$, we extend Xx by $x+1$ (the "old" $x+1$ becomes $(x+2)$). In either of the cases considered above, Theorem 1 guarantees that no prohibitions will be created. So, Xx can be extended to the right to form a uniquely k-determined $(n+1)$-permutation, and thus Xx is not a crucial n-permutation. ☐

2.4 The case $k = 3$

In this subsection we take a closer look to the graph $\mathcal{P}_4(\mathcal{L}_3)$ whose paths give all uniquely 3-determined permutations (we read marked arcs of a path to form the permutation corresponding to it). It turns out that $\mathcal{P}_4(\mathcal{L}_3)$ has a nice structure (see Figure 2).

Suppose w' denotes the *complement* to an n-permutation $w = w_1w_2\cdots w_n$. That is, $w'_i = n - w_i + 1$ for $1 \leq i \leq n$. $\mathcal{P}_4(\mathcal{L}_3)$ has the following 12 nodes (those are all uniquely 3-determined 4-permutations):

$$\begin{array}{ll} a = 1234 & a' = 4321 \\ b = 1324 & b' = 4231 \\ c = 1243 & c' = 4312 \\ d = 3421 & d' = 2134 \\ e = 1423 & e' = 4132 \\ f = 3241 & f' = 2314 \end{array}$$

In Figure 2 we draw 20 arcs corresponding to the 20 uniquely 3-determined 5-permutations. Notice that $\mathcal{P}_4(\mathcal{L}_3)$ is not strongly connected: for example, there is no directed path from c to f.

To find the generating function $A_3(x) = \sum_{n\geq 0} A_{3,n}x^n$ for the number of uniquely 3-determined permutations one can build a 12x12 matrix corresponding to $\mathcal{P}_4(\mathcal{L}_3)$ and to proceed with the transfer matrix method. However, we do not do that since, as it was mentioned in Subsection 2.2, the

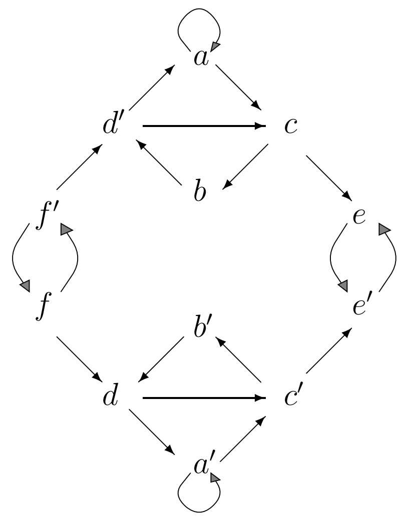

Figure 2: Graph $\mathcal{P}_4(\mathcal{L}_3)$ (the case $k = 3$).

generating function for these numbers is known [12, A003274]:

$$A_3(x) = \frac{1 - 2x + 2x^2 + x^3 - x^5 + x^6}{(1 - x - x^3)(1 - x)^2}.$$

3 Open problems

It is clear that any n-permutation is uniquely n-determined, whereas for $n \geq 2$ no n-permutation is uniquely 1-determined. Moreover, for any $n \geq 2$ there are exactly two uniquely 2-determined permutations, namely the monotone permutations. For a permutation π, we define its *index* $IR(\pi)$ *of reconstructibility* to be the minimal integer k such that π is uniquely k-determined.

Problem 1. Describe the distribution of $IR(\pi)$ among all n-permutations.

Problem 2. Study the set of uniquely k-determined permutations in the case when a set of nodes is removed from $\mathcal{P}_k$, that is, when some of patterns of length k are prohibited.

An n-permutation π is *m-k-determined*, $m, k \geq 1$, if there are exactly m (different) n-permutations having the same path in $\mathcal{P}_k$ as π has. In particular, the uniquely k-determined permutations correspond to the case $m = 1$.

Problem 3. Find the number of m-k-determined n-permutations.

Problem 3 is directly related to finding the number of *linear extensions* of a *poset*. Indeed, to any path w in $\mathcal{P}_k$ there naturally corresponds a poset $\mathcal{W}$. In particular, any factor of length k in w consists of comparable to each other elements in $\mathcal{W}$. For example, if $k = 3$ and $w = 134265$ then $\mathcal{W}$ is the poset in Figure 3.

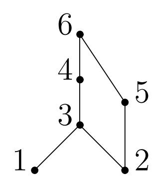

Figure 3: The poset associated with the path $w = 134265$ in $\mathcal{P}_3$ ($k = 3$).

If all the elements are comparable to each other in w, then $\mathcal{W}$ is a linear order and w gives a uniquely k-determined permutation. If $\mathcal{W}$ contains exactly one pair of incomparable elements, then w gives (two) 2-k-determined permutations. In the example in Figure 3, there are 4 pairs of incomparable elements, (1,2), (1,5), (3,5), and (4,5), and this poset can be extended to a linear order in 7 different ways giving (seven) 7-3-determined permutations.

Problem 4. Which posets on n elements appear while considering paths (of length $n - k + 1$) in $\mathcal{P}_k$? Give a classification of the posets (different from the classification by the number of pairs of incomparable elements).

Problem 5. How many linear extensions can a poset (associated to a path in $\mathcal{P}_k$) on n elements with t pairs of incomparable elements have?

Problem 6. Describe the structure of $\mathcal{L}_k$ (see Subsection 2.3 for definitions) that consists of irreducible prohibitions. Is there a nice way to generate $\mathcal{L}_k$? How many elements does $\mathcal{L}_k$ have?

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