GEOMETRIC PROPERTIES OF QUASICONFORMAL MAPS AND SPECIAL FUNCTIONS

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Abstract

Our goal is to provide a survey of some topics in quasiconformal analysis of current interest. We try to emphasize ideas and leave proofs and technicalities aside. Several easily stated open problems are given. Most of the results are joint work with several coauthors. In particular, we adopt results from the book authored by Anderson-Vamanamurthy-Vuorinen [AVV6].

Part 1. Quasiconformal maps and spheres
Part 2. Conformal invariants and special functions
Part 3. Recent results on special functions
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1 Quasiconformal maps and spheres

Some current trends in multi-dimensional quasiconformal analysis are reviewed in [G6], [G8], [I2], [V6], [V7], [Vu5].

1.1. Categories of homeomorphisms. Below we shall discuss homeomorphisms of a domain of $\mathbb{R}^n$ onto another domain in $\mathbb{R}^n$, $n \geq 2$. Conformal maps provide a well-known subclass of general homeomorphisms. By Riemann's mapping theorem this class is very flexible and rich for $n = 2$ whereas Liouville's theorem shows that, for $n \geq 3$, conformal maps are the same as Möbius transformations, i.e., their class is very narrow. Thus the unit ball $B^n = \{x \in \mathbb{R}^n : \ |x| < 1\}$ can be mapped conformally only onto a half-space or a ball if the dimension is $n \geq 3$. Quasiconformal maps constitute a convenient interpolating category of maps, much wider than conformal maps, and less general than locally Hölder-continuous homeomorphisms. We also note that bilipschitz maps are a subclass of quasiconformal maps. Deferring the definition of a quasisymmetric map to 1.30, we note that bilipschitz maps are a subclass of quasisymmetric maps, which in turn are a subclass of quasiconformal maps.

1.2. Modulus of a curve family. Now follows perhaps the most technical part of this paper, the definition of the modulus of a curve family. The nonspecialist reader may be relieved to hear that this notion will be used later only in the definition of quasiconformal mappings and that an alternative definition of quasiconformal mappings can be given in terms of the geometric notion of linear dilatation (see 1.8). Let G be a domain in $\mathbb{R}^n$ and let Γ be a curve family in G. For $p > 1$ the *p-modulus* $M_p(\Gamma)$ is defined by

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_G \rho^p dm \,, \tag{1.3}$$

where $F(\Gamma) = \{\rho : G \to R \cup \{\infty\},\ \rho \geq 0 \text{ Borel: } \int_\gamma \rho ds \geq 1 \text{ for all locally rectifiable } \gamma \in \Gamma\}$. The most important case is $p = n$ and we set $M(\Gamma) =$

$M_n(\Gamma)$—in this case we just call $M(\Gamma)$ the modulus of Γ. The *extremal length* of Γ is $M(\Gamma)^{1/(1-n)}$. The modulus is *a conformal invariant*, i.e. $M(\Gamma) = M(h\Gamma)$ if h is a conformal map and $h\Gamma = \{h \circ \gamma : \ \gamma \in \Gamma\}$. For the basic properties of the modulus we refer the reader to [V1], [Car], [Oh], [Vu2].

1.4. Modulus and relative size. For a domain $G \subset \mathbb{R}^n$ and $E, F \subset G$ denote

$$\Delta(E, F; G) = \{\text{all curves joining } E \text{ and } F \text{ in } G\}.$$

We define the *relative size* of the pair E, F by

$$r(E, F) = \min\{d(E), d(F)\}/d(E, F)\,,$$

where $d(E) = \sup\{|x-y| : x, y \in E\}$ and

$$d(E, F) = \inf\{|x-y| : x \in E, \text{and}\, y \in F\}.$$

If E and F are disjoint continua then $M(\Delta(E, F; \mathbb{R}^n))$ and $r(E, F)$ are simultaneously small or large. In fact, there are increasing homeomorphisms $h_j : [0, \infty) \to [0, \infty)$ with $h_j(0) = 0,\ j = 1, 2$, such that

$$h_1(r(E, F)) \leq M(\Delta(E, F; \mathbb{R}^n)) \leq h_2(r(E, F)) \tag{1.5}$$

(see [V1], [Vu2]). The explicit expressions for h_j in [Vu2, 7.41-7.42] involve special functions.

1.6. Quasiconformal maps. Let $K \geq 1$. A homeomorphism $f : G \to G'$ is termed *K-quasiconformal* if for all curve families Γ in G

$$M(f\Gamma)/K \leq M(\Gamma) \leq K M(f\Gamma). \tag{1.7}$$

The least constant K in (1.7) is called *the maximal dilatation* of f.

Note that conformal invariance is embedded in this definition: for $K = 1$ equality holds throughout in (1.7).

This definition resembles the bilipschitz condition, but it should be noted that quasiconformal maps can transform distances in a highly nonlinear and totally unlipschitz manner.

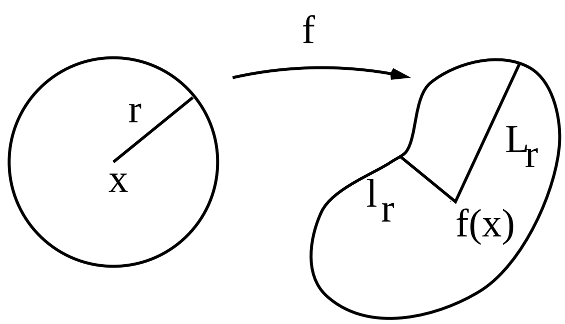
There are numerous equivalent ways of characterizing quasiconformal maps [Car]. It often happens that a mapping K_1-quasiconformal in the sense of one definition is K_2-quasiconformal in the sense of another definition, where K_2 depends from K_1 in an explicit way and, what is most important, $K_2 \to 1$ if $K_1 \to 1$. We shall next consider in 1.8 an equivalent definition based on the linear dilatation. We shall see that in the case of this definition, finding such a constant K_2 explicitly has required a time span as long as the history of higher-dimensional quasiconformal maps.

1.8. Linear dilatation. For a homeomorphism $f : G \to G'$, $x_0 \in G$, $r \in (0, d(x_0, \partial G))$, let

$$H(x_0, f, r) = \sup\{\frac{|f(x) - f(x_0)|}{|f(y) - f(x_0)|} : \ |x - x_0| = |y - y_0| = r\},$$

$$H(x_0, f) = \limsup_{r \to 0} H(x_0, f, r).$$

Then $H(x_0, f)$ is called the *linear dilatation* of f at x_0.



There is an alternative characterization of quasiconformal maps, to the effect that a homeomorphism with bounded linear dilatation

$$\sup\{H(x, f) : x \in G\} \le L < \infty$$

is quasiconformal [V1]. We shall next review the known estimates for the constant L in terms of the maximal dilatation.

Consider first the case $n = 2$. A. Mori proved in [Mor2] that if $f : G \to G'$, with $G, G' \subset \mathbb{R}^2$, is K-quasiconformal, then for all $x_0 \in G$

$$H(x_0, f) \le e^{\pi K}. \tag{1.9}$$

This bound is not sharp when $K \to 1$. The sharp bound

$$H(x_0, f) \le \lambda(K) = \frac{u^2}{1 - u^2}, \quad u = \varphi_K(1/\sqrt{2})\,, \tag{1.10}$$

is due to Lehto, Virtanen, and Väisälä [LVV] in the particular case $G = \mathbb{R}^2$ and due to Shah Dao-Shing and Fan Le-Le [SF] in the general case of a proper subdomain $G \subset \mathbb{R}^2$. For the definition of the special function $\varphi_K \equiv \varphi_{K,2}$, see 2.28.

Next we consider the case $n \geq 2$. If $f : G \to G'$, with $G, G' \subset \mathbb{R}^n$, is K-quasiconformal then, by a 1962 result of F.W. Gehring [G1, Lemma 8, pp. 371-372],

$$H(x_0, f) \leq d(n,K) \equiv \exp\left[\left(\frac{K\omega_{n-1}}{\tau_n(1)}\right)^{1/(n-1)}\right] \tag{1.11}$$

for all $x_0 \in G$, where $\omega_{n-1} = n\pi^{n/2}/\Gamma(1+\frac{n}{2})$ is the $(n-1)$-dimensional surface area of the unit sphere ∂B^n, and τ_n is the capacity of the Teichmüller condenser (see 2.12). For $n = 2$, the earlier result of A. Mori (1.9) is recovered as a particular case of (1.11), that is, $d(2,K) = e^{\pi K}$. Unfortunately $d(n,K) \not\to 1$ as $K \to 1$. In 1986 M. Vuorinen sharpened the bound (1.11) to

$$H(x_0, f) \leq c(n,K) \equiv 1 + \tau_n^{-1}(\tau_n(1)/K) < \frac{1}{10} d(n,K)\,. \tag{1.12}$$

Note that $c(n,K) \to 2$ as $K \to 1$ [Vu2, 10.22, 10.32]. In 1990 Vuorinen proved for a K-quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ of the whole space $\mathbb{R}^n$ [Vu3]

$$H(0, f) \leq \exp(6(K+1)^2\sqrt{K-1}) \equiv s(K) \tag{1.13}$$

with the desirable property $s(K) \to 1$ as $K \to 1$. In 1996 P. Seittenranta [Se2] was able to prove a similar result for maps of proper subdomains G of $\mathbb{R}^n$: a K-quasiconformal mapping $f : G \to G'$ satisfies

$$H(x_0, f) \leq s(K) \tag{1.14}$$

for all $x_0 \in G$ with the same $s(K)$ as in (1.13). Note that (1.14) would easily follow from (1.13) if we could solve a local structure problem stated below in 2.47. In fact, slightly better bounds than (1.13) and (1.14), involving the special function τ_n are known. Note also that for $n = 2$ a sharper form of (1.14) holds by (1.10) and [AVV2] since, for $K > 1$,

$$\exp(\pi(K-1)) \leq \lambda(K) \leq \exp(\pi(K-1/K)). \tag{1.15}$$

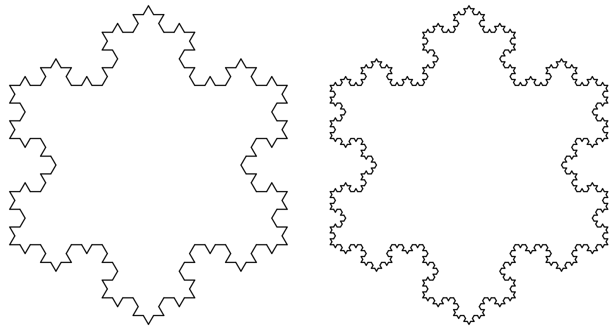
1.16. Open problem. Can the upper bound (1.14) be replaced by $s(n,K)$ with $\lim_{n\to\infty} s(n,K) = 1$ for each fixed $K > 1$?

1.17. Quasispheres and quasicircles. If $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, is K-quasiconformal, then the set fS^{n-1} is called a K-*quasisphere* or, if $n = 2$, a K-*quasicircle*. Here, as usual, $S^{n-1} = \partial B^n$ and $B^n = \{x \in \mathbb{R}^n : \ |x| < 1\}$.

Plane domains that are bounded by quasicircles, called quasidisks, have been studied extensively. See the surveys of Gehring [G5], [G7]. Compared to what is known for the dimension $n = 2$, very little is known in higher dimensions $n \geq 3$. We shall formulate below some open problems, both for the plane and the higher-dimensional case.

Part of the interest of quasispheres derives from the fact that these sets can have interesting geometric structure of fractal type. In fact, some of the differences between the categories of bilipschitz and quasiconformal maps can be understood if one studies the geometric structure of the images of spheres under these maps.

1.18. Examples of quasicircles. (1) Perhaps the most widely known example of a nonrectifiable quasicircle is the *snowflake curve* (also called *von Koch curve*), which is constructed in the following way. Take an equilateral triangle. To each side adjoin an equilateral triangle whose base agrees with the middle-third segment of the side; then remove this middle-third segment. Iterating this procedure recursively ad infinitum we get as a result a nonrectifiable Jordan curve of Hausdorff dimension > 1. Other similar examples are given in [GV2], [G5, p. 25], and [LV, p. 110].



(2) The Julia set J_f of an iteration $z \mapsto f(z)$ is the set of all those points that remain bounded under the repeated iterations. As a rule, J_f has an interesting fractal type structure, and for suitable f, J_f is a quasicircle. For the case of quadratic f, see [GM] and for rational f see [St].

(3) Images of circles under bilipschitz maps are always rectifiable (and hence of Hausdorff dimension 1) but they may fail to have tangents at some points. In fact, bilipschitz maps are differentiable only almost everywhere and if this "bad set" of zero measure is nonempty peculiar things may happen. See [VVW] for a construction of a bilipschitz circle which is $(q, 2)$- thick in the sense of definition 1.52 below.

(4) There are examples of Jordan domains with rectifiable boundaries which are not bounded by quasispheres. For instance, the "rooms and corridors"-type domains violating the Ahlfors condition in (1.23) can be used.

(5) We next give a construction of a bilipschitz map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(0) = 0$ which carries rays passing through 0 to "logarithmic spirals" through 0. We first fix an integer $p \geq 5$ and note that there exists $L \geq 1$ and an $L-$bilipschitz mapping of the annulus $\overline{B}^2(p) \setminus B^2$ which is identity on $S^1(p)$ and a restriction of the rotation $z \mapsto e^{i\theta} z, \theta \in (0, \pi/(2p))$, on $S^1(1)$. The boundary values of this map guarantee that this mapping can be extended to an $L-$bilipschitz map of the whole plane, which in the annuli $B(p^{k+1}) \setminus \overline{B}(p^k)\,, k \in Z$, agrees with our original map up to conjugations by suitable rotations and dilations. For a similar construction, see Luukkainen and Väisälä [LuV, 3.10 (4), 4.11].

(6) The univalent function

$$f(z) = \int_0^z \exp\{ib \sum_{k=0}^{\infty} \zeta^{2^k}\} d\zeta, \quad b < \frac{1}{4}\,,$$

defined in the unit disk B^2, provides an analytic representation of a quasicircle $\Gamma = f(\partial D)$ that fails to have a tangent at each of its points. For details see Ch. Pommerenke [Po, pp.304-305].

1.19. Particular classes of domains. The unit ball in $\mathbb{R}^n$ is the standard domain for most applications in quasiconformal analysis. Since the early 1960's several classes of domains have been introduced in studies on quasiconformal maps. It is not our goal to review such studies, but we note that at least the following two types of domain classes have been studied:

(1) domains satisfying a geometric condition;

(2) domains characterized by conditions involving moduli of curve families, capacities, or other analytic conditions.

Domains of type (1) include so-called uniform domains and their various generalizations. Domains of type (2) include, e.g., so-called QED-domains. A domain $G \subset \mathbb{R}^n$ is called $c-$QED, $c \in (0,1]$ if, for each pair of disjoint continua $F_1, F_2 \subset G$, it is true that $M(\Delta(F_1,F_2;G)) \geq cM(\Delta(F_1,F_2;\mathbb{R}^n))$. There is a useful survey of some of these classes by J. Väisälä [V6].

Let us look at a property of the unit ball. For nondegerate continua $E, F \subset B^n$ we have

$$M(\Delta(E,F;\mathbb{R}^n)) \geq M(\Delta(E,F;B^n)) \geq$$

$$M(\Delta(E,F;\mathbb{R}^n))/2 \geq \frac{1}{2}h_1(r(E,F))$$

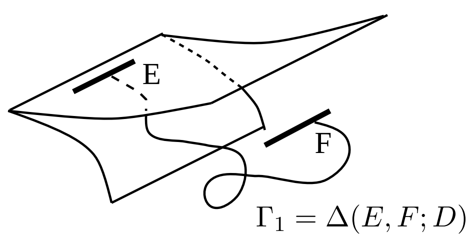
by [G4] and (1.5). (In particular, the unit ball is 1/2-QED.) For a domain $D \subset \mathbb{R}^n$ and $r_0 > 0$ we set

$$L(D,r_0) = \inf_{r(E,F)\geq r_0} M(\Delta(E,F;D))\,, \tag{1.20}$$

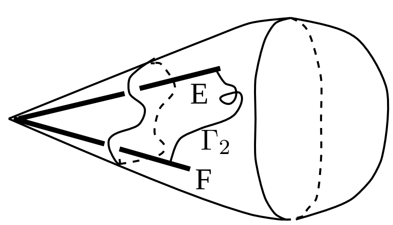
where E and F are continua. For all dimensions $n \geq 2$ it is easy to construct "rooms and corridors" type Jordan domains with $L(D,r_0)=0$ (only simplest estimates of moduli are needed from [V1, pp. 20-24]). For dimensions $n \geq 3$ one can construct such domains also in the form

$$D_g = \{(x,y,z) \in \mathbb{R}^3 :\ x>0,\ |y| < g(x)\}$$

for a suitable homeomorphism $g:[0,\infty) \to [0,\infty),\ g(0)=0,\ g'(0)=0$; now the access to the "ridge" $A \equiv \{(0,y,0):\ y \in \mathbb{R}\}$ of the domain gets narrower and narrower as we approach A from within D_g.



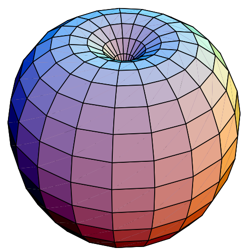
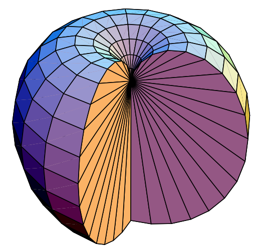
It is not difficult to show with the help of (1.5) that the class of domains with $L(D,r_0) > 0$ is invariant under quasiconformal maps of $\mathbb{R}^n$. Hence we see that boundaries of domains with $L(D,r_0)=0$ cannot be quasispheres.



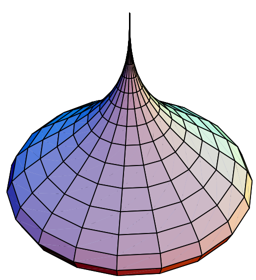
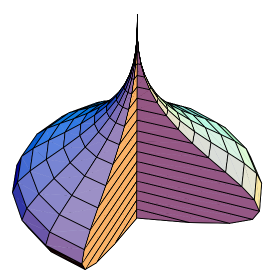
One can also construct domains $D \subset \mathbb{R}^n$ such that for a pair of disjoint continua $E, F \subset D$ with $r(E,F) = \infty$ we have $M(\Delta(E,F;D)) < \infty$.

1.21. Quasiconformal images of B^3. By Liouville's theorem, the unit ball B^n, $n \geq 3$, can be mapped conformally only onto another ball or a half-space. Gehring and Väisälä [GV1] created an extensive theory which gives necessary (and, in certain cases, sufficient) conditions for a domain to be of the form fB^n where $f : B^n \to \mathbb{R}^n$ is quasiconformal. They also exhibited several interesting domains illuminating their results which we shall now discuss.

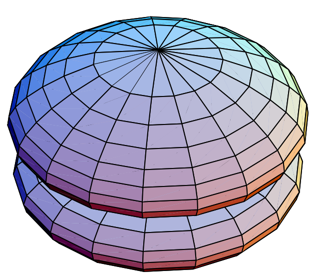
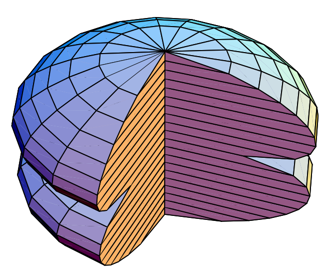
(1) The first example is an apple-shaped domain (cf. picture). By [GV1] such a domain cannot, in general, be mapped quasiconformally onto B^3.

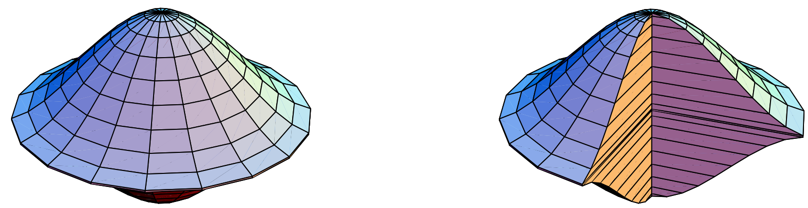
(2) On the other hand, there are onion-shaped domains that can be so mapped.

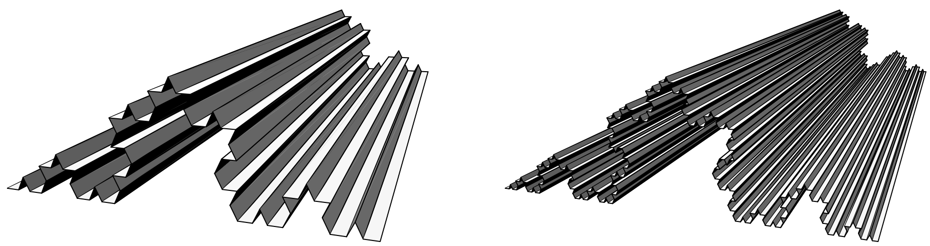
(3) In examples (1) and (2) above, the critical behavior takes place near one boundary point at the tip of a spire. In the case of an onion-shaped domain the spire is outwards-directed and for apple-shaped domains it is inwards-directed. In this and the following example the critical set consists of the edge of a boundary "ridge". An example of a domain with inward-directed ridge is shown ("yoyo-domain") in the picture below. The shape of the yoyo can be so chosen that the domain is a quasiconformal image of B^3.

(4) Consider now a "ufo-shaped" domain where the ridge is outward-directed (cf. the picture below). In this case the shape can be so chosen that the domain is not quasiconformally equivalent to B^3.



(5) P. Tukia [Tu2] used an example of S. Rickman to construct a domain whose boundary is the Cartesian product $K \times \mathbb{R}$ where K is a snowflake-style curve with a periodic structure. The domain underneath the surface fails to be quasiconformally equivalent to B^3.



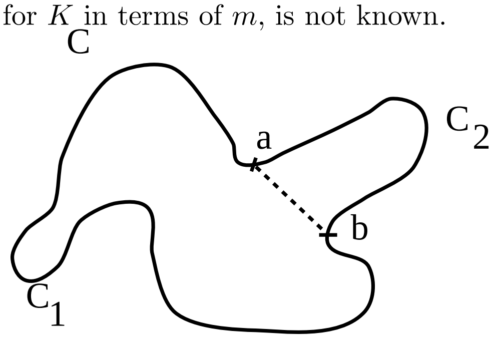
(6) Note that for dimensions $n \geq 3$ it is possible that a Jordan domain can be quasiconformally mapped onto B^n but that its complement fails to have this property.

1.22. Ahlfors' condition for quasicircles. Quasicircles have been studied extensively and many characterizations for them given by many authors. For interesting surveys, see [G5], [G7]. Chronologically, one of the first characterizations was given by L. V. Ahlfors in [Ah1] and this result still continues to be the most popular one and it reads as follows: A Jordan curve $C \subset \overline{\mathbb{R}}^2$ is a quasicircle if and only if there exists a constant $m \geq 1$ such that for all finite points $a, b \in C$

$$\min\{d(C_1), d(C_2)\} \leq m|a-b|\,, \tag{1.23}$$

where C_1 and C_2 are the components of $C \setminus \{a, b\}$ and where d stands for the Euclidean diameter.

Note that this formulation shows that (1.23) guarantees the existence of a K-quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $C = fS^1$. However, the least upper bound for K in terms of m, is not known.



1.24. Open problem. Generalize Ahlfors' condition to quasispheres.

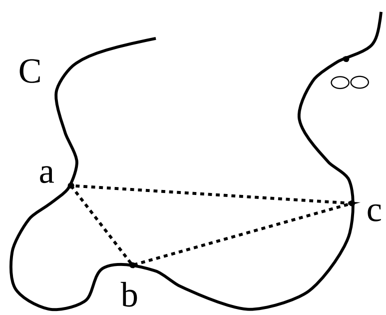
1.25. Bilipschitz circles and spheres. In harmony with our hierarchy of the categories of maps in 1.1, it is natural to ask if a criterion similar to (1.23) exists also for bilipschitz circles or surfaces. The case $n = 2$ was settled by P. Tukia [Tu1] in 1980 and also by D. Jerison- C. Kenig [JK] in 1982. The case $n \geq 3$ is open. Some results of this type were obtained by S. Semmes [S1], [S2] and T. Toro [To1], [To2].

1.26. Open problem. Find the least K for which a quadrilateral with given dimensions is a K-quasicircle. A particular case is the rectangle. R. Kühnau [Küh2, p. 104] has proved that a triangle with the least angle $\alpha\pi (< \pi/3)$ is a K-quasicircle with $K^2 \geq (1+d)/(1-d), d = |1-\alpha|$, with equality for the equilateral triangle ($\alpha = 1/3$). (In fact, equality holds for all $\alpha \in (0, 1/3)$ by S. Werner [We].)

1.27. Open problem - triangle condition. We say that a Jordan curve $C \subset \overline{\mathbb{R}}^2$ with $\infty \in C$ satisfies a triangle condition if there exists a constant $M \geq 1$ such that for all successive finite points $a, b, c \in C$ we have

$$|a-b| + |b-c| \leq M|a-c| \tag{1.28}$$

Show that there exists a constant $K \geq 1$ such that $C = f\mathbb{R}$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is K-quasiconformal. Give $K = K(M)$ explicitly in terms of M with $K(M) \to 1$ as $M \to 1$.



1.29. Remarks. (1) From a result of S. Agard - F.W. Gehring [AG] it follows that $K(M) \geq 1 + 0.25(M-1)$ for $M \in (1, 2)$.

(2) D. Trotsenko has informed the author (1996) about an idea to settle the open problem 1.27 with $K(M) \leq 1 + c_1\sqrt{M-1},\ c_1 = 10^5$, for $M < 1 + 10^{-13}$. See also [Tr].

1.30. Quasisymmetric maps. Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism with $\eta(0) = 0$ and let $f : G \to G'$ be a homeomorphism, where $G, G' \subset \mathbb{R}^n$. We say [TV1] that f is *η-quasisymmetric* if, for all $a, b, c \in G$ with $a \neq c$,

$$\frac{|f(a) - f(b)|}{|f(a) - f(c)|} \leq \eta\left(\frac{|a-b|}{|a-c|}\right) \tag{1.31}$$

1.32. Beurling - Ahlfors extension result. A. Beurling and L. Ahlfors [BAh] introduced the class of homeomorphisms $h : \mathbb{R} \to \mathbb{R}$ satisfying

$$\frac{1}{M} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M \tag{1.33}$$

for all $x \in \mathbb{R}, t > 0$, and for some $M > 1$. Such homeomorphisms were later termed quasisymmetric. Note that, for maps of the real axis, condition (1.33) agrees with (1.31) under the additional constraint $|a-b| = |a-c|$. Beurling and Ahlfors also proved that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ of the real axis can be extended to a K-quasiconformal map $f^* : \mathbb{R}^2 \to \mathbb{R}^2$ iff f satisfies (1.33). We remark that again there is a problem of finding the optimal constant K if $M > 1$ is given. It is known by [L, p. 34] that one can choose $K \le \min\{M^{3/2}, 2M-1\}$.

1.34. Quasisymmetry - quasiconformality. If $f : G \to G'$ satisfies (1.31) it follows easily that $H(x_0, f) \le \eta(1)$ for all $x_0 \in G$. By the alternative characterization of quasiconformality in terms of the linear dilatation 1.8, we thus see that quasisymmetric maps constitute a subclass of quasiconformal maps. As a rule, these two classes of maps are different. However, if $G = \mathbb{R}^n$ then quasiconformal maps are η-quasisymmetric, by a result of P. Tukia and J. Väisälä [TV1]. Much more delicate is the question of finding for a given $K > 1$ an explicit η_K which is "asymptotically sharp" when $K \to 1$. In [Vu3] it was shown, for the first time, that an explicit $\eta_{K,n}(t)$ exists which tends to t as $K \to 1$: If $f : \mathbb{R}^n \to \mathbb{R}^n, n \ge 2$, is K-quasiconformal, then f is $\eta_{K,n}$-quasisymmetric with

$$(1.35)\qquad \begin{cases} \eta_{K,n}(1) \le \exp(6(K+1)^2\sqrt{K-1}), & \\ \eta_{K,n}(t) \le \eta_{K,n}(1)\varphi_{K,n}(t), & 0 < t < 1, \\ \eta_{K,n}(t) \le \eta_{K,n}(1)/\varphi_{1/K,n}(1/t), & t > 1. \end{cases}$$

Here $\varphi_{K,n}(t)$ is the distortion function in the quasiconformal Schwarz lemma (cf. Theorem 2.28) with

$$(1.36)\qquad \lambda_n^{1-\beta} r^\beta \le \varphi_{1/K,n}(r) \le \varphi_{K,n}(r) \le \lambda_n^{1-\alpha} r^\alpha\,,$$

$\alpha = K^{1/(1-n)} = 1/\beta,\ \lambda_n \in [4, 2e^{n-1})$. A K-quasiconformal map of B^n need not be quasisymmetric, but its restriction to $\overline{B}^n(s),\ s \in (0,1)$, is quasisymmetric. In fact, P. Seittenranta [Se2] proved that for prescribed $K > 1$ and $n \ge 2$, there exists an explicit $s \in (0,1)$ such that $f|\overline{B}^n(s)$ is $\overline{\eta}_{K,n}$-quasisymmetric where $\overline{\eta}_{K,n}$ is of the same type as in (1.35).

1.37. Linear approximation property. Our examples of quasicircles in 1.18 show that quasicircles need not have tangents at any point. On the other hand, when $K \to 1$, we expect that K-quasicircles become more like usual circles. We next introduce a definition which enables us to quantify such a passage to the limit:

Given integers $n \ge 2,\ p \in \{1, ..., n-1\}$, and positive numbers $r_0 > 0,\ \delta \in (0,1)$, we say that a compact set $E \subset \mathbb{R}^n$ satisfies the *linear approximation property* with parameters (p, δ, r_0) if for every $x \in E$ and every $r \in (0, r_0)$ there exists a p-dimensional hyperplane $V_r \ni x$ such that

$$E \cap B^n(x,r) \subset \{w \in \mathbb{R}^n :\ d(w, V_r) \le \delta r\}.$$

P. Mattila and M. Vuorinen proved in 1990 [MatV] that quasispheres satisfy this property.

1.38. Theorem. *Let $K_2 > 1$ be such that*

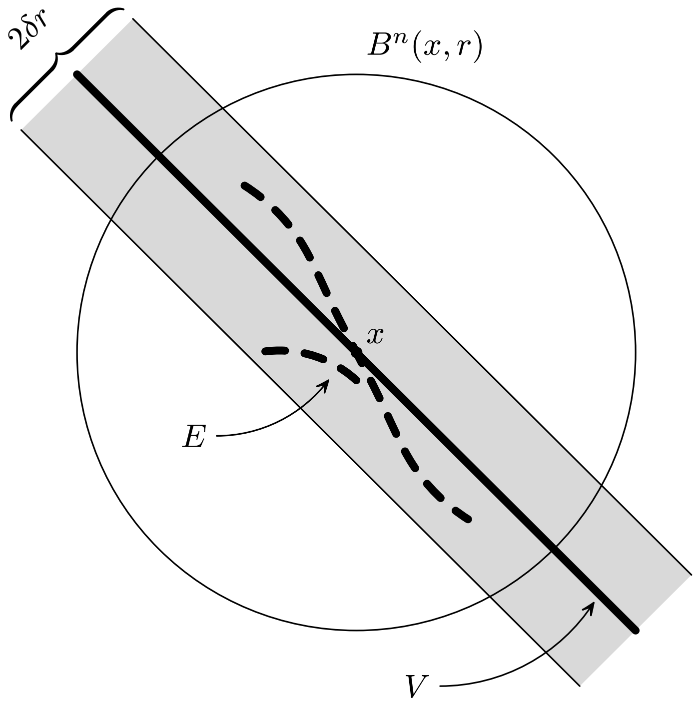
$$c = \eta_{K,n}(1)^{-2}/2 > 15/32$$

for all $K \in (1, K_2]$. Then a K-quasisphere $E = fS^{n-1}$ satisfies the linear approximation property with parameters

$$(n-1, 4g(K), d(E)g(K)), \text{ where } g(K) = \sqrt{1-2c}\,. \tag{1.39}$$

Observe that here $\delta = 4g(K) \to 0$ as $K \to 1$.

This limit behavior shows that, the closer $K-1$ is to 0, the better K-quasispheres can be locally approximated by $(n-1)$-dimensional hyperplanes. Note that at a point $x \in E$ the approximating hyperplanes V_r may depend on r: they will very strongly depend on r if x is a "bad" point. An example of such bad behavior is a quasicircle which logarithmically spirals in a neighborhood of a point x.



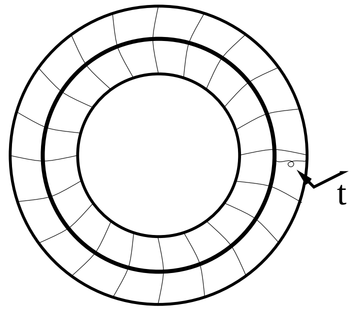
1.40. Jones' β-parameters. In the same year as [MatV] appeared, P. Jones [Jo] introduced "β-parameters" for the analysis of geometric properties of plane sets. In fact, the particular case $n = 2$, $p = 1$, of the linear approximation property is very close to the condition used by Jones in his investigations. Later on, Jones' β-parameters were used extensively by C. Bishop - P. Jones [BJ1], G. David - S. Semmes [DS], K. Okikiolu [Ok], and H. Pajot [Paj].

1.41. Open problem. For $n = 2$ the parameter δ of the linear approximation property in (1.39) is roughly $\sqrt{K-1}$. Can this be reduced, say to $K-1$, when K is close to 1?

1.42. Open problem. The Hausdorff dimension of a K-quasicircle has a majorant of the form $1 + 10(K-1)^2$ (see [BP2], [MatV, 1.8]). Is there a similar bound for the Hausdorff dimension of a K-quasisphere in $\mathbb{R}^n$, e.g. in the form $n - 1 + c(K-1)^2$ where c is a constant?

1.43. Rectifiability of quasispheres. Snowflake-type quasicircles provide examples of locally nonrectifiable curves. We now briefly review conditions under which quasicircles will be rectifiable. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal and $t \in (0, 1/2)$, then for convenience of notation we set

$$K(t) = K(f|A(t)),\ A(t) = \bigcup_{x \in S^{n-1}} B^n(x,t)\,. \tag{1.44}$$



A natural question is this: Does $K(t) \to 1$ as $t \to 0$ imply that fS^{n-1} is rectifiable? For $n = 2$, J. Becker and Ch. Pommerenke [BP1] have shown that the answer is in the negative. Imposing a stronger condition for the convergence $K(t) \to 1$, we have a positive result [MatV]:

1.45. Theorem. *If*

$$\int_0^{1/2} \frac{1-\alpha(t)}{t} dt < \infty,\ \alpha(t) = K(t)^{1/(1-n)}\,, \tag{1.46}$$

then fS^{n-1} is rectifiable.

An alternative proof of Theorem 1.45 was given by Yu. G. Reshetnyak in [Re2, pp. 378-382]. For some related results see also [GuV]. For $n = 2$ one can replace condition (1.46) by a slightly weaker one, as shown in [ABL], [Carle].

1.47. Quasiconformal maps of S^{n-1}. Many of the peculiarities of quasiconformal maps exhibited above are connected with the interesting geometric structure of quasispheres. We will now briefly discuss the simplest case when $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal map with $fS^{n-1} = S^{n-1}$. Let $g = f|S^{n-1}$. Then $H(x, g) \le H(x, f)$ for every $x \in S^{n-1}$. By the alternative characterization mentioned in 1.8, we see that if $n-1 \ge 2$, then g is quasiconformal [note: we have not defined quasiconformality in dimension 1]. Thus for $n \ge 3$ the restriction g satisfies all the properties of a quasiconformal map. In particular, g is absolutely continuous with respect to $(n-1)$-dimensional Hausdorff measure on S^{n-1}. For $n = 2$ the situation is drastically different, as the following result of Beurling and Ahlfors shows.

1.48. Beurling - Ahlfors' singular function. In [BAh] Beurling and Ahlfors constructed a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ satisfying the condition (1.33) for some $M > 1$ such that h is not absolutely continuous with respect to 1-dimensional Lebesgue measure. By their extension result mentioned in 1.32, h is the restriction of a quasiconformal mapping h^* of $\mathbb{R}^2$. If g is a Möbius transformation with $g(S^1) = \mathbb{R}$, then the conjugation $g^{-1} \circ h^* \circ g$ is the required counterexample.

1.49. Tukia's quasisymmetric function. Answering a question of W.K. Hayman and A. Hinkkanen, P. Tukia constructed in [Tu3] an example showing that a quasisymmetric map f of $\mathbb{R}$ can map a set E, with H-dim $E < \varepsilon$ onto a set with H-dim$(\mathbb{R} \setminus fE) < \varepsilon$. See also [BS] and [Ro].

1.50. Thick sets. We conclude this section with a discussion of a property opposite to the linear approximation property. Let $c > 0,\ p \in \mathbb{N}$. We say that $A \subset \mathbb{R}^n$ is (c,p)-*thick* if, for every $x \in A$ and for all $r \in (0, d(A)/3)$, there exists a p-simplex Δ with vertices in $A \cap B^n(x,r)$ with $m_p(\Delta) \geq cr^p$ [VVW], [V5].

Snowflake-type curves are examples of $(c,2)$-thick curves. One can even show that for every $K > 1$ there are $(\frac{\sqrt{K}-1}{768}, 2)$-thick K-quasicircles. For this purpose one uses a snowflake-style construction, but replaces the angles $\frac{\pi}{3}$ by smaller ones that tend to 0 as $K \to 1$ [VVW].

A condition similar to thickness is the notion of wiggly sets [BJ2].

1.51. Open problem. Are there quasispheres in $\mathbb{R}^n,\ n \geq 3$, which are (c,n)-thick for some $c > 0$?

1.52. Open problem. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism and let fS^{n-1} be (c_1, n)-thick. Is it true that

$$\text{H-dim}(fS^{n-1}) \geq n - 1 + c_2 > n - 1?$$

For $n = 2$ the answer is known to be in the affirmative [BJ2].

1.53. Additional references. The change of Hausdorff dimension under quasiconformal maps has been studied recently in [IM2] and [Ast]. A subclass of quasicircles, so-called *asymptotically conformal curves*, has been studied, for instance, in [BP1], [ABL], [GuR].

1.54. More open problems. Some open problems can be found in [Vu2, p. 193], [AVV3].

1.55. Books. The existing books on quasiconformal maps include [Car], [KK], [L], [LV], [V1]. Generalizations to the case of noninjective mappings, so-called *quasiregular mappings*, are studied in [HKM], [I1], [IM2], [Re2], [Ri], [V2], [Vu2].

2 Conformal invariants and special functions

In this section we try to answer some fundamental questions such as:

a. Why are conformal invariants used in geometric function theory?

b. Why are special functions important for conformal invariants?

c. What are some of the open problems of the field?

In what follows we will provide some answers to these questions, as well as pointers to the literature for further information. In a nutshell our answer to "a" is provided by the developments in geometry and analysis that emerged from Klein's Erlangen Program and to "b" by the fact that the solution to some conformally invariant extremal problems involve special functions.

2.1. Klein's Erlangen Program. The genesis of F. Klein's Erlangen Program is attached usually to the year 1872 when Klein became a professor at the University of Erlangen. In this program, the idea of using group theory to study geometry was crystallized into a form where the following conceptions played a crucial role

- use of isometries ("rigid motions") and invariants

- two configurations are regarded equivalent if one can be carried to the other by a rigid motion (group element)

- the basic "models" of geometry are

(a) Euclidean geometry

(b) hyperbolic geometry (Bolyai-Lobachevskii)

(c) spherical geometry

The main examples of rigid motions are provided by various subgroups of Möbius transformations of $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. The group of Möbius transformations is generated by reflections in $(n-1)$-dimensional spheres and hyperplanes.

2.2. Geometric invariants. In each of the models of Klein's geometries, there are natural metrics that are invariant under "rigid motions". For spherical geometry, such a metric is the chordal metric, defined in terms of the stereographic projection $\pi : \overline{\mathbb{R}}^n \to S^{n-1}(\frac{1}{2}e_n, \frac{1}{2})$ by

$$q(a,b) = \frac{|a-b|}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}} = |\pi a - \pi b|\,,$$

$$\pi x = e_{n+1} + (x - e_{n+1})/|x - e_{n+1}|^2,$$

for $a, b, x \in \mathbb{R}^n$. The absolute (cross) ratio is defined as follows

$$|a,b,c,d| = \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)} = \frac{|a-c||b-d|}{|a-b||c-d|}.$$

Its most important property is invariance under Möbius transformations.

We shall next consider a few examples of geometric invariants in the sense of Klein.

2.3. Hyperbolic geometry. For distinct points $a, b \in B^n$ let $a_*,\ b_* \in \partial B^n$ be distinct points such that the quadruple a_*, a, b, b_* can be moved by a rigid motion T (=Möbius selfmap of B^n) to $(-e_1, 0, \lambda e_1, e_1)$, $\lambda \in (0,1)$. Then $T^{-1}(-e_1, e_1)$ is an arc of an orthogonal circle through a and b. We define the *hyperbolic metric* ρ by

$$\rho(a,b) = \log|a_*, a, b, b_*|. \tag{2.4}$$

By Möbius invariance of the absolute ratio we see that ρ is invariant under Möbius selfmaps of B^n.

In addition to a metric, another fundamental notion of hyperbolic geometry is the hyperbolic volume of a polyhedron. For $n = 2$ and $a, b, c \in B^2$ let α, β, γ be the angles of a triangle with vertices a, b, c. Then the *hyperbolic area* of the triangle is

$$v(a,b,c) = \pi - (\alpha + \beta + \gamma). \tag{2.5}$$

Also hyperbolic area is invariant under Möbius selfmaps of B^2.

A natural question is whether it is possible to define similar geometries in domains not homeomorphic to B^n. In every proper subdomain of $\mathbb{R}^n$ one can define the *absolute ratio metric* by

$$\begin{cases} \delta_G(a,b) = \log(1 + r_G(a,b)) \\ r_G(a,b) = \sup\{|a,c,b,d| : c, d \in \partial G\} \end{cases} \tag{2.6}$$

(see [Se1]). Clearly this is a Möbius-invariant metric, and it can be shown that $\delta_{B^n} \equiv \rho$ for $G = B^n$. Some of the basic properties of δ_G are proved in [Se1]. Another metric, the so-called *Apollonian metric*, defined by

$$\alpha_G(a,b) = \sup\{\log |c,a,b,d| : c,d \in \partial G\} \tag{2.7}$$

was studied recently in [Be] and [Se1] (strictly speaking, α_G is only a pseudometric).

2.8. Origin of quasiconformal maps. Klein's Erlangen Program received wide acclaim, and similar ideas proved fruitful also in geometric function theory. H.A. Schwarz (Schwarz lemma), H. Poincaré, and C. Carathéodory were some of the eminent promoters of these ideas.

It is in this stage of the mathematical evolution that H. Grötzsch wrote his now famous 1928 paper which was to become the first paper on plane quasiconformal maps. It is sometimes pointed out that, a century earlier in his theory of surfaces, Gauss had studied notions that were close to quasiconformal maps. One of the important tools introduced by Grötzsch was a new conformal invariant, the modulus of a quadrilateral. Remarkable progress took place in 1950 when L. Ahlfors and A. Beurling found a new conformal invariant, the extremal length of a curve family (cf. 1.1) which soon became a popular tool in geometric function theory ([G6], [J1], [Kuz], [Rod]). Higher-dimensional quasiconformal maps entered the stage first in a note by M.A. Lavrentiev in 1938 but the systematic study was started only in 1959 by C. Loewner, F.W. Gehring, B. Shabat, and J. Väisälä.

2.9. Liouville's theorem. Soon after the publication of Riemann's famous mapping theorem concerning conformal maps of simply-connected plane domains, Liouville proved that, in striking contrast to the two-dimensional case, the only C^3 conformal maps of subdomains of $\mathbb{R}^n, n \geq 3$, are restrictions of Möbius transformations. Under weaker differentiability hypotheses this result was proved by F. W. Gehring [G2] and Yu. G. Reshetnyak [Re1]. See also B. Bojarski and T. Iwaniec [BI1]. Yu. G. Reshetnyak has created so-called *stability theory,* which is a study of properties of K-quasiconformal and K-quasiregular maps with small $K-1$. The main goal of this theory is to find quantitative ways to measure the distance of these mapping classes from Möbius maps. The fundaments of this theory are presented in [Re2]. In spite of the many results in [Re2], some very basic questions are still open, see 2.11 below. Significant results on stability theory were proved by V. I. Semenov [Sem1], [Sem2] and others.

2.10. Main problem of quasiconformal mapping theory. We recall from Section 1 the definition of a K-quasiconformal map $f : G \to G'$ where G and G' are domains in $\overline{\mathbb{R}}^n$: a homeomorphism f is K-quasiconformal if, for all curve families Γ in G',

$$(*) \quad M(f\Gamma)/K \leq M(\Gamma) \leq KM(f\Gamma).$$

A main problem of quasiconformal mapping theory in $\mathbb{R}^n$ is how to extract explicit "geometric information" from (*) preserving "asymptotic sharpness" as $K \to 1$.

It should be noted that the vast majority of results on quasiconformal mappings are not sharp in this sense. Of the results below, the Schwarz lemma 2.28 is an example of an asymptotically sharp result.

We next list three simple ideas that might be used when studying this problem.

Idea 1. Use "canonical situations", where the modulus of Γ can be computed explicitly, as comparison functions.

Idea 2. Idea 1 and the basic inequality (*) lead to (nonlinear) constraints which we need to simplify.

Idea 3. Try to relate (*) to "geometric notions" distances, metrics, etc. This leads to conformally invariant extremal problems, whose solutions can often be expressed in terms of *special functions*.

2.11. Open problem. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $K-$quasiconformal map normalized by $f(0) = 0$, $f(e_1) = e_1$ and let I stand for the class of all isometries h of $\mathbb{R}^n$ with $h(0) = 0$, $h(e_1) = e_1$. Find an explicit and concrete upper bound for

$$\varepsilon(K, n) \equiv \inf_{A \in I} \sup\{|f(x) - A(x)| : |x| \le 1\}$$

such that the bound tends to 0 when $K \to 1$.

2.12. Canonical ring domains. There are two ring domains in $\mathbb{R}^n$ whose capacities are frequently used as comparison functions. These canonical ring domains are the Grötzsch ring $R_G(s),\ s > 1$, with complementary components $\overline{B}^n$ and $\{te_1 : t \ge s\}$ and the Teichmüller ring $R_T(t),\ t > 0$, with complementary components $[-e_1, 0]$ and $[te_1, \infty)$. For the Grötzsch (Teichmüller) ring the capacity is the modulus of the curve family joining the complementary components, denoted by $\gamma_n(s)$ and $\tau_n(t)$, respectively. These capacities are related by

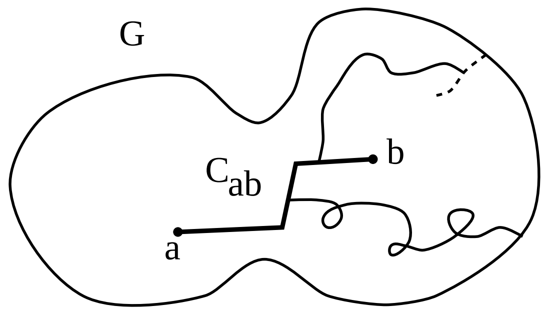
$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1) \tag{2.13}$$

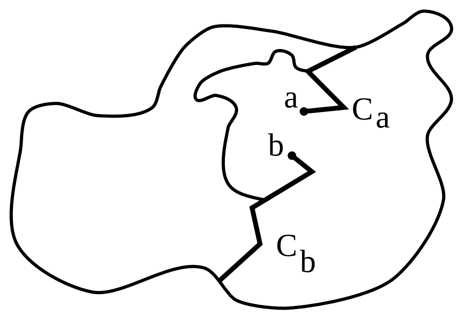
for $s > 1$. There are several estimates for $\gamma_n(s)$ and $\tau_n(t)$, for all $n \ge 3$; see [G1], [A1], [AVV6], [Vu2, Section 7]. When $n = 2$ both functions can be expressed in terms of elliptic integrals; see (2.25) and (2.26) below.

2.14. Conformal invariants μ_G and λ_G. Since we are seeking invariant formulations, the absolute ratio is a natural tool. Another possibility is to use point-pair invariants of a domain $G \subset \mathbb{R}^n$, such as $\mu_G(a, b)$ or $\lambda_G(a, b)$, $a, b \in G$, defined as follows

$$\begin{cases} \mu_G(a,b) = \inf_{C_{ab}} M(\Delta(C_{ab}, \partial G; G)), \\ \lambda_G(a,b) = \inf_{C_a, C_b} M(\Delta(C_a, C_b; G)), \end{cases} \tag{2.15}$$

where the infima are taken over all continua C_{ab} (pairs of continua C_a, C_b)) in G joining a and b (a to ∂G and b to ∂G, resp.) Both μ_G and λ_G are solutions of the respective conformal invariant extremal problems, and they both have proved to be efficient tools in the study of distortion theory of quasiconformal maps [LF], [Vu2]. Both μ_G and $\lambda_G^{-1/n}$ ([LF]) are metrics for most subdomains of $\mathbb{R}^n$ (for μ_G we must require that cap $\partial G > 0$ and, for λ_G, that card $(\overline{\mathbb{R}}^n \setminus G) \ge 2$).





2.16. Bounds for μ_G and λ_G. When comparing the mutual advantages of the absolute ratio and the extremal quantities (2.15) we note that the former is more explicit. On the other hand—and this is the most important property of μ_G and λ_G—the transformation rules of μ_G and λ_G under K-quasiconformal maps of the proper subdomain G of $\overline{\mathbb{R}}^n$ are simple: quasiconformal maps are bilipschitz in the respective metrics.

2.17. Theorem (Transformation rules). *If $f : G \to G'$ is K-quasiconformal, then*

$$(1)\ \mu_{fG}(f(a), f(b))/K \le \mu_G(a,b) \le K\mu_{fG}(f(a), f(b))\,,$$

$$(2)\ \lambda_{fG}(f(a), f(b))/K \le \lambda_G(a,b) \le K\lambda_{fG}(f(a), f(b))\,,$$

for all $a, b \in G$.

For all applications of these transformation rules we need estimates or explicit formulas for μ_G and λ_G. Below we review what is currently known about these invariants and point out some open problems. Some applications of these invariants are given in [F1] -[F4], [Pa1], [Pa2], [Vu2], [AVV6]. Note that the important Schwarz lemma for quasiconformal maps, Theorem 2.28, follows from Theorem 2.17.

If $G = B^n$, then there is an explicit formula for μ_{B^n} as well as one for λ_{B^n} [Vu2]. For $n = 2$, $G = \mathbb{R}^2 \setminus \{0\}$, there is an explicit formula for λ_G which follows easily from the formula for the solution of the Teichmüller extremal problem [Kuz, p. 192].

Next, the general chart of inequalities among various metrics is given in [Vu1]. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal and $G = fB^n$, then there are upper and lower bounds for both λ_G and μ_G [Se1]. Next, if $G = \mathbb{R}^n \setminus \{0\}$, then there are upper and lower bounds for λ_G [Vu2, Section 8]. If ∂G is uniformly perfect, then there are lower bounds for μ_G in terms of the metric δ_G in (2.6)—such a bound can be derived from the results of [JV]. Finally, for $G = B^2 \setminus \{0\}$ there are upper and lower bounds for λ_G [LeVu].

2.18. Open problem. Find an explicit formula for $\lambda_{B^2\setminus\{0\}}$. Improve the upper and lower bounds for λ_G, $G = \mathbb{R}^n \setminus \{0\}$.

2.19. Remark. J. Ferrand proved in [LF] that $\lambda_G^{-1/n}$ is a metric. In [AVV3] it was shown that $\lambda_{B^n}(x,y)^{1/(1-n)}$ is a metric and asked whether $\lambda_{B^n}(x,y)^{1/(1-n)}$ is a metric for more general domains. Affirmative solutions were subsequently found by A. Yu. Solynin [Sol], J. Jenkins [J2], and J. Ferrand [F3].

2.20. Lipschitz conditions with respect to μ_G and λ_G. The transformation rules of Theorem 2.17 are just special case of the more general inequality (*). However, in many cases it is enough to use Theorem 2.17 instead of (*). Therefore, the following question is natural. Consider homeomorphisms $f : B^n \to f(B^n) = B^n$ satisfying the property 2.17(1) (or 2.17(2)). Are such maps quasiconformal? This question was raised by J. Ferrand [LF] and a negative answer was given in [FMV], where it was also shown that such maps are Hölder-continuous.

2.21. Heuristic principle. The practitioners of quasiconformal mapping theory have observed the following heuristic principle: estimates for the modulus of a curve family associated with a geometric configuration often lead to information about quasiconformal mappings. Unfortunately explicit formulas are available only in the simplest cases. Symmetrization has proved to be a very useful method for finding lower bounds for the solutions of extremal problems such as the minimization of the capacities of some suitable class of ring domains; see [Ba2], [Ba3], [Dub], [SolV]. It should be noted that for dimensions $n \geq 3$ there is not even a simple algoritm for the numerical computation of the Grötzsch capacity $\gamma_n(s)$. For $n = 3$ some computations were carried out [SamV]. For the dimension $n = 2$ there is an explicit formula for the Grötzsch capacity in terms of elliptic integrals, as we shall see below. We shall also see that the same special functions will occur in several function-theoretic extremal problems, and some most beautiful identities for these special functions can be derived from Ramanujan's work on modular equations.

In harmony with the above heuristic principle we now start a review of special functions that will occupy a considerable part of Sections 2 and 3.

2.22. Hypergeometric functions. For $a, b, c \in \mathbb{R},\ c \neq 0, -1, -2,..$ the (Gaussian) *hypergeometric function* is defined by the series

$$F(a,b;c;r) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} r^n$$

for $|r| < 1$, where $(a,0) = 1,\ (a,n+1) = (a,n)(a+n),\ n = 0,1,2,\ldots$. The hypergeometric function, one of the most important special functions, was studied extensively by several eminent nineteenth century mathematicians such as K.F. Gauss, E. Kummer, B. Riemann, H.A. Schwarz, E. Goursat, and F. Klein [Ask1], [Dut], [Kl2]. Its importance is, in part, connected with its numerous particular cases: there are lists in [PBM] with hundreds of special cases of $F(a,b;c;r)$ for rational triples (a,b,c). Another reason for the importance of $F(a,b;c;r)$ is its frequent occurrence in several different contexts in the 1990's, see [Ao], [Ask2], [CC], [CH], [DM], [GKZ], [Var], [Va], [WZ1],

[WZ2]. For our purposes, the main particular case of the hypergeometric function is the *complete elliptic integral* $\mathcal{K}(r)$ [AS], [C3], [WW]

$$\mathcal{K}(r) = \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}; 1; r^2),\ 0 \le r < 1. \tag{2.23}$$

2.24. Conformal map onto a disk minus a radial slit. A conformal mapping of a concentric annulus onto a disk minus a radial segment starting from the origin is provided by an elliptic function. The length of such a segment depends on the ratio of the radii in a nonelementary fashion. In fact, if the inner and outer radius of the annulus are $t \in (0,1)$ and 1, then the length $r \in (0,1)$ of the radial segment satisfies the following transcendental equation, obtained by equating the capacities of these two ring domains;

$$\frac{2\pi}{\log \frac{1}{t}} = \frac{2\pi}{\mu(r)};\ \mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \tag{2.25}$$

where $r' = \sqrt{1-r^2}$ and we set $\mu(1) = 0$. For $n = 2$ the Grötzsch capacity can be expressed as

$$\gamma_2(s) = 2\pi/\mu(1/s),\ \ s > 1. \tag{2.26}$$

2.27. Schwarz lemma for quasiconformal maps. The Schwarz lemma for analytic functions is one of the basic results of complex analysis. A counterpart of this result also holds for quasiconformal maps in the following form.

2.28. Theorem. *Let $f : B^n \to fB^n \subset B^n$ be K-quasiconformal and $f(0) = 0$. Then, for $x \in B^n$,*

$$(1)\ |f(x)| \le \varphi_{K,n}(|x|) \le \lambda_n^{1-\alpha}|x|^\alpha,\ \ \alpha = K^{1/(1-n)},$$

$$(2)\ |f(x)| \le \psi_{K,n}(|x|) \equiv \sqrt{1 - \varphi_{1/K,n}(\sqrt{1-|x|^2})^2},$$

where $\varphi_{K,n}(r) \equiv 1/\gamma_n^{-1}(K\gamma_n(1/r))$ and $\varphi_{K,2}(r) = \mu^{-1}(\mu(r)/K)$. If, moreover, $fB^n = B^n$, then

$$(3)\ |f(x)| \ge \varphi_{1/K,n}(|x|) \ge \lambda_n^{1-\beta}|x|^\beta,$$

$$(4)\ |f(x)| \ge \psi_{1/K,n}(|x|).$$

Note that in Theorem 2.28 both (1) and (2) are asymptotically sharp when $K \to 1$. Here $\lambda_2 = 4$, $\lambda_n \in [4, 2e^{n-1})$ is a constant [A2]. It can be shown that, in (1) and (2), $\varphi_{K,n}(r)$ and $\psi_{K,n}(r)$ are different for $n > 2$ and identically equal for $n = 2$.

2.29. Corollary. *If $f : B^n \to fB^n \subset B^n$ is K-quasiconformal, then, for all $a, b \in B^n$,*

$$\tanh \frac{\rho(f(a), f(b))}{2} \le \varphi_{K,n}(\tanh \frac{\rho(a,b)}{2}).$$

It should be observed that in Corollary 2.29 we do not require the normalization $f(0) = 0$. Corollary 2.29 can be extended also to domains G of the form $f_1 B^n$ where $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ is K_1-quasiconformal [Se1].

2.30. Open problem. Let $f : B^2 \to B^2 = fB^2$ be K-quasiconformal, $a, b, c \in B^2$ and α, β, γ the angles of the hyperbolic triangle with vertices a, b, c. Find bounds for the hyperbolic area of the triangle with vertices $f(a), f(b), f(c)$.

2.31. Quasisymmetric maps $n = 2$. We recall that quasisymmetric maps already were defined and briefly discussed in Section 1, where we pointed out that quasiconformal maps of $\mathbb{R}^n$ are η-quasisymmetric with an explicit $\eta_{K,n}$ given there. For $n = 2$ one can sharpen this result considerably, since a simple expression for $\eta_{K,2}$ is available by the following result of S. Agard [Ag].

2.32. Theorem. *A K-quasiconformal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $\eta_{K,2}$-quasisymmetric with*

$$\eta_{K,2}(t) = \frac{u^2}{1-u^2}; \; u = \varphi_{K,2}\left(\sqrt{\frac{t}{1+t}}\right).$$

Some sharp growth estimates for quasisymmetric maps were found by J. Zając in [Za1], [Za2] in terms of the function $\varphi_{K,2}(r)$. A related topic is the so-called Douady-Earle extension problem, where sharp bounds were recently found by D. Partyka [Par3] in terms of the function $\varphi_{K,2}(r)$. The function $\varphi_{K,2}$ satisfies many inequalities, which are sometimes used in these studies. Some inequalities are given, e.g., in [AVV2], [QVV2], [QVu1].

2.33. Schottky's theorem. Schottky's classical result asserts the existence of a function $\psi : (0,1) \times (0,\infty) \to (0,\infty)$ such that

$$\sup\{|f(z)| : \; |z| = r, \; f \in \mathcal{A}(t)\} \equiv \psi(r,t)\,,$$

where

$$\mathcal{A}(t) = \{f : B^2 \to \mathbb{R}^2 \setminus \{0,1\} : f \text{ analytic}, |f(0)| = t\}.$$

Numerous explicit bounds for $\psi(r,t)$ have been found. W.K. Hayman proved that

$$\log \psi(r,t) \le (\pi + \log^+ t)\frac{1+r}{1-r}, \tag{2.34}$$

and J. Hempel [Hem1], [Hem2] proved, using some results of S. Agard [Ag], that

$$\psi(r,t) = \eta_{M,2}(t)\,, M = \frac{1+r}{1-r}. \tag{2.35}$$

G. Martin [Ma] found a new proof of (2.35) based on holomorphic motions. One can use Theorem 2.32 and (2.35) to find sharper forms of Hayman's result (2.34) as shown in [QVu2]. In [QVu2] references to related work by Jenkins, Lai, and Zhang are given. Perhaps the best explicit estimate known today is due to S.-L. Qiu [Q3]: with $B = \exp(2\mu(1/\sqrt{1+t}))$,

$$16\eta_{K,2}(t) \le \min\{16t + B^K - B, \; (16t+8)^K - 8\}, \tag{2.36}$$

for all $t \ge 0$, $K \ge 1$. Here equality holds if $K = 1$ or $t = 0$.

2.37. Implementation of the heuristic principle. We now give an explicit example of the implementation of our heuristic principle to the Schwarz lemma 2.28. Indeed, the property that $\mu(r) + \log r$ is monotone decreasing on $(0,1)$ implies the upper bound $\varphi_{K,2}(r) \le 4^{1-1/K} r^{1/K}$ for $K > 1$,

$r \in (0,1)$ (this is the bound in Theorem 2.28(1) with $n = 2$). Several other inequalities for $\varphi_{K,2}(r)$ can be proved in the same way, if one uses other monotone functions involving $\mu(r)$ in place of $\mu(r) + \log r$. A natural question is now: how do we find such monotone functions? There is no simple answer to this question beyond the obvious one: by studying $\mathcal{K}(r)$ and related functions (recall that $\mu(r) = \pi\,\mathcal{K}(r')/(2\,\mathcal{K}(r))$). Many monotonicity properties of $\mathcal{K}(r)$ were found in the 1990's with help of ad hoc techniques from classical analysis. What is missing is a unified approach for proving such monotonicity results. Since the publication of [AVV2] in 1988 many inequalities and properties for $\varphi_{K,n}(r)$ were obtained; see e.g. [QVV1], [QVV2], [QVu1], [QVu2], [Par1]-[Par3], [Q3], [QV], [Za1]-[Za2]. These papers solve many of the open problems stated in [AVV3], [AVV5], and elsewhere. The above heuristic principle has also found many applications there. Another application of this principle occurs in [AVV1], where it was shown that, for dimensions $n \geq 3$, one can prove many results for quasiconformal maps in a dimension-free way.

2.38. Open problem ([QVV2]). Show, for fixed $K > 1$, $K \neq 2$, that the function

$$g(K,r) = \frac{\text{artanh}\varphi_{K,2}(r)}{\text{artanh}(r^{1/K})}$$

is monotone from $(0,1)$ onto $(c(K), d(K))$ with $c(K) = \min\{K, 4^{1-1/K}\}, d(K) = \max\{K, 4^{1-1/K}\}$. Note that $g(2,r) \equiv 2$ since $\varphi_2(r) = 2\sqrt{r}/(1+r)$.

2.39. Mori's theorem. A well-known theorem of A. Mori [Mor1] states that a K-quasiconformal map f of the unit disk B^2 onto itself, normalized by $f(0) = 0$, satisfies

$$|f(x) - f(y)| \leq M|x-y|^{1/K}\,, \tag{2.40}$$

for all $x, y \in B^2$ where $M = 16$ is the smallest constant independent of K. Clearly, this result is far from sharp if K is close to 1. O. Lehto and K. I. Virtanen [LV] asked whether (2.40) holds with the constant $M = 16^{1-1/K}$. This problem has been studied by several authors, including R. Fehlmann and M. Vuorinen [FV] (the case $n \geq 2$), V. I. Semenov [Sem1], S.-L. Qiu [Q1], G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen [AVV4]. See also [BP3]. Currently it is known that we can choose $M \leq 64^{1-1/K}$ [AVV4, 5.8]. Estimates of the function $\varphi_K(r)$ play a crucial role in such studies.

2.41. Open problem (from [LV]). Show that inequality (2.40) holds with $M \leq 16^{1-1/K}$. (Even the particular case when the points x and y are on the same radius is open.)

Our next goal is to describe the classical method of computing $\mathcal{K}(r)$ in terms of the arithmetic-geometric mean. This procedure naturally brings forth the question of finding inequalities for $\mathcal{K}(r)$ in terms of mean values, which we will also touch upon.

2.42. Arithmetic-geometric mean. For $x, y > 0$, the *arithmetic* and *geometric means* are denoted by

$$A(x,y) = (x+y)/2,\ G(x,y) = \sqrt{xy},$$

respectively, and the *logarithmic mean* is defined by

$$L(x,y) = \frac{x-y}{\log(x/y)},\ x \neq y,\ L(x,x) = x.$$

Next, for $a > b > 0$ let

$$a_0 = a,\ b_0 = b,\ a_{n+1} = A(a_n, b_n),\ b_{n+1} = G(a_n, b_n).$$

Then $a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$ and

$$AG(a,b) \equiv \lim a_n = \lim b_n$$

is the *arithmetic-geometric mean* of a and b. See [BB1], [AlB], [ACJP]. Recently, the arithmetic-geometric mean has been studied, in particular, in connection with the high-precision computations of the decimal places of π [BBBP], [Lei]. The next theorem was proved by Lagrange and Gauss (independently) some time between 1785 and 1799; see [C1] and [Co].

2.43. Theorem. *For $r \in (0,1)$, $r' = \sqrt{1-r^2}$, we have*

$$\mathcal{K}(r) = \frac{\pi}{2AG(1,r')}.$$

For the approximation of $\mathcal{K}(r)$ in terms of mean values it will be expedient to have notation as follows:

$$M_t(x,y) = M(x^t, y^t)^{1/t},\ t > 0, \tag{2.44}$$

for the modification of the means $M = A, G, L, AG$. These increase with t. There are numerous inequalities among the above mean values, see [BB1], [BB2], [C2], [CV], [VV1], [Sán]. The inequality $L(x,y) \leq AG(x,y)$ for $x, y > 0$ occurs in [CV]. In the opposite direction, the following theorem was proved by J. and P. Borwein in 1994 [BB2].

2.45. Theorem. $AG(x,y) \leq L_{3/2}(x,y)$ *for* $x, y > 0$.

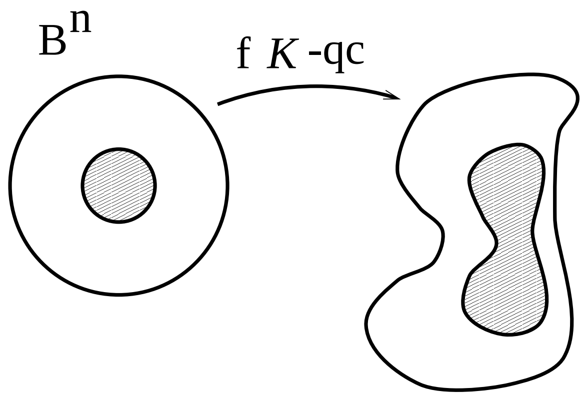
Some approximations for $\mathcal{K}(r)$ in terms of elementary functions can be obtained if we use Theorem 2.45 or Theorem 2.43 and carry out a few steps of the AG-iteration. In the next theorem two such approximations are given. Part (1) is due R. Kühnau [Küh3], and part (2) is due to B.C. Carlson and J.L. Gustafson [CG1]. See also [QV].

2.46. Theorem. *For $r \in [0,1)$ we have*

$$(1)\ \ \mathcal{K}(r) > \frac{9}{8+r^2} \log \frac{4}{r'},$$

$$(2)\ \ \mathcal{K}(r) < \frac{4}{3+r^2} \log \frac{4}{r'}.$$

For some recent inequalities for elliptic integrals, see [AQV].



2.47. Open problem (from [Vu2, p. 193]) . Prove or disprove the following assertion. For each $n \ge 2$, $r \in (0,1)$, and $K \ge 1$ there exists a number $d(n,K,r)$ with $d(n,K,r) \to d(n,K)$ as $r \to 0$ and $d(n,K) \to 1$ as $K \to 1$ such that whenever $f : B^n \to \mathbb{R}^n$ is K–qc, then $fB^n(r)$ is a $d(n,K,r)$–quasiball. More precisely, the representation $fB^n(r) = gB^n$ holds where $g : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is a $d(n,K,r)$–qc mapping with $g(\infty) = \infty$. (Note: It was kindly pointed out by J. Becker that we can choose $d(2,1,r) = (1+r)/(1-r)$ either by [BC, pp. 39–40] or by a more general result of S. L. Krushkal' [KR].)

Here we want an explicit constant $d(n,K)$—the existence follows e.g. from the work of Tukia and Väisälä [TV2]. An affirmative solution to this local structure problem would have interesting applications.

3 Recent results on special functions

In this last section we shall mainly discuss recent results related to special functions. We also mention a few geometric questions on quasiconformal maps.

Some of the results below are related to the work of the Indian mathematical genius S. Ramanujan 1887-1920. His published work has had a deep impact on number theory, combinatorics, and special functions (Hardy, Selberg, Dyson, Deligne).

Ramanujan left numerous unpublished results in his notebooks at the time of his premature death. It is estimated that the total number of his results is 3000-4000. The publication of the edited notebooks with reconstructed proofs by B. Berndt in 1985-1996 (5 volumes) made these results widely accessible. B. Berndt was awarded Steele Prize for Mathematical Exposition in 1996 for this extraordinary achievement.

3.1. Asymptotic behavior of hypergeometric functions . The behavior of the hypergeometric function $F(a,b;c;r)$, $a,b,c > 0$, at $r = 1$ can be classified into three cases:

Case A. $c > a+b$. Now (Gauss) $F(a,b;c;1) < \infty$.

Case B. $c = a+b$. By Gauss' result as $r \to 1$,

$$F(a,b;a+b;r) \sim \frac{1}{B(a,b)} \log \frac{1}{1-r}\,; B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Case C. $c < a+b$. In this case the asymptotic relation is

$$F(a,b;a+b;r) \sim D(1-r)^{c-a-b}, D = B(c,a+b-c)/B(a,b),$$

as $r \to 1$.

The case $c = a + b$ is called the *zero-balanced case.* The hypergeometric function satisfies numerous identities [AS], [Bat1]-[Bat3]. Perhaps the most famous are those due to Kummer [Pr], [AS, 15.5].

3.2. Ramanujan asymptotic formula. The Gauss asymptotic formula in Case B was considerably refined by Ramanujan, who proved [Ask1] that if $\psi(a) = \Gamma'(a)/\Gamma(a)$ and

$$R(a,b) = -\psi(a) - \psi(b) - 2\gamma, \quad R(1/2,1/2) = \log 16,$$

where γ stands for the Euler-Mascheroni constant and $B(a,b)$ for the beta function, then

$$B(a,b)F(a,b;a+b;r) + \log(1-r) =$$
$$R(a,b) + O((1-r)\log(1-r))$$

as $r \to 1$.

Ramanujan's result was extended recently, in [ABRVV] where it was shown e.g. that for $a, b \in (0,1)$, $B = B(a,b)$,

$$BF(a,b;a+b;r) + (1/r)\log(1-r)$$

is increasing on $(0,1)$ with range $(B-1, R)$. See also [PV1] where $a+b$ is replaced by c. Convexity properties of the hypergeometric function of the unit disk have been studied recently e.g. in [PV2], [PV3], [PSa].

3.3. Perturbation of identity . By continuity, small changes of argument lead to small changes of the values of the function. For example, we expect that

$$F(a_1,b_1;c_1;r) \text{ and } F(a_2,b_2;c_2;r)$$

are close if the parameters (a_1,b_1,c_1) and (a_2,b_2,c_2) are close. In view of the asymptotic behavior at $r = 1$ considered in 3.1 above, it seems natural to require that $c_1 - a_1 - b_1 = c_2 - a_2 - b_2$. Two natural questions are: When are $\mathcal{K}(r)$ and $(\pi/2)F(a,b;a+b;r^2)$ close to each other (recall that $\mathcal{K}(r) = (\pi/2)F(1/2,1/2;1;r^2)$)? Can we extend the many properties of $\mathcal{K}(r)$ to $F(a,b;a+b;r^2)$?

3.4. Landen inequality . Recall first that one of the most important properties of $\mathcal{K}(r)$ is given by the *Landen identity* (1771) (see [AlB], [Hou]), which states that for all $r \in (0,1)$

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\,\mathcal{K}(r). \tag{3.5}$$

The next theorem, an extension of (3.5), might be called a *Landen inequality* [QVu3].

3.6. Theorem. *For $a, b \in (0,1)$ with $a+b \le 1$ we have, for all $r \in (0,1)$,*

$$F\left(a,b;a+b;\left(\frac{2\sqrt{r}}{1+r}\right)^2\right) \le (1+r)F(a,b;a+b;r^2).$$

In [AVV5] there is a list of monotonicity properties of $\mathcal{K}(r)$. For instance, $\mathcal{K}(r)/\log(4/r')$ is monotone decreasing. It is natural to ask if such properties have a generalization for hypergeometric functions. The next open problem is of this type.

3.7. Open problem . Let $a, b \in (0, 1]$ with $a + b \le 1$,

$$Q(r) = B(a, b)F(a, b; a + b; r)/\log \frac{c}{1 - r}, \quad c = e^{R(a,b)},$$

and $G(r) = (Q(r) - 1)/(1 - r)$. Is it true that the Maclaurin coefficients of $G(r)$ are positive?

An affirmative answer would give a refinement of Ramanujan's asymptotic formula 3.2 and also imply that G is strictly increasing and convex. Computer experiments suggest that the answer is in the affirmative.

The last topic of this section deals with the algebraic identities for the function

$$\varphi_K(r) = \varphi_{K,2}(r) = \mu^{-1}(\mu(r)/K)$$

that follow from Ramanujan's work on modular equations [Bern1], [Bern2], [Bern3]. Of these [Bern3, pp. 8-9] contains a very helpful list of Ramanujan's numerous contributions in the field. Because Ramanujan's work in this field became widely accessible only with the publication of [Bern3] in 1991, the derivation of these results as corollaries to Ramanujan's work could not have been possible before 1991.

3.8. Modular equations of degree p. The argument $r \in (0, 1)$ of the complete elliptic integral $\mathcal{K}(r)$ is sometimes called the *modulus* of $\mathcal{K}$. A *modular equation* of *degree* $p > 0$ is the relation

$$\frac{\mathcal{K}(s')}{\mathcal{K}(s)} = p\frac{\mathcal{K}(r')}{\mathcal{K}(r)} \Leftrightarrow \mu(s) = p\mu(r). \tag{3.9}$$

The solution of this equation is $s = \varphi_{1/p}(r)$. Modular equations were studied by several mathematicians in the nineteenth century. The most remarkable progress was made, however, by Ramanujan in 1900-1920. We first record a few basic properties of $\varphi_K(r)$ which will be handy for the discussion of modular equations:

$$\begin{cases} \varphi_K(r)^2 + \varphi_{1/K}(r')^2 = 1, \\ \varphi_A(\varphi_B(r)) = \varphi_{AB}(r), \ A, B > 0, \\ \varphi_{1/K}(r) = \varphi_K^{-1}(r), \\ \varphi_2(r) = \dfrac{2\sqrt{r}}{1 + r}, \end{cases} \tag{3.10}$$

for all $r \in [0, 1]$. The classical *Legendre-Jacobi modular equation of order 3*

$$\sqrt{rs} + \sqrt{r's'} = 1, \ s = \varphi_{1/3}(r), \tag{3.11}$$

can be solved for s. We can easily find the solution if we use a symbolic computation program such as Mathematica. The solution was worked out (by hand!) in [KZ].

3.12. Ramanujan modular equations. We use the term modular equation not only for the transcendental equation (3.9) but also for an algebraic equation that follows from (3.9), as in [Bern3]. An example of such an algebraic equation is the Legendre-Jacobi modular equation (3.11). We now rewrite (3.11) using Ramanujan's notation:

$$\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = 1, \quad \alpha = r^2,\, \beta = \varphi_{1/3}(r)^2,$$

for all $r \in (0,1)$. Following [Bern3] and [Vu6] we now give a few of Ramanujan's modular equations.

3.13. Theorem. *The function φ_K satisfies the following identities:*

(1) For $\alpha = r^2,\ \beta = \varphi_{1/5}(r)^2$, we have

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1.$$

(2) For $\alpha = r^2,\ \beta = \varphi_{1/7}(r)^2$, we have

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1.$$

(3) For $\alpha = r^2,\ \beta = \varphi_{1/3}(r)^2,\ \gamma = \varphi_{1/9}(r)^2$ we have

$$\{\alpha(1-\gamma)\}^{1/8} + \{\gamma(1-\alpha)\}^{1/8} = 2^{1/3}\{\beta(1-\beta)\}^{1/24}.$$

(4) For $\alpha = r^2,\ \beta = \varphi_{1/23}(r)^2$, we have

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1.$$

(5) For $\alpha = r^2,\ \beta = \varphi_{1/7}(r)^2$, or for $\alpha = \varphi_{1/3}(r)^2,\ \beta = \varphi_{1/5}(r)^2$, we have

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} =$$
$$\{\frac{1}{2}(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)})\}^{1/2}.$$

Proof. All of these identities are from [Bern2]: (1) is [Bern2, p.280, Entry 13 (i)]; (2) is p. 314, Entry 19 (i); (3) is p. 352, Entry 3 (vi); (4) is p. 411, Entry 15 (i); and (5) is p. 435, Entry 21 (i).

3.14. Theorem. *The function φ_K satisfies the following identities for $s \in (0,1)$:*

(1) $xy + x'y' + 2^{5/3}\{xyx'y'\}^{1/3} = 1$, where $x = \varphi_{\sqrt{5}}(s),\ y = \varphi_{1/\sqrt{5}}(s)$,

(2) $(xy)^{1/4} + (x'y')^{1/4} = 1$, where $x = \varphi_{\sqrt{7}}(s),\ y = \varphi_{1/\sqrt{7}}(s)$,

(3) $(xy)^{1/4} + (x'y')^{1/4} = 2^{1/3}(s^2(1-s^2))^{1/24}$ where $x = \varphi_3(s),\ y = \varphi_3(s')$,

(4) $(xy)^{1/4} + (x'y')^{1/4} + 2^{2/3}(xx'yy')^{1/12} = 1$ where $x = \varphi_{1/\sqrt{23}}(s),\ y = \varphi_{\sqrt{23}}(s')$,

(5) $(xy)^{1/4} + (x'y')^{1/4} - \{xx'yy'\}^{1/4} = \{\frac{1}{2}(1+xy+x'y')\}^{1/2}$ where $x = \varphi_{\sqrt{5/3}}(s),\ y = \varphi_{\sqrt{3/5}}(s)$.

Proof. All parts follow from Theorem 3.13 above in the same way after r is chosen appropriately. For this reason we give here the details only for (5). For (5) set $r = \varphi_{\sqrt{15}}(s)$. By (2) and (3) we see that

$$\alpha = \varphi_{1/3}(r)^2 = \varphi_{\sqrt{5/3}}(s)^2, \quad 1-\alpha = \varphi_{\sqrt{3/5}}(s')^2\,,$$

and thus the proof follows from Theorem 3.13 (5).

3.15. Corollary. *We have the following identities:*

(1) $$2uu' + 2^{5/3}(uu')^{2/3} = 1; \quad u = \varphi_{\sqrt{5}}(1/\sqrt{2}),$$

(2) $$2(uu')^{1/4} = 1; \quad u = \varphi_{\sqrt{7}}(1/\sqrt{2}),$$

(3) $$\sqrt{u} + \sqrt{u'} = 2^{1/4}; \quad u = \varphi_3(1/\sqrt{2}),$$

(4) $$2(uu')^{1/4} + 2^{2/3}(uu')^{1/6} = 1; \quad u = \varphi_{\sqrt{23}}(1/\sqrt{2}),$$

(5) $$2(uu')^{1/4} - (uu')^{1/2} = \{\frac{1}{2}(1+2uu')\}^{1/2};$$

$$u = \varphi_{\sqrt{5/3}}(1/\sqrt{2}).$$

Proof. All parts follow from Theorem 3.14 and (3.10) in the same way. We give here the details only for (5). Set $s = 1/\sqrt{2}$ in Theorem 3.14 (5) and observe that then, $x' = y,\ y' = x$ and thus (5) follows as desired.

3.16. Generalized modular equations . A *generalized modular equation with signature* $1/a$ *and order (or degree)* p is

(3.17) $$\frac{F(a,1-a;1;1-s^2)}{F(a,1-a;1;s^2)} = p\frac{F(a,1-a;1;1-r^2)}{F(a,1-a;1;r^2)}.$$

Such equations were studied extensively by Ramanujan, who also gave a great number of algebraic identities for the solutions. Many of his results were proved in 1995 by Berndt, Bhargava, and Garvan in a long paper [BBG] (see also [Gar]). The main cases they studied are:

$$a = 1/6, 1/4, 1/3, \quad p = 2, 3, 5, 7, 11, ...$$

With

$$\mu_a(r) = \frac{\pi}{2\sin(\pi a)}\frac{F(a,1-a;1;1-r^2)}{F(a,1-a;1;r^2)}$$

the solution of (3.17) is given by

$$s = \mu_a^{-1}(p\mu_a(r)) \equiv \varphi^a_{1/p}(r).$$

Note that $\mu_a(r) = \mu_{1-a}(r)$ for $a \in (0,1/2)$ and $\mu(r) = \mu_{1/2}(r)$.

For generalized modular equations the Ramanujan notation is

$$\alpha \equiv r^2, \quad \beta \equiv \varphi^a_{1/p}(r)^2 \quad .$$

3.18. Theorem[BBG, Theorem 7.1]. *If β has degree 2 in the theory of signature 3, then, with $a = 1/3, \alpha = r^2, \beta = \varphi^a_{1/2}(r)^2$,*

$$(\alpha\beta)^{\frac{1}{3}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{3}} = 1.$$

3.19. Theorem[BBG, Theorem 7.6]. *If β has degree 5 then, with $a = 1/3, \alpha = r^2, \beta = \varphi^a_{1/5}(r)^2$,*

$$(\alpha\beta)^{\frac{1}{3}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{3}} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}} = 1. \tag{3.20}$$

3.21. Theorem[BBG, Theorem 7.8]. *If β has degree 11 then, with $a = 1/3, \alpha = r^2, \beta = \varphi^a_{1/11}(r)^2$,*

$$\begin{aligned}(\alpha\beta)^{\frac{1}{3}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{3}} + 6\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}}+ \\ 3\sqrt{3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{12}}\{(\alpha\beta)^{\frac{1}{6}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{6}}\} = 1.\end{aligned} \tag{3.22}$$

Several open problems are now immediate.

3.23. Open problem . To what extent can the properties of $\mu(r)$ be extended for $\mu_a(r)$?

3.24. Open problem . To what extent can the properties of $\varphi_K(r)$ be extended for $\varphi^a_K(r)$?

Solving these problems will require very extensive studies. A basic tool is the *Ramanujan derivative formula* [Bern2, p. 86]:

$$\frac{d\mu_a(r)}{dr} = -\frac{1}{r(1-r^2)}\frac{1}{F(a,1-a;1;r^2)^2}. \tag{3.25}$$

A direct application of the $F(a,b;c;r)$ derivative formula

$$\frac{d}{dr}F(a,b;c;r) = \frac{ab}{c}F(a+1,b+1;c+1;r)$$

from [AS, 15.2.1] leads to a more complicated form than (3.25) so the formula (3.25) is specific for the case $b = 1-a, c = 1$. By equating this more complicated form and (3.25) we obtain the following interesting identity for $a, r \in (0,1)$:

$$\begin{cases} F(1+a,2-a;2;1-r)F(a,1-a;1;r)+ \\ F(1+a,2-a;2;r)F(a,1-a;1;1-r) = \dfrac{\sin(\pi a)}{\pi a(1-a)r(1-r)}. \end{cases} \tag{3.26}$$

3.27. Theorem[BPV]. *For $0 < a \le 1/2, r, s \in (0,1)$, we have*

$$\mu_a(r) + \mu_a(s) \le 2\mu_a(\sqrt{\frac{2rs}{1+rs+r's'}}) \le 2\mu_a(\sqrt{rs}),$$

with equality for $r = s$.

The above inequality $\mu_a(r) + \mu_a(s) \le 2\mu_a(\sqrt{rs})$ resembles the multiplicative property of the logarithm $\log a + \log b = 2 \log \sqrt{ab}\,, a, b > 0$, and hence $\mu_a(r)$ behaves, to some extent, like a logarithm.

3.28. Theorem[QVu4]. *For $a \in (0, 1/2]$ and $r, t \in (0, 1)$,*

$$2\mu_a(\frac{r+t}{1+rt+r't'}) \le \mu_a(r) + \mu_a(t)\,,$$

with equality for $t = r$.

3.29. Open problem . In view of these results it is natural to ask if there is an addition formula for μ_a.

3.30. Duplication inequality . The Landen identity yields the following duplication formula for $\mu(r)$

$$\mu(r) = 2\mu(\frac{2\sqrt{r}}{1+r}).$$

The next theorem from [QVu4] could be called a *duplication inequality* for $\mu_a(r)$.

3.31. Theorem. *For $a \in (0, 1/2]$ let $R = R(a, 1-a)$,*

$$C \equiv \left(1 + \frac{\sin \pi a}{\pi}(R - \log 16)\right)^2 ,$$

and $C_1 = \min\{2, C\}$. Then, for $r \in (0, 1)$,

$$\mu_a(r) \le 2\mu_a(\frac{2\sqrt{r}}{1+r}) \le C_1 \mu_a(r).$$

Jacobi's work yields dozens of infinite product expansions for elliptic functions. A representative identity is the following one, where $q = \exp(-2\mu(r))$

$$\exp(\mu(r) + \log r) = 4 \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4 .$$

Observe that $\mu(r)$ occurs on both sides!

3.32. Theorem. *For $a \in (0, 1/2]$ let $R = R(a, 1-a)$. For $r \in (0, 1)$ let*

$$r_0 = \sqrt{1-r^2}, r_n = \frac{2\sqrt{r_{n-1}}}{1+r_{n-1}}, \quad p = \prod_{n=0}^{\infty} (1+r_n)^{2^{-n}} .$$

Then

$$p \le \exp(\mu_a(r) + \log r) \le \frac{\exp R}{16} p,$$

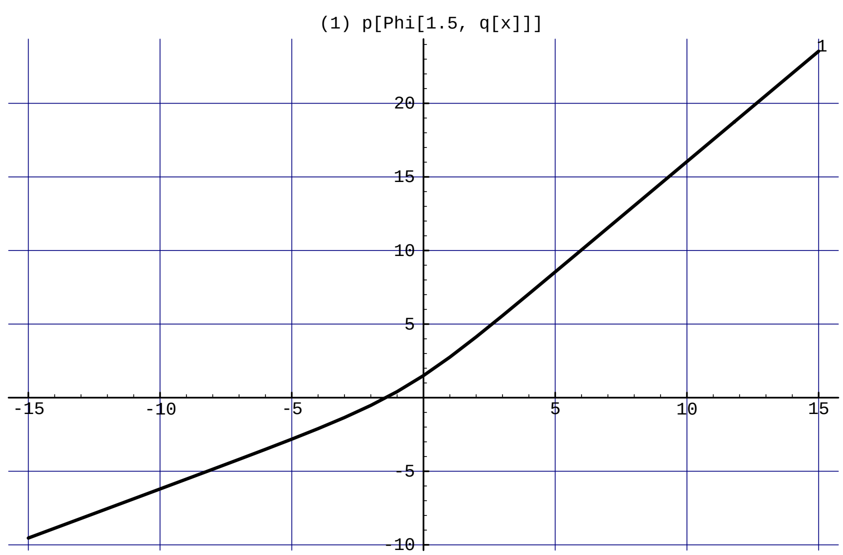
with equality for $a = 1/2$.

In this theorem from [QVu5] even the case $a = 1/2$ is new. Note that $\mu_a(r)$ occurs only in the middle term!

3.33. Linearization for $\varphi_K(r)$. It was observed in [SamV] that the functions $p(x) = \log(x/(1-x)), q(x) = e^x/(1+e^x)$,

$$p : (0,1) \to \mathbb{R}, \quad q : \mathbb{R} \to (0,1) \quad p = q^{-1}$$

have a regulating effect on $\varphi_K(r)$:



3.34. Theorem([AsVV]). *The function $g : \mathbb{R} \to \mathbb{R}, g(x) = p(\varphi_K(q(x))$ has increasing derivative with range $(1/K, K)$.*

We call such a transformation a *linearization*.

3.35. Open problem . Find a similar result for $\varphi_K^a(r)$.

3.36. Conformal Invariants Software . We shall discuss here briefly the C, Mathematica, and MATLAB language software supplement for the monograph [AVV6], which will soon be completed. This software will fill one disk (1.4M), and it provides, in these languages, algorithms for the special functions mentioned earlier. A useful survey of special function computation is [LOl]. See also [AS], [Bak], [Mosh].

Nine example programs are used in [AVV6] to summarize the key procedures needed for computer experiments, function tabulation, and graphing. Although computation of most special functions is in principle "well known," in practice finding algorithms requires much work and occasionally one has to implement algorithms. Mathematica contains as built-ins many of the functions we need.

A preliminary version of the software and the manual are available. The software runs on both PC and Unix machines.

3.37. Newton algorithm for $\mu^{-1}(y), y > \frac{\pi}{2}$. Set $x_0 = 1/\cosh y$ and

$$x_{n+1} = x_n - \frac{\mu(x_n) - y}{\mu'(x_n)} = x_n - \frac{(\mu(x_n) - y)(x_n - x_n^3)}{AG(1, x_n')^2}.$$

In practice, this algorithm always converges, but the proof of convergence is missing.

3.38. Open problems (for numerical analysis students).
(a) Does the above Newton algorithm converge to $\mu^{-1}(y)$ for $y > \pi/2$?
(b) Is it true that $x_n < x_{n+1} < 1$ for all n if $y > \pi$?

D. Partyka [Par1]-[Par2] has also devised algorithms for computing functions related to $\mu^{-1}(y)$.

3.39. Remark. The open problems 3.35 and 3.38 have been recently studied in [AVV7].

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References

[AS] M. Abramowitz and I. A. Stegun, editors: *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, 1965.

[Ag] S. Agard: *Distortion theorems for quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I 413 (1968), 1–12.

[AG] S. Agard and F. W. Gehring: *Angles and quasiconformal mappings*, Proc. London Math. Soc. 14 (1965), 1–21.

[Ah1] L. V. Ahlfors: *Quasiconformal reflections*, Acta Math. 109 (1963), 291–301.

[Ah2] L. V. Ahlfors: *Collected Papers (R. M. Short, asst. ed.)*, Birkhäuser, Boston, 1982.

[AhB] L. Ahlfors and A. Beurling: *Conformal invariants and function-theoretic null-sets*, Acta Math. 83 (1950), 101–129.

[AlB] G. Almkvist and B. Berndt: *Gauss, Landen, Ramanujan, the arithmetic-geometric mean, ellipses, π, and the Ladies Diary*, Amer. Math. Monthly 95 (1988), 585–608.

[A1] G. D. Anderson: *Extremal rings in n-space for fixed and varying n*, Ann. Acad. Sci. Fenn. Ser. A I 575 (1974), 1–21.

[A2] G. D. Anderson: *Dependence on dimension of a constant related to the Grötzsch ring*, Proc. Amer. Math. Soc. 61 (1976), 77–80.

[ABRVV] G. D. Anderson, R. W. Barnard, K. C. Richards, M. K. Vamanamurthy, and M. Vuorinen: *Inequalities for zero-balanced hypergeometric functions*, Trans. Amer. Math. Soc. 347 (1995), 1713–1723.

[ALV] G. D. Anderson, M. Lehtinen, and M. Vuorinen: *Conformal invariants in the punctured unit disk*, Ann. Acad. Sci. Fenn. Ser. A I 19 (1994), 133–146.

[AQ] G. D. Anderson and S.-L. Qiu: *A monotoneity property of the gamma function*, Proc. Amer. Math. Soc. (to appear).

[AQV] G. D. Anderson, S.-L. Qiu, and M. K. Vamanamurthy: *Elliptic integral inequalities, with applications*, Constr. Approx. (to appear).

[AVV1] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Dimension-free quasiconformal distortion in n-space*, Trans. Amer. Math. Soc. 297 (1986), 687–706.

[AVV2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Distortion functions for plane quasiconformal mappings*, Israel J. Math. 62 (1988), 1–16.

[AVV3] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Special functions of quasiconformal theory*, Exposition. Math. 7 (1989), 97–138.

[AVV4] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Conformal invariants, quasiconformal maps, and special functions,* in Quasiconformal Space Mappings: A Collection of Surveys, ed. by M. Vuorinen, Lecture Notes in Math., Vol. 1508, Springer-Verlag, 1992, pp. 1–19.

[AVV5] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Hypergeometric functions and elliptic integrals*, in Current Topics in Analytic Function Theory, ed. by H. M. Srivastava and S. Owa, World Scientific Publ. Co., Singapore – London, 1992, 48–85.

[AVV6] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Conformal invariants, inequalities, and quasiconformal maps*, J. Wiley, 1997.

[AVV7] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Generalized elliptic integrals and modular equations,* Manuscript, October 1997.

[ABL] J. M. Anderson, J. Becker, and F. D. Lesley: *Boundary values of asymptotically conformal mapping*, J. London Math. Soc. 38 (1988), 453-462.

[Ao] K. Aomoto: *Hypergeometric functions, the past, today, and possible future (Japanese)*, Sugaku Expositions 9 (1996), 99-116.

[ACJP] J. Arazy, T. Claesson, S. Janson, and J. Peetre: *Means and their iterations*, Proc. of the 19th Nordic Congress of Mathematicians, Reykjavik 1984, pp. 191–212, Icelandic Math. Soc., 1985.

[AsVV] V. V. Aseev, M.K. Vamanamurthy, and M. Vuorinen: *Quasiadditive properties and bilipschitz conditions,* Aequationes Math. (to appear).

[Ask1] R. Askey: *Ramanujan and hypergeometric and basic hypergeometric series*, Ramanujan Internat. Symposium on Analysis, December 26–28, 1987, ed. by N. K. Thakare, 1–83, Pune, India, Russian Math. Surveys 451 (1990).

[Ask2] R. Askey: *Handbooks of Special Functions, A Century of Mathematics in America*, ed. by P. Duren, Part III, 369–391, Amer. Math. Soc., 1989.

[Ast] K. Astala: *Area distortion of quasiconformal mappings*, Acta Math. 173 (1994), 37-60.

[Ba1] A. Baernstein II: *Ahlfors and conformal invariants*, Ann. Acad. Sci. Fenn. Ser. A I 13 (1988), 289–312.

[Ba2] A. Baernstein II: *Some topics in symmetrization*, in Harmonic Analysis and Partial Differential Equations, ed. by J. Garsia-Cuerra, Lecture Notes in Math. Vol. 1384, pp. 111–123, Springer-Verlag, 1989.

[Ba3] A. Baernstein II: *A unified approach to symmetrization*, Symposia Math. 35 (1995), 47–91.

[BBBP] D. H. Bailey, J. M. Borwein, P. B. Borwein, and S. Plouffe: *The Quest for Pi*, Math. Intelligencer 19 (1997), 50–57.

[Bak] L. Baker: *C Mathematical Function Handbook*, McGraw-Hill, New York, 1992.

[BPV] R. Balasubramanian, S. Ponnusamy and M. Vuorinen: *Functional inequalities for the quotients of hypergeometric functions*, J. Math. Anal. Appl. (to appear).

[Bat1] Bateman Manuscript Project, (ed. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi): *Higher Transcendental Functions, Vol. I*, McGraw-Hill, New York, 1953.

[Bat2] Bateman Manuscript Project, (ed. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi): *Higher Transcendental Functions, Vol. II*, McGraw-Hill, New York, 1953.

[Bat3] Bateman Manuscript Project, (ed. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi): *Higher Transcendental Functions, Vol. III*, McGraw-Hill, New York, 1955.

[Be] A. F. Beardon: *The Apollonian metric of a domain in* R^n, Manuscript, 1996.

[BC] J. Becker: *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. 255 (1972), 23–43.

[BP1] J. Becker and Ch. Pommerenke: *Über die quasikonforme Fortsetzung schlichter Funktionen*, Math. Z. 161 (1978), 69–80.

[BP2] J. Becker and Ch. Pommerenke: *On the Hausdorff dimension of quasicircles*, Ann. Acad. Sci. Fenn. Ser. A I Math. 12 (1987), 329–333.

[BP3] J. Becker and Ch. Pommerenke: *Hölder continuity of conformal maps with quasiconformal extension* , Complex Variables Theory Appl. 10 (1988), 267–272.

[Bern1] B. C. Berndt: *Ramanujan's Notebooks, Vol. I*, Springer-Verlag, Berlin – Heidelberg – New York, 1985.

[Bern2] B. C. Berndt: *Ramanujan's Notebooks, Vol. II*, Springer-Verlag, Berlin – Heidelberg – New York, 1989.

[Bern3] B. C. Berndt: *Ramanujan's Notebooks, Vol. III*, Springer-Verlag, Berlin – Heidelberg – New York, 1991.

[Bern4] B. C. Berndt: *Ramanujan's Notebooks, Vol. IV*, Springer-Verlag, Berlin – Heidelberg – New York, 1993.

[Bern5] B. C. Berndt: *Ramanujan's theory of theta functions*, CRM Proc. and Lecture Notes 1, 1–63, Amer. Math. Soc., 1993.

[BBG] B. C. Berndt, S. Bhargava, and F. G. Garvan: *Ramanujan's theories of elliptic functions to alternative bases*, Trans. Amer. Math. Soc. 347 (1995), 4163–4244.

[BAh] A. Beurling and L. Ahlfors: *The boundary correspondence under quasiconformal mappings*, Acta Math. 96 (1956), 125–142.

[BJ1] C. J. Bishop and P. Jones: *Hausdorff dimension and Kleinian groups,* Acta Math. (to appear)

[BJ2] C. J. Bishop and P. Jones: *Wiggly sets and limit sets*, Manuscript 1996.

[BS] C. J. Bishop and T. Steger: *Representation theoretic rigidity in $PSL(2,R)$,* Acta Math. 170 (1993), 121–149.

[BI1] B. Bojarski and T. Iwaniec: *Another approach to Liouville theorem*, Math. Nachr. 107 (1982), 253–262.

[BI2] B. Bojarski and T. Iwaniec: *Analytical foundations of the theory of quasiconformal mappings in R^n*, Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983), 257–324.

[BB1] J. M. Borwein and P. B. Borwein: *Pi and the AGM*, John Wiley & Sons, New York, 1987.

[BB2] J. M. Borwein and P. B. Borwein: *Inequalities for compound mean iterations with logarithmic asymptotes*, J. Math. Anal. Appl. 177 (1993), 572–582.

[Car] P. Caraman: *n-Dimensional Quasiconformal (QCf) Mappings*, Abacus Press, Tunbridge Wells, Kent, England, 1974.

[Carle] L. Carleson: *On mappings conformal at the boundary*, J. Analyse Math. 19 (1967), 1–13.

[C1] B. C. Carlson: *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly 78 (1971), 496–505.

[C2] B. C. Carlson: *The logarithmic mean*, Amer. Math. Monthly 79 (1972), 615–618.

[C3] B. C. Carlson: *Special Functions of Applied Mathematics*, Academic Press, New York, 1977.

[CG1] B. C. Carlson and J. L. Gustafson: *Asymptotic expansion of the first elliptic integral*, SIAM J. Math. Anal. 16 (1985), 1072–1092.

[CG2] B. C. Carlson and J. L. Gustafson: *Asymptotic approximations for symmetric elliptic integrals*, SIAM J. Math. Anal. 25 (1994), 288–303.

[CV] B. C. Carlson and M. Vuorinen: *An inequality of the AGM and the logarithmic mean*, SIAM Rev. 33 (1991), Problem 91-117, 655.

[CC] D. V. Chudnovsky and G. V. Chudnovsky: *Hypergeometric and modular function identities, and new rational approximations to a continued fraction expansions of classical constants and functions*, in A Tribute to Emil Grosswald - Number Theory and Related Analysis, ed. by M. Knopp and M. Sheingorn, Contemporary Math., Vol. 143 (1993), 117–162.

[CH] P. B. Cohen and F. Hirzebruch: *A survey of [DM]*, Bull. Amer. Math. Soc. 32 (1995), 88–105.

[Co] D. A. Cox: *Gauss and the arithmetic-geometric mean*, Notices Amer. Math. Soc. 32 (1985), 147–151.

[DS] G. David and S. Semmes: *Fractured fractals and broken dreams: self-similar geometry through metric and measure*, Bookmanuscript, 1996.

[DM] P. Deligne and G. D. Mostow: *Commensurabilities among lattices in $PU(1,n)$*, Annals of Mathematics Studies, Princeton University Press, Princeton, New Jersey, 1993.

[Dub] V. N. Dubinin: *Symmetrization in the geometric theory of functions of a complex variable*, Russian Math. Surveys 49 (1994), 1–79.

[Dut] J. Dutka: *The early history of the hypergeometric function*, Archieve for History of Exact Sciences, 31 (1984), 15–34.

[FV] R. Fehlmann and M. Vuorinen: *Mori's theorem for n-dimensional quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I 13 (1988), 111–124.

[F1] J. Ferrand: *Invariants conformes dans l'espace de Möbius et caracterizations des applications quasiconformes*, Rev. Roumaine Math. Pures Appl. 33 (1988), 47–55.

[F2] J. Ferrand: *A characterization of quasiconformal mappings by the behavior of a function of three points*, Proc. 13th Rolf Nevanlinna Colloquium (Joensuu, 1987), ed. by I. Laine, S. Rickman, and T. Sorvali, Lecture Notes in Math., Vol. 1351, pp. 110–123, Springer-Verlag, Berlin – Heidelberg – New York, 1988.

[F3] J. Ferrand: *Conformal capacity and conformally invariant metrics*, Pacific J. Math. (to appear).

[F4] J. Ferrand: *Conformal capacities and conformally invariant functions on Riemannian manifolds*, Geom. Dedicata 61 (1996), 103–120.

[FMV] J. Ferrand, G. Martin, and M. Vuorinen: *Lipschitz conditions in conformally invariant metrics*, J. Analyse Math. 56 (1991), 187–210.

[Fu] B. Fuglede: *Extremal length and functional completion*, Acta Math. 98 (1957), 171–219.

[Gar] F. Garvan: *Cubic modular identities of Ramanujan, hypergeometric functions and analogues of the arithmetic-geometric mean iteration*, in The Rademacher Legacy to Mathematics, ed. by G. E. Andrews, D. M. Bressoud, and L. A. Parson, Contemporary Math., Vol. 166, Amer. Math. Soc., Providence, RI, 1994, 245–264.

[G1] F. W. Gehring: *Symmetrization of rings in space*, Trans. Amer. Math. Soc. 101 (1961), 499–519.

[G2] F. W. Gehring: *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. 103 (1962), 353–393.

[G3] F. W. Gehring: *Quasiconformal mappings*, in Complex Analysis and its Applications, Vol. II, International Atomic Energy Agency, Vienna, 1976, pp. 213–268.

[G4] F. W. Gehring: *A remark on domains quasiconformally equivalent to a ball*, Ann. Acad. Sci. Fenn. Ser. A I 2 (1976), 147–155.

[G5] F. W. Gehring: *Characteristic Properties of Quasidisks*, Les Presses de l'Université de Montréal, Montréal, 1982.

[G6] F. W. Gehring: *Topics in quasiconformal mappings*, in Proc. 1986 Internat. Cong. Math. (Berkeley, CA), Vol. I (1987), pp. 62–80.

[G7] F. W. Gehring: *Uniform domains and the ubiquitous quasidisk*, Jber. d. Dt. Math.-Verein. 89 (1987), 88–103.

[G8] F. W. Gehring: *Quasiconformal mappings and quasidisks*, Lectures in the Vth Finnish-Polish-Ukrainian Summer School in Complex Analysis, Lublin, 15-21 August 1996.

[GM] F. W. Gehring and G. Martin: *Holomorphic motions, Schottky's Theorem and an inequality for discrete groups* , Computational Methods and Function Theory, ed. by R. M. Ali, St. Ruscheweyh, and E. B. Saff, pp. 173-181, World Scientific Publ. Co. 1995.

[GV1] F. W. Gehring and J. Väisälä: *The coefficients of quasiconformality of domains in space*, Acta Math. 114 (1965), 1–70.

[GV2] F. W. Gehring and J. Väisälä: *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. 6(2) (1973), 504–512.

[GKZ] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky: *Generalized Euler Integrals and A-hypergeometric functions*, Adv. Math., 84 (1990), 255–271.

[GuR] V. Ya. Gutlyanskii and V. I. Ryazanov: *On the local behaviour of quasi-conformal mappings,* Izv. RAN Ser. Mat. 59 (1995), 31-58.

[GuV] V. Ya. Gutlyanskii and M. Vuorinen: *On maps almost conformal at the boundary*, Complex Variables Theory Appl. (to appear)

[H] G. H. Hardy: *Ramanujan*, Cambridge Univ. Press, London, 1940.

[He] C.-Q. He: *Distortion estimates of quasiconformal mappings*, Sci. China Ser. A 27 (1984), 225–232.

[HKM] J. Heinonen, T. Kilpeläinen, and O. Martio: *Nonlinear potential theory of degenerate elliptic equations*, Clarendon Press 1993, Oxford.

[Hem1] J. A. Hempel: *Precise bounds in the theorems of Schottky and Picard*, J. London Math. Soc. 21 (1980), 279–286.

[Hem2] J. A. Hempel: *The Poincaré metric of the twice punctured plane and the theorems of Landau and Schottky*, J. London Math. Soc. 2 (1979), 435–445.

[Hou] Ch. Houzel: *Elliptische Funktionen und Abelsche Integrale,* in J. Dieudonné ed., Geschichte der Matematik 1700–1900, VEB Deutscher Verlag der Wissenschaften, Berlin, 1985, 422–539.

[I1] T. Iwaniec: *Some aspects of partial differential equations and quasiregular mappings*, Proc. 1982 Int. Cong. Math. (Warsaw, Poland), PWN, Polish Scientific Publishers, Warzawa 1984, Vol. 2, 1193–1208.

[I2] T. Iwaniec: *Quasiconformality geometry and the governing PDE's*, Lectures in the Vth Finnish-Polish-Ukrainian Summer School in Complex Analysis, Lublin, 15-21 August 1996.

[IM1] T. Iwaniec and G. Martin: *Quasiconformal mappings and capacity*, Indiana Univ. Math. J. 40 (1991), 101–122.

[IM2] T. Iwaniec and G. Martin: Nonlinear analysis and quasiconformal maps, bookmanuscript.

[Ja] C. G. J. Jacobi: *Fundamenta Nova Theoriae Functionum Ellipticarum (1829)*, Jacobi's Gesammelte Werke, Vol. 1 (Berlin, 1881–1891).

[JV] P. Järvi and M. Vuorinen: *Uniformly perfect sets and quasiregular mappings,* J. London Math. Soc. (2) (1996), 515-529.

[J1] J. A. Jenkins: *The Method of the Extremal Metric*, Proc. Congr. on the Occasion of the solution of the Bieberbach conjecture, ed. by A. Baernstein II, D. Drasin, P. Duren, and A. Marden, Mathematical Surveys and Monographs 21, Amer. Math. Soc., Providence, RI, 1986, 95–104.

[J2] J. A. Jenkins: *On metrics defined by modules*, Pacific J. Math. 167 (1995), 289–292.

[JK] D. S. Jerison and C. E. Kenig: *Hardy spaces, A_∞, and singular integrals on chord-arc domains*, Math. Scand. 50 (1982), 221–247.

[Jo] P. W. Jones: *Rectifiable sets and the travelling salesman problem,* Invent. Math. 102 (1990), 1–15.

[KZ] W. Kawa and J. Zając (Jr.): *Dynamical approximation of the distortion function Φ_K,* Bull. Soc. Sci. Lettres Łodz 45 (1995), 39-48; Serié: Reserches sur les deformations 20.

[Kl1] F. Klein: *Vorlesungen über die hypergeometrische Funktionen*, B. G. Teubner, Berlin, 1933.

[Kl2] F. Klein: *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Springer-Verlag, Berlin – Heidelberg.

[Knu] D. E. Knuth: *The Art of Computer Programming Vol. II*, 2nd ed., Addison-Wesley Publ. C., 1981.

[KR] S. L. Krushkal': *On an interpolating family of a univalent analytic function,* (Russian), Preprint No 18 (1986), Institute of Mathematics of the Sibirian Branch of the USSR Academy of Sciences, Novosibirsk.

[KK] S. L. Krushkal' and R. Kühnau: *Quasiconformal mappings—new methods and applications.* (Russian), Izdat. "Nauka", Sibirsk. Otdelenie, Novosibirsk, 1984.

[Küh1] R. Kühnau: *Möglichst konforme Spiegelung an einer Jordankurve*, Jber. d. Dt. Math.-Verein. 90 (1988), 90–109.

[Küh2] R. Kühnau: *Einige neuere Entwicklungen bei quasikonformen Abbildungen*, Jber. d. Dt. Math.-Verein. 94 (1992), 141–169.

[Küh3] R. Kühnau: *Eine Methode, die Positivität einer Funktion zu prüfen*, Zeitschrift f. angew. Math. u. Mech. 74 (1994), 140–142.

[Kuz] G. V. Kuz'mina: *Moduli of Families of Curves and Quadratic Differentials*, Proc. Steklov Inst. Math., 1982, issue 1 (Russian original: Tom 139, 1980).

[LeVu] M. Lehtinen and M. Vuorinen: *On Teichmüller's modulus problem in the plane*, Rev. Roumaine Math. Pures Appl. 33 (1988), 97–106.

[L] O. Lehto: *Univalent Functions and Teichmüller Spaces*, Graduate Texts in Math. Vol. 109, Springer-Verlag, New York – Heidelberg – Berlin, 1987.

[LV] O. Lehto and K. I. Virtanen: *Quasiconformal Mappings in the Plane*, 2nd ed., Die Grundlehren der math. Wissenschaften, Band 126, Springer-Verlag, New York – Heidelberg – Berlin, 1973.

[LVV] O. Lehto, K. I. Virtanen, and J.Väisälä: *Contributions to the distortion theory of quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I 273 (1959), 1–14.

[Lei] D. C. van Leijenhorst: *Algorithms for the approximation of π,* Nieuw Arch. Wiskunde 14 (1996), 255-274.

[LF] J. Lelong-Ferrand: *Invariants conformes globaux sur les varietes riemanniennes*, J. Differential Geom. 8 (1973), 487–510.

[LOl] D. W. Lozier and F. W. J. Olver: *Numerical evaluation of special functions*, Proc. Symps. Appl. Math. Vol. 48, "Mathematics of Computation 1943–1993: A Half-Century of Computational Mathematics" (Walter Gautschi, ed.), pp. 79–125, Amer. Math. Soc., Providence, RI, 1994.

[LuV] J. Luukkainen and J. Väisälä: *Elements of Lipschitz topology*, Ann. Acad. Sci. Fenn. Ser. A I Math. 3 (1977), 85–122.

[Ma] G. J. Martin: *The distortion theorem for quasiconformal mappings, Schottky's theorem and holomorphic motions*, Proc. Amer. Math. Soc. 125 (1997), 1095–1103.

[MatV] P. Mattila and M. Vuorinen: *Linear approximation property, Minkowski dimension and quasiconformal spheres,* J. London Math. Soc. (2) 42 (1990), 249-266.

[Mil] J. W. Milnor: *Hyperbolic geometry: the first 150 years*, Bull. Amer. Math. Soc. 6 (1982), 9–24.

[Mor1] A. Mori: *On an absolute constant in the theory of quasiconformal mappings*, J. Math. Soc. Japan 8 (1956), 156–166.

[Mor2] A. Mori: *On quasi-conformality and pseudo-analyticity*, Trans. Amer. Math. Soc. 84 (1957), 56–77.

[Mosh] S. L. Moshier: *Methods and Programs for Mathematical functions,* Ellis Horwood Ltd. Chichester, 1989.

[Mos1] G. D. Mostow: *Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. 34 (1968), 53–104.

[Mos2] G. D. Mostow: *Braids, hypergeometric functions, and lattices*, Bull. Amer. Math. Soc. 16 (1987), 225–246.

[N] R. Näkki: *Boundary behavior of quasiconformal mappings in n–space*, Ann. Acad. Sci. Fenn. Ser. A I 484 (1970), 1–50.

[Oh] M. Ohtsuka: *Extremal length and precise functions*, Manuscript, 1996.

[Ok] K. Okikiolu: *Characterization of subsets of rectifiable curves in R^n*, J. London Math. Soc. 46 (1992), 336–348.

[Paj] H. Pajot: *Etude des propriertes de rectifiabilite des sous-ensembles de R^n*, These, Universite de Paris-Sud, 1996.

[Pa1] P. Pansu: *Quasiconformal mappings and manifolds of negative curvature*, Curvature and topology of Riemann manifolds. Proc. 17th Intern. Taniguchi symposium held in Katata, Japan, Aug. 26–31, 1985, Lecture Notes in Math. Vol. 1201, Springer-Verlag, Berlin–Heidelberg–New York, 1986.

[Pa2] P. Pansu: *Difféomorphismes de p-dilatation bornée,* Ann. Acad. Sci. Fenn. Math. 22 (1997), 475–506.

[Par1] D. Partyka: *Approximation of the Hersch-Pfluger distortion function. Applications*, Ann. Univ. Mariae Curie-Skłodowska Sect. A., 45 (1992), 99–111.

[Par2] D. Partyka: *Approximation of the Hersch-Pfluger distortion function*, Ann. Acad. Sci. Fenn. Ser. A I 18 (1993), 343–354.

[Par3] D. Partyka: *On the maximal dilatation of the Douady-Earle extension*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 48 (1994), 80–97.

[Po] Ch. Pommerenke: *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

[PSa] S. Ponnusamy and S. Sabapathy: *Geometric properties of generalized hypergeometric functions* ,The Ramanujan Journal, Vol. 1, No.2, 1997, 187–210.

[PV1] S. Ponnusamy and M. Vuorinen: *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika 44 (1997), 278–301.

[PV2] S. Ponnusamy and M. Vuorinen: *Univalence and convexity properties of Gaussian hypergeometric functions*, Preprint 82, University of Helsinki, 1995, 34pp.

[PV3] S. Ponnusamy and M. Vuorinen: *Univalence and convexity properties of confluent hypergeometric functions* , Preprint 103, University of Helsinki, January 1996, 20pp.

[Pr] R. T. Prosser: *On the Kummer solutions of the hypergeometric equation,* Amer. Math. Monthly 101 (1994), 535–543.

[PBM] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev: *Integrals and Series, Vol. 3: More Special Functions*, trans. from the Russian by G. G. Gould, Gordon and Breach Science Publishers, New York, 1988.

[Q1] S.-L. Qiu: *Distortion properties of K-qc maps and a better estimate of Mori's constant*, Acta Math. Sinica 35(1992), 492–504.

[Q2] S.-L. Qiu: *The proof of a conjecture on the first elliptic integrals*, J. Hangzhou Inst. of Elect. Eng. 3(1993), 29–36.

[Q3] S.-L. Qiu: *Agard's η-distortion function and Schottky's theorem*, Sci. Sinica Ser. A, 26(8) (1996)

[QV] S.-L. Qiu and M. K. Vamanamurthy: *Sharp estimates for complete elliptic integrals*, SIAM J. Math. Anal. 27 (1996), 823–834.

[QVV1] S.-L. Qiu, M. K. Vamanamurthy, and M. Vuorinen: *Bounds for quasiconformal distortion functions*, J. Math. Anal. Appl. 205 (1997), 43–64.

[QVV2] S.-L. Qiu, M. K. Vamanamurthy, and M. Vuorinen: *Some inequalities for the Hersch-Pfluger distortion function*, Manuscript, (1995).

[QVV3] S.-L. Qiu, M. K. Vamanamurthy, and M. Vuorinen: *Some inequalities for the growth of elliptic integrals*, SIAM J. Math. Anal. (to appear).

[QVu1] S.-L. Qiu and M. Vuorinen: *Submultiplicative properties of the φ_K-distortion function*, Studia Math. 117 (1996), 225–242.

[QVu2] S.-L. Qiu and M. Vuorinen: *Quasimultiplicative properties of the η-distortion function*, Complex Variables Theory Appl. 30 (1996), 77-96.

[QVu3] S.-L. Qiu and M. Vuorinen: *Landen inequalities for hypergeometric functions,* Preprint 116, Department of Mathematics, University of Helsinki, 1996.

[QVu4] S.-L. Qiu and M. Vuorinen: *Duplication inequalities for the ratios of hypergeometric functions,* Preprint 118, Department of Mathematics, University of Helsinki, 1996.

[QVu5] S.-L. Qiu and M. Vuorinen: *Infinite products and inequalities for the normalized quotients of hypergeometric functions,* Preprint 140, Department of Mathematics, University of Helsinki, 1997.

[Ra] E. D. Rainville: *Special Functions*, Macmillan, New York, 1960.

[Ram] S. Ramanujan: *The lost notebook and other unpublished papers*, Springer-Verlag, New York, 1988.

[Re1] Yu. G. Reshetnyak: *The Liouville theorem with minimal regularity conditions* (Russian), Sibirsk. Mat. Zh. 8 (1967), 835–840.

[Re2] Yu. G. Reshetnyak: *Stability theorems in geometry and analysis*, Mathematics and its Applications, 304, Kluwer Academic Publishers Group, Dordrecht, 1994.

[Ri] S. Rickman: *Quasiregular mappings*, Ergebnisse der Math. Band 26, Springer-Verlag, Berlin, 1993.

[Rod] B. Rodin: *The method of extremal length*, Bull. Amer. Math. Soc. 80 (1974), 587–606.

[Ro] S. Rohde: *On conformal welding and quasicircles,* Michigan Math. J. 38 (1991), 111–116.

[SamV] K. Samuelsson and M. Vuorinen: *Computation of capacity in 3D by means of a posteriori estimates for adaptive FEM*, Preprint, Royal Institute of Technology, Stockholm, August 1995.

[Sán] J. Sándor: *On certain inequalities for means II*, J. Math. Anal. Appl. 199 (1996), 629–635.

[Se1] P. Seittenranta: *Möbius-invariant metrics* , Math. Proc. Cambridge Philos. Soc. (to appear)

[Se2] P. Seittenranta: *Linear dilatation of quasiconformal maps in space*, Duke Math. J. (to appear).

[SeC] A. Selberg and S. Chowla: *On Epstein's zeta function*, J. Reine Angew. Math. 227 (1967), 87–110.

[Sem1] V. I. Semenov: *On some dynamic systems and quasiconformal mappings* (Russian), Sibirsk. Mat. Zh. 28 (1987), 196–206.

[Sem2] V. I. Semenov: *Certain applications of the quasiconformal and quasiisometric deformations*, Rev. Roumaine Math. Pures Appl. 36 (1991), 503–511.

[S1] S. W. Semmes : *Chord-arc surfaces with small constant II: good parameterizations*, Manuscript

[S2] S. W. Semmes : *On the nonexistence of bilipschitz parameterizations and geometric problems about A_∞-weights*, Rev. Mat. Iberoamericana 12 (1996), 337-410.

[SF] D.-S. Shah and L.-L. Fan: *On the modulus of quasiconformal mappings*, Science Record, New Ser. Vol. 4 (5) (1960), 323–328.

[Sol] A. Yu. Solynin: *Moduli of doubly-connected domains and conformally invariant metrics (Russian)*, Zap. Nautsh. Semin. LOMI, tom 196 (1991), 122–131, Sankt Peterburg "Nauka", 1991.

[SolV] A. Yu. Solynin and M. Vuorinen: *Extremal problems and symmetrization for plane ring domains*, Trans. Amer. Math. Soc. 348 (1996), 4095-4112.

[St] N. Steinmetz: *Jordan and Julia,* Math. Ann. 307 (1997), 531–541.

[T1] O. Teichmüller: *Untersuchungen über konforme und quasikonforme Abbildung*, Deutsche Math. 3 (1938), 621–678.

[T2] O. Teichmüller: *Gesammelte Abhandlungen, herausgegeben von L. V. Ahlfors und F. W. Gehring*, Berlin, New York, Springer-Verlag, 1982.

[To1] T. Toro: *Geometric conditions and existence of bi-lipschitz parametrizations,* Duke Math. J. 77 (1995), 193-227.

[To2] T. Toro: *Surfaces with generalized second fundamental form in L^2 are Lipschitz manifolds,* J. Differential Geom. 39 (1994), 65-101.

[Tr] D. A. Trotsenko: *Fractal lines and quasisymmetries (Russian),* Sibirsk. Mat. Zh. 36 (1996), 1399-1415.

[Tu1] P. Tukia: *The planar Schönflies theorem for Lipschitz maps*, Ann. Acad. Sci. Fenn. Ser. A I 5 (1980), 49–72.

[Tu2] P. Tukia: *A quasiconformal group not isomorphic to a Möbius group*, Ann. Acad. Sci. Fenn. Ser. A I 6 (1981), 149–160.

[Tu3] P. Tukia: *Hausdorff dimension and quasisymmetric mappings*, Math. Scand. 65 (1989), 152–160.

[TV1] P. Tukia and J. Väisälä: *Quasisymmetric embeddings of metric spaces*, Ann. Acad. Sci. Fenn. Ser. A 1 (1980), 97–114.

[TV2] P. Tukia and J. Väisälä: *Extension of embeddings close to isometries or similarities*, Ann. Acad. Sci. Fenn. Ser. A 9 (1984), 153–175.

[V1] J. Väisälä: *Lectures on n-Dimensional Quasiconformal Mappings*, Lecture Notes in Math. Vol. 229, Springer-Verlag, Berlin – Heidelberg – New York, 1971.

[V2] J. Väisälä: *A survey of quasiregular maps in R^n*, in Proc. 1978 Int. Cong. Math. (Helsinki, Finland), Vol. II, Academia Scientiarum Fennica, Helsinki, 1980, pp. 685–691.

[V3] J. Väisälä: *Quasisymmetric embeddings in euclidean spaces*, Trans. Amer. Math. Soc. 264 (1981), 191–204.

[V4] J. Väisälä: *Quasimöbius maps*, J. Analyse Math. 44 (1984/85), 218–234.

[V5] J. Väisälä: *Bilipschitz and quasisymmetric extension properties,* Ann. Acad. Sci. Fenn. Ser. 11 (1986), 239-274.

[V6] J. Väisälä: *Domains and Maps*, in Quasiconformal Space Mappings: A Collection of Surveys, ed. by M. Vuorinen, Lecture Notes in Math., Vol. 1508, Springer-Verlag, 1992, pp. 119–131.

[V7] J. Väisälä: *The free quasiworld*, Lectures in the Vth Finnish-Polish-Ukrainian Summer School in Complex Analysis, Lublin, 15-21 August 1996.

[VVW] J. Väisälä, M. Vuorinen, and H. Wallin: *Thick sets and quasisymmetric maps*, Nagoya Math. J. 135 (1994), 121-148.

[VV1] M. K. Vamanamurthy and M. Vuorinen: *Inequalities for means*, J. Math. Anal. Appl. 183 (1994),155–166.

[VV2] M. K. Vamanamurthy and M. Vuorinen: *Functional inequalities, Jacobi products, and quasiconformal maps*, Illinois J. Math. 38 (1994), 394–419 .

[Var] V. S. Varadarajan: *Linear meromorphic differential equations: A modern point of view*, Bull. Amer. Math. Soc. 33 (1996), 1–42.

[Va] A. Varchenko: *Multidimensional hypergeometric functions and their appearance in conformal field theory, algebraic K-theory, algebraic geometry, etc.*, Proc. Internat. Congr. Math. (Kyoto, Japan, 1990), 281–300.

[Vu1] M. Vuorinen: *Conformal invariants and quasiregular mappings*, J. Analyse Math. 45 (1985), 69–115.

[Vu2] M. Vuorinen: *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math., Vol. 1319, Springer-Verlag, Berlin – Heidelberg – New York, 1988.

[Vu3] M. Vuorinen: *Quadruples and spatial quasiconformal mappings*, Math. Z. 205 (1990), 617–628.

[Vu4] M. Vuorinen: *Conformally invariant extremal problems and quasiconformal maps*, Quarterly J. Math. Oxford Ser. (2), 43 (1992), 501–514.

[Vu5] M. Vuorinen (ed.): *Quasiconformal Space Mappings*, A Collection of Surveys 1960–90, Lecture Notes in Math. 1508, Springer-Verlag, Berlin – Heidelberg – New York, 1992.

[Vu6] M. Vuorinen: *Singular values, Ramanujan modular equations, and Landen transformations*, Studia Math. 121 (1996), 221-230.

[We] S. Werner: *Spiegelungskoeffizient und Fredholmscher Eigenwert für Polygone,* Ann. Acad. Sci. Fenn. Ser. A I Math. 22 (1997), 165-186.

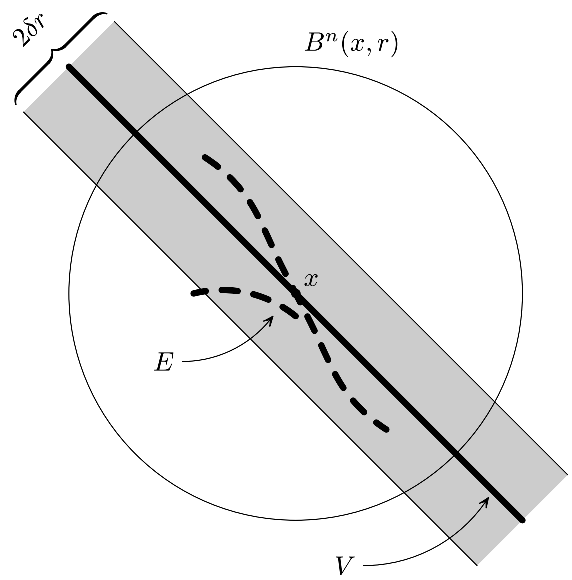
[WW] E. T. Whittaker and G. N. Watson: *A Course of Modern Analysis, 4th ed.*, Cambridge Univ. Press, London, l958.

[WZ1] H. S. Wilf and D. Zeilberger: *Rational functions certify combinatorial identities*, J. Amer. Math. Soc. 3 (1990), 147–158.

[WZ2] H. S. Wilf and D. Zeilberger: *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math. 108 (1992), 575–633.

[Za1] J. Zając: *The distortion function Φ_K and quasihomographies*, in Current Topics in Analytic Function Theory, ed. by H. M. Srivastava and S. Owa, World Scientific Publ. Co., Singapore – London (1992), 403–428.

[Za2] J. Zając: *Quasihomographies in the theory of Teichmüller spaces*, Dissertationes Math. (Rozprawy Mat.) CCCLVII (1996), 1–102.



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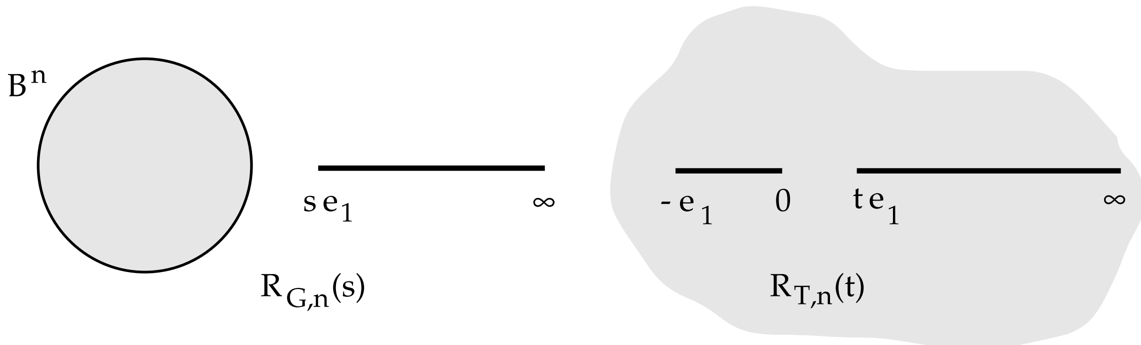

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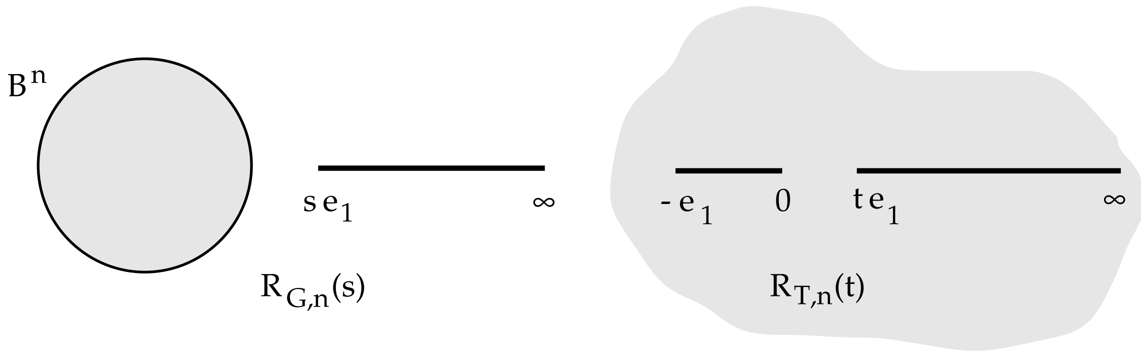

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