Some Integer Sequences Based on Derangements

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Abstract

Sequences whose terms are equal to the number of functions with specified properties are considered. Properties are based on the notion of derangements in a more general sense. Several sequences which generalize the standard notion of derangements are thus obtained. These sequences generate a number of integer sequences from the well-known Sloane's encyclopedia.

Let A be an $m \times n$ rectangular area whose elements are from a set Ω, and let $c_1, \ldots, c_m$ be from Ω. Following the paper [1], we call each column of A which is equal to $[c_1, \ldots, c_m]^T$ an i-column of A. As usual by $[n]$ will be denoted the set $\{1, 2, \ldots, n\}$, and by $|X|$ the number of elements of a finite set X. Mutually disjoint subsets are called blocks. A block with k elements is called k-block. We also denote by $n^{(m)}$ the falling factorials, that is, $n^{(m)} = n(n-1)\cdots(n-m+1)$. Stirling numbers of the second kind will be denoted by $S(m, n)$.

We start with the following:

Theorem 1 *Suppose that $X_1, X_2, \ldots, X_k$ are blocks in $[m]$ and $Y_1, Y_2, \ldots, Y_k$ are subsets in $[n]$. Label all functions $f : [m] \to [n]$ by $1, 2, \ldots, n^m$ arbitrary and form a $k \times n^m$ matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if $f_j(X_i) \subseteq Y_i$, and $a_{ij} = 0$ otherwise. The number D_1 of i-columns of A consisting of 0's is equal*

$$D = \sum_{I \subseteq [k]} (-1)^{|I|} A(I), \tag{1}$$

where

$$A(I) = n^{|[m] \setminus \cup_{i \in I} X_i|} \cdot \prod_{i \in I} |Y_i|^{|X_i|}, \tag{2}$$

and I runs over all subsets of $[m]$.

Proof. According to Theorem 1.1 in [1], the number D is equal to the right side of (1) if $A(I)$ is the maximal number of columns j of A such that $a_{ij} = 1$ for all $i \in I$. It follows that $A(I)$ is equal to the number of functions $f : [m] \to [n]$ such that $f(X_i) \subseteq Y_i,\ (i \in I)$. This number is clearly equal to the number on the right side of (2).

In a similar way we obtain the following:

Theorem 2 *Suppose that $X_1, X_2, \ldots, X_k$ are blocks in $[m]$ and $Y_1, Y_2, \ldots, Y_k$ are subsets of $[n]$. Label all functions $f : [m] \to [n]$ by $1, 2, \ldots, n^m$ arbitrary, and form a $k \times n^m$ matrix $B = (b_{ij})$ such that $b_{ij} = 1$ if $f_j(X_i) = Y_i$, and $a_{ij} = 0$ otherwise. The number N of i-columns of A consisting of 0's is equal*

$$N = \sum_{I \subseteq [k]} (-1)^{|I|} B(I),$$

where

$$B(I) = n^{|[m] \setminus \cup_{i \in I} X_i|} \cdot \prod_{i \in I} |Y_i|! S(|X_i|, |Y_i|),$$

and I runs over all subsets of $[m]$.

Depending on the number of elements of $X_1, \ldots, X_k; Y_1, \ldots, Y_k$ it is possible to obtain a number of different sequences. Consider first the simplest case when each $X_1, \ldots, X_k;\ Y_1, \ldots, Y_k$ consists of one element. Then

$$A(I) = n^{m-|I|},$$

so that Theorem 1.2 of [1] may be applied. We thus obtain the following consequence of Theorem 1.

Corollary 1 *Given distinct $x_1, \ldots, x_k$ in $[m]$ and arbitrary $y_1, \ldots, y_k$ in $[n]$, then the number $D_{11}(m, n, k)$ of functions $f : [m] \to [n]$ such that*

$$f(x_i) \neq y_i,\ (i = 1, 2, \ldots, k),$$

is equal

$$D_{11}(m, n, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} n^{m-i} \left(= n^{m-k}(n-1)^k\right).$$

A number of sequences in [2] is generated by this simple function. Some of them are stated in the following:

Table 1.

1. $A001477(n) = D_{11}(1, n, 1)$,	2. $A002378(n) = D_{11}(2, n, 1)$,
3. $A045991(n) = D_{11}(3, n, 1)$,	4. $A085537(n) = D_{11}(4, n, 1)$,
5. $A085538(n) = D_{11}(5, n, 1)$,	6. $A085539(n) = D_{11}(6, n, 1)$,
7. $A000079(n) = D_{11}(n, 2, 1)$,	8. $A008776(n) = D_{11}(n, 3, 1)$,
9. $A002001(n) = D_{11}(n, 4, 1)$,	10. $A005054(n) = D_{11}(n, 5, 1)$,
11. $A052934(n) = D_{11}(n, 6, 1)$,	12. $A055272(n) = D_{11}(n, 7, 1)$,
13. $A055274(n) = D_{11}(n, 8, 1)$,	14. $A055275(n) = D_{11}(n, 9, 1)$,
15. $A052268(n) = D_{11}(n, 10, 1)$,	16. $A055276(n) = D_{11}(n, 11, 1)$,
17. $A000290(n) = D_{11}(2, n, 2)$,	18. $A011379(n) = D_{11}(3, n, 2)$,
19. $A035287(n) = D_{11}(4, n, 2)$,	20. $A099762(n) = D_{11}(5, n, 2)$,
21. $A000079(n) = D_{11}(n, 2, 2)$,	22. $A003946(n) = D_{11}(n, 3, 2)$,
23. $A002063(n) = D_{11}(n, 4, 2)$,	24. $A055842(n) = D_{11}(n, 5, 2)$,
25. $A055846(n) = D_{11}(n, 6, 2)$,	26. $A055270(n) = D_{11}(n, 7, 2)$,
27. $A055847(n) = D_{11}(n, 8, 2)$,	28. $A055995(n) = D_{11}(n, 9, 2)$,
29. $A055996(n) = D_{11}(n, 10, 2)$,	30. $A056002(n) = D_{11}(n, 11, 2)$,
31. $A056116(n) = D_{11}(n, 12, 2)$,	32. $A076728(n) = D_{11}(n, n, 2)$,
33. $A000578(n) = D_{11}(3, n, 3)$,	34. $A005051(n) = D_{11}(n, 3, 3)$,
35. $A056120(n) = D_{11}(n, 4, 3)$,	36. $A000583(n) = D_{11}(4, n, 4)$,
37. $A101362(n) = D_{11}(5, n, 4)$,	38. $A118265(n) = D_{11}(n, 4, 4)$.

Suppose that

$$|X_1| = |X_2| = \ldots = |X_k| = 1,\ |Y_1| = |Y_2| = \cdots = |Y_k| = 2.$$

Then

$$A(I) = 2^i n^{m-|I|}.$$

We may again apply Theorem 1.2 in [2] to obtain the following:

Corollary 2 *Given distinct $x_1, \ldots, x_k$ in $[m]$ and arbitrary 2-sets $Y_1, \ldots, Y_k$ in $[n]$, then the number $D_{12}(m, n, k)$ of functions $f : [m] \to [n]$ such that*

$$f(x_i) \notin Y_i,\ (i = 1, 2, \ldots, k),$$

is equal

$$D_{12}(m, n, k) = \sum_{i=0}^{k} (-2)^i \binom{k}{i} n^{m-i} \left(= n^{m-k}(n-2)^k\right).$$

This function also generates a number of sequences in [2].The following table contains some of them.

Table 2.

1. $A000027(n) = D_{12}(1, n, 1)$,	2. $A005563(n) = D_{12}(2, n, 1)$
3. $A027620(n) = D_{12}(3, n, 1)$,	4. $A000244(n) = D_{12}(n, 3, 1)$,
5. $A004171(n) = D_{12}(n, 4, 1)$,	6. $A005053(n) = D_{12}(n, 5, 1)$,
7. $A067411(n) = D_{12}(n, 6, 1)$,	8. $A000290(n) = D_{12}(2, n, 2)$,
9. $A0002444(n) = D_{12}(n, 3, 2)$,	10. $A000578(n) = D_{12}(3, n, 3)$,
11. $A081294(n) = D_{12}(n, 4, 3)$,	12. $A000583(n) = D_{12}(4, n, 4)$,

If, in the conditions of Theorem1, hold

$$|X_1| = \cdots = |X_k| = 2;\ |Y_1| = \cdots |Y_k| = 1,$$

then

$$A(I) = n^{m-2|I|},$$

so that we have the following:

Corollary 3 *Suppose that $X_1, \ldots, X_k$ are 2-blocks in $[m]$, and $y_1, \ldots, y_k$ arbitrary elements in $[n]$, then the number $D_{21}(m, n, k)$ of functions $f : [m] \to [n]$ such that*

$$f(X_i) \neq \{y_i\},\ (i = 1, 2, \ldots, k)$$

is equal

$$D_{21}(m, n, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} n^{m-2i} \left(= n^{m-2k}(n^2 - 1)^k\right).$$

We also state some sequences in [2] generated by this function.

Table 3.

1. $A005563(n) = D_{21}(2, n, 1)$,	2. $A007531(n) = D_{21}(3, n, 1)$
3. $A047982(n) = D_{21}(4, n, 1)$,	4. $A005051(n) = D_{21}(n, 3, 1)$,
5. $A005010(n) = D_{21}(n, 2, 2)$,	

Take finally the case $|X_i| = |Y_i| = 2,\ (i = 1, 2, \ldots, k))$. We have now

$$A(I) = 4^{|I|} \cdot n^{m-2|I|}.$$

We thus obtain the following consequence of Theorem1.

Corollary 4 *Let $X_1, \ldots, X_k$ in $[m]$ be 2-blocks, and $Y_1, \ldots, Y_k$ in $[n]$ be arbitrary 2-sets. Then the number $D_{22}(m,n,k)$ of functions $f : [m] \to [n]$ such that*

$$f(X_i) \not\subseteq Y_i, \ (i = 1, 2, \ldots, k)$$

is equal

$$D_{22}(m,n,k) = \sum_{i=0}^{k} (-4)^i \binom{k}{i} n^{m-2i} \left(= n^{m-2k}(n^2-4)^k\right).$$

A few sequences in [2], given in the next table, is defined by this function.

Table 4.

1. $A005030(n) = D_{22}(n,3,1)$, 2. $A002001(n) = D_{22}(n,4,1)$
3. $A002063(n) = D_{22}(n,4,2)$,

Take now the case $|X_i| = |Y_i| = 2, \ (i = 1, 2, \ldots, k))$ in the conditions of Theorem 2. We have

$$B(I) = 2^{|I|} \cdot n^{m-2|I|}.$$

Thus we have the next:

Corollary 5 *Let $X_1, \ldots, X_k$ be 2-blocks in $[m]$ and $Y_1, \ldots, Y_k$ arbitrary 2-sets in $[n]$. Then the number $S_{22}(m,n,k)$ of functions $f : [m] \to [n]$ such that*

$$f(X_i) \neq Y_i, \ (i = 1, 2, \ldots, k)$$

is equal

$$S_{22}(m,n,k) = \sum_{i=0}^{k} (-2)^i \binom{k}{i} n^{m-2i} \left(= n^{m-2k}(n^2-2)^k\right).$$

The sequence A005032 in [2] is generated by this function.

We shall now consider injective functions from $[m]$ to $[n]$, $(m \leq n)$. We start with the following:

Theorem 3 *Let $X_1, X_2, \ldots, X_k$ be blocks in $[m]$ and $Y_1, Y_2, \ldots, Y_k$ blocks in $[n]$ such that*

$$|X_i| = |Y_i|, \ (i = 1, 2, \ldots, k).$$

If a $k\times n^{(m)}$ matrix A is defined such that $a_{ij}=1$ if $f_j(X_i)=Y_i$ and $a_{ij}=0$ otherwise, then the number $I(m,n,k)$ of i-columns of A consisting of 0's is equal

$$I_k(m,n)=\sum_{I\subseteq[k]}(-1)^{|I|}(n-|\cup_{i\in I}X_i|)^{(m-|\cup_{i\in I}X_i|)}\cdot\prod_{i\in I}|X_i|!.$$

Proof. In this case we have

$$A(I)=(n-|\cup_{i\in I}X_i|)^{(m-|\cup_{i\in I}X_i|)}\cdot\prod_{i\in I}|X_i|!,$$

so that theorem follows from Theorem 1.1. in [1].

We shall also state some particular cases of this theorem. Suppose first that

$$|X_i|=|Y_i|=1,\ (i=1,\dots,k).$$

The number $A(I)$ in this case is equal

$$(n-|I|)^{(m-|I|)}.$$

We thus obtain the following:

Corollary 6 *For disjoint $x_1,\dots,x_k$ in $[m]$ and disjoint $y_1,\dots,y_k$ in $[m]$, the number $I_1(m,n,k)$ of injections $f:[m]\to[n]$ such that*

$$f(x_i)\neq y_i,\ (i=1,2,\dots,k)$$

is equal

$$I_1(m,n,k)=\sum_{i=0}^{k}(-1)^i\binom{k}{i}(n-i)^{(m-i)}.$$

Note 1 *Since obviously holds $D(n)=I(n,n,n)$, where $D(n)$ is the number of derangements of n elements, this function is an extension of derangements.*

There are a number of sequences in [2] that are generated by this function. We state some of them in the next table.

Table 5.

1. $A000290(n) = I(2, n, 1)$,
2. $A045991(n) = I(3, n, 1)$
3. $A114436(n) = I(3, n, 1)$
4. $A047929(n) = I(4, n, 1)$,
5. $A001563(n) = I(n, n, 1)$,
6. $A001564(n) = I(n, n, 2)$,
7. $A001565(n) = I(n, n, 3)$,
8. $A002061(n) = I(2, n, 2)$,
9. $A027444(n) = I(3, n, 2)$,
10. $A058895(n) = I(4, n, 2)$,
11. $A027444(n) = I(3, n, 2)$,
12. $A074143(n) = I(n-1, n, 1)$,
13. $A001563(n) = I(n-1, n, 1)$,
14. $A094304(n) = I(n-1, n, 1)$,
15. $A109074(n) = I(n-1, n, 1)$,
16. $A094258(n) = I(n-1, n, 1)$,
17. $A001564(n) = I(n-1, n, 2)$,
18. $A001565(n) = I(n-1, n, 3)$,
19. $A001688(n) = I(n-1, n, 4)$
20. $A001689(n) = I(n-1, n, 5)$,
21. $A023043(n) = I(n-1, n, 6)$,
22. $A023044(n) = I(n-1, n, 7)$,
23. $A023045(n) = I(n-1, n, 8)$,
24. $A023046(n) = I(n-1, n, 9)$,
25. $A023407(n) = I(n-1, n, 10)$,
26. $A001563(n) = I(n-2, n, 1)$,
27. $A001564(n) = I(n-2, n, 2)$,
28. $A061079(n) = I(n, 2n, 1)$.

As a special case we also have the following generalization of derangements.

Corollary 7 *If $X_1, X_2, \ldots, X_n$ is a partition of $[kn]$ such that*

$$|X_i| = k,\ (i = 1, 2, \ldots, n)),$$

then the number $D(n, k)$ of permutations f of $[kn]$ such that $f(X_i) \neq X_i,\ (i = 1, 2, \ldots, n)$ is equal

$$D(n, k) = \sum_{i=0}^{n} (-1)^i (k!)^i (nk - ik)!.$$

For $k = 1$ we obtain the standard formula for derangements.

Note 2 *From the preceding formula the following sequences in [2] are derived:*
$A128805, A127888, A116221, A116220, A116219$.

References

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[2] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*

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A001477, A002378, A045991, A085537, A085538, A085539, A000079, A008776, A002001, A005054, A052934, A055272, A055274, A055275, A052268, A055276, A000290, A011379, A035287, A099762, A000079, A003946, A002063, A055842, A055846, A055270, A055847, A055995, A055996, A056002, A056116, A076728, A000578, A005051, A056120, A000583, A101362, A118265, A000027, A005563, A027620, A000244, A004171, A005053, A067411, A000290, A002444, A000578, A081294, A000583, A005563, A007531, A047982, A005051, A005010, A005032, A005030, A002001, A002063, A005032, A000290, A045991, A114436, A047929, A001563, A001564, A001565, A002061, A027444, A058895, A027444, A074143, A001563, A094304, A109074, A094258, A001564, A001565, A001688, A001689, A023043, A023044, A023045, A023046, A023407, A001563, A001564, A061079, A128805, A127888, A116221, A116220, A116219.