# DETERMINANT FORMULAS RELATING TO
# TABLEAUX OF BOUNDED HEIGHT

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ABSTRACT. Chen et al. recently established bijections for $(d+1)$-noncrossing/ nonnesting matchings, oscillating tableaux of bounded height $d$, and oscillating lattice walks in the $d$-dimensional Weyl chamber. Stanley asked what is the total number of such tableaux of length $n$ and of any shape. We find a determinant formula for the exponential generating function. The same idea applies to prove Gessel's remarkable determinant formula for permutations with bounded length of increasing subsequences. We also give short algebraic derivations for some results of the reflection principle.

## 1. INTRODUCTION

For a partition $\lambda = (\lambda_1, \ldots, \lambda_d)_\geq$ of length (or height) at most $d$, we associate it with a $\bar{\lambda} := \lambda + (d, d-1, \ldots, 1)$. Then $\bar{\lambda}$ belongs to the $d$-dimensional *Weyl chamber* defined by $W^d = \{ (x_1, \ldots, x_d) : x_1 > \cdots > x_d > 0, x_i \in \mathbb{Z} \}$. In particular, we denote by $\bar{\mathbf{0}} = (d, d-1, \ldots, 1)$ the associate of the empty partition $\varnothing$. For $\bar{\lambda}, \bar{\mu} \in W^d$, let $b_n(\bar{\lambda}; \bar{\mu})$ be the number of *Weyl oscillating lattice walks* of length $n$, from $\bar{\lambda}$ to $\bar{\mu}$, staying within $W^d$, with steps *positive* or *negative* unit coordinate vectors.

**Theorem 1** (Grabiner-Magyar [8], Equation 26)**.** *For fixed $\bar{\lambda}, \bar{\mu} \in W^d$, we have a determinant formula for the exponential generating function:*

$$g_{\bar{\lambda}\bar{\mu}}(t) = \sum_{n \geq 0} b_n(\bar{\lambda}; \bar{\mu}) \frac{t^n}{n!} = \det \left( I_{\bar{\mu}_i - \bar{\lambda}_j}(2t) - I_{\bar{\mu}_i + \bar{\lambda}_j}(2t) \right)_{1 \leq i,j \leq d}, \tag{1}$$

*where*

$$I_s(2t) = [z^s] \exp(t(z + z^{-1})) = \sum_{n \geq 0} \frac{1}{n!(n+s)!} t^{2n+s} \tag{2}$$

*is the hyperbolic Bessel function of the first kind of order $s$.*

Chen et al. [3] recently established bijections showing that $(d+1)$-noncrossing (nonnesting) matchings and oscillating tableaux are in bijection with certain Weyl oscillating lattice walks. Then Stanley asked (by private communication) the following question: How many Weyl oscillating lattice walks of length $n$ are there if we start at $\bar{\mathbf{0}}$ but may end anywhere? Our main result answers this question:

**Theorem 2.** *The exponential generating function for the number of oscillating lattice walks in $W^d$ starting at $\bar{\mathbf{0}} = (d, d-1, \ldots, 1)$, and with no restriction on the*

*end points is*

$$G(t) := \sum_{n \geq 0} \sum_{\mu \in W^d} b_n(\bar{\mathbf{0}}; \mu) \frac{t^n}{n!} = \det(J_{i-j}(2t))_{1 \leq i,j \leq d}, \qquad (3)$$

*where* $J_s(2t) = [z^s] \, (1+z) \exp((z + z^{-1})t) = I_s(2t) + I_{s-1}(2t)$.

Terminologies not presented here will be given in section 2, where we will explore the connection of oscillating tableaux with the Brauer algebra and symplectic group, just as that of standard Young tableaux (SYTs for short) with the symmetric group and general linear group. We will see that Theorem 2 actually gives a determinant formula for oscillating tableaux of bounded height, which is an analogy of Gessel's formula for SYTs of bounded height.

Section 3 is for completeness of section 4, but is of some independent interest. We describe a simple algebraic derivation of the hook-length formula (Theorem 6) and Theorem 1, as well as some notations. The method is easily seen to apply to many other results of the reflection principle. One can see from the proof a reason why using exponential generating function is preferable in this context.

Section 4 includes the derivation of Theorem 2. Starting from the Grabiner-Magyar formula, one can obtain a constant term expression that can be used to do algebraic calculation. The theorem is then derived in three key steps: we first apply the Stanton-Stembridge trick (a kind of symmetrization), then a classical formula for symmetric functions, and finally reversely apply the Stanton-Stembridge trick. The same idea applies to prove a generalized form (Theorem 11) of Gessel's remarkable determinant formula [5]:

**Theorem 3** (Gessel). *Let $u_d(n)$ be the number of permutations on $\{1, 2, \ldots, n\}$ with longest increasing subsequences of length at most $d$. Then*

$$\sum_{n \geq 0} u_d(n) \frac{t^2}{n!^2} = \det(I_{i-j}(2t))_{1 \leq i,j \leq d}. \qquad (4)$$

Our starting point is the well-known hook-length formula.

## 2. NOTATIONS, CONNECTIONS, AND APPLICATIONS

In this section, we will introduce many objects and try to explain their connections with the classical objects for the symmetric group, in the view of enumeration. Some of the connections are in [12, Section 9], whose notations we shall closely follow. Finally, we will give some applications of Theorem 2.

2.1. **Notations.** We assume basic knowledge of the symmetric group $\mathfrak{S}_n$ and its representation. See, e.g., [11, Chapter 7]. Now we introduce some objects.

The *Brauer algebra* $\mathfrak{B}_n$ (depending on a parameter $x$ which is irrelevant here) is a certain semisimple algebra with the underlying space the linear span (say over $\mathbb{C}$) of (complete) matchings on $[2n] = \{1, 2, \ldots, 2n\}$. The dimension of $\mathfrak{B}_n$ is

$$\dim \mathfrak{B}_n = (2n-1)!! = (2n-1)(2n-3) \cdots 3 \cdot 1.$$

Its irreducible representations are indexed by partitions of $n - 2r$, for $0 \leq r \leq \lfloor n/2 \rfloor$. The dimension of the irreducible representation $\mathfrak{B}^\mu$ is equal to $\tilde{f}_n^\mu$, that we are going to introduce.

An *oscillating tableau* (or *up-down tableau*) of shape $\mu$ and length $n$ is a sequence $(\varnothing = \mu^0, \mu^1, \ldots, \mu^n = \mu)$ of partitions such that for all $1 \leq i \leq n-1$, the diagram

of $\mu^i$ is obtained from $\mu^{i-1}$ by either adding or removing one square. Denote by $\tilde{f}_n^\mu$ the number of such tableaux. It is known that if $\mu$ is a partition of $n - 2r$ for some nonnegative integer $r$, then

$$\tilde{f}_n^\mu = \binom{n}{2r}(2r - 1)!!f^\mu, \qquad \mu \vdash (n - 2r),$$

where $f^\mu$ is the number of standard Young tableaux of shape $\mu$. See, e.g., [2, Appendix B6] for further information.

Denote by $\mathfrak{M}_n$ the set of matchings on $[2n]$. A *matching* $M \in \mathfrak{M}_n$ is a partition of $[2n]$ into $n$ two-blocks, written in the form $\{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\}$. We also write $(i_k, j_k)$ for $\{i_k, j_k\}$ if $i_k < j_k$. We represent $M$ by a diagram obtained by identifying $i$ with $(i, 0)$ in the plane for $i = 1, \ldots, 2n$, and drawing arcs, called edges, from $i_k$ to $j_k$ above the horizontal $x$-axis for all $k$. For $d \geq 2$, a $d$-crossing of a matching $M$ is a set of $d$ arcs $(i_{r_1}, j_{r_1}), (i_{r_2}, j_{r_2}), \ldots, (i_{r_d}, j_{r_d})$ of $M$ such that $i_{r_1} < i_{r_2} < \cdots < i_{r_d} < j_{r_1} < j_{r_2} < \cdots < j_{r_d}$. A matching without any $d$-crossing is called a $d$-noncrossing matching. We omit here the similar definition of $d$-nesting. Figure 1 shows the diagram corresponding to the matching

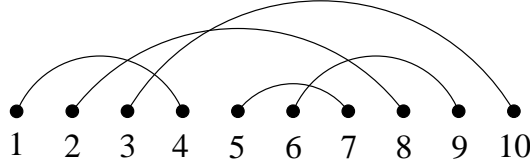$$M = \{\{1, 4\}, \{2, 8\}, \{3, 10\}, \{5, 7\}, \{6, 9\}\}.$$



FIGURE 1. A matching on [10], in which the edges $\{1, 4\}, \{2, 8\}, \{3, 10\}$ form a 3-crossing.

Now we introduce apparently new objects. For an oscillating tableau $O$ of shape $\varnothing$ (hence of even length), reading $O$ backwardly still gives an oscillating tableau of shape $\varnothing$, denoted by $O^{rev}$. We say that $O$ is *palindromic* if $O = O^{rev}$. For a matching $M$ of $[2n]$, let $M^{refl}$ denote the matching obtained from $M$ by reflecting in the vertical line $x = n + 1/2$. Figure 2 shows the diagram corresponding to $M^{refl}$. Then $M$ is said to be *bilaterally symmetric* if $M = M^{refl}$. Equivalently, $(i, j)$ is an edge of $M$ if and only if so is $(2n + 1 - j, 2n + 1 - i)$.
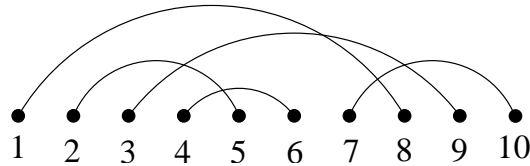


FIGURE 2. The reflection of the matching in Figure 1.

**Proposition 4.** *The exponential generating function of the number* $\mathrm{bsm}_n$ *of bilaterally symmetric matchings on* $[2n]$ *is*

$$\sum_{n \geq 0} \mathrm{bsm}_n \frac{t^n}{n!} = \exp(t + t^2).$$

*Proof.* For a bilaterally symmetric matching $M$ on $[2n]$, identify it with the graph $M'$ obtained from $M$ by adding the (dashed) edges $(i, 2n+1-i)$ for $i = 1, 2, \ldots, n$. Then every vertex of $M'$ has degree 2, so that $M'$ can be uniquely decomposed into connected components, each being a cycle. The cycles can be of only three types, as drawn in Figure 3. Therefore, the lemma follows by the well-known exponential
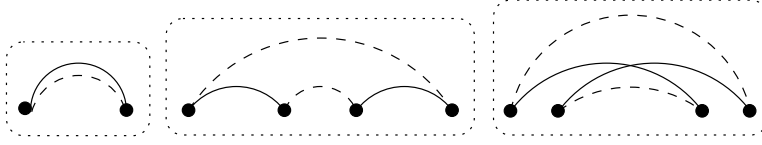


FIGURE 3. The types of connected components of bilaterally symmetric matchings.

formula for generating functions. See, e.g., [11, Corollary 5.1.6]. □

2.2. **Connections.** We first give a list of the classical objects and their analogies:

| Classical Objects | Their analogies |
|---|---|
| the symmetric group $\mathfrak{S}_n$ | the Brauer algebra $\mathfrak{B}_n$ |
| the general linear group $GL(d)$ | the symplectic group $Sp(2d)$ |
| standard Young tableaux | oscillating tableaux |
| permutations on $[n]$ | matchings on $[2n]$ |
| involutions | bilaterally symmetric matchings |

Next we give connections in the view of enumeration. By a well-known result in representation theory, we have

$$\sum_{\mu \vdash (n-2r)} (\tilde{f}_n^\mu)^2 = (2n-1)!!, \qquad \sum_{\mu \vdash n} (f^\mu)^2 = n!, \tag{5}$$

where the first sum ranges over all nonnegative integers $r$ with $0 \leq r \leq \lfloor n/2 \rfloor$ and partitions $\mu$ of $n - 2r$. The former equation of (5) is for $\mathfrak{B}_n$ and the latter one is for $\mathfrak{S}_n$; $\tilde{f}_n^\mu$ and $f^\mu$ are the dimension of the corresponding irreducible representations. We shall always list the analogous formula before the classical one. Equation (5) suggests a RSK-correspondence for matchings just as that for permutations. Observe that a pair of oscillating tableaux of the same shape of length $n$ can be naturally combined as one oscillating tableau of shape $\varnothing$ of length $2n$. To be precise, the decomposition $\gamma(O) = (P, Q)$ is given by

$$\gamma : (\varnothing = \mu^0, \mu^1, \ldots, \mu^{2n} = \varnothing) \longmapsto ((\varnothing = \mu^0, \mu^1, \ldots, \mu^n), (\varnothing = \mu^{2n}, \mu^{2n-1}, \ldots, \mu^n)).$$

Thus it is sufficient to construct a bijection from the set $\mathfrak{M}_n$ of matchings to the set $\mathcal{O}_n$ of oscillating tableaux of the empty shape and length $2n$. Such a bijection was first given by Stanley (unpublished), and was extended by Sundaram [13] to arbitrary shapes to give a combinatorial proof of the Cauchy identity for

the symplectic group $Sp(2d)$, and was recently extended by Chen et al. [3] for partitions.

Let $\Phi$ be the bijection from $\mathfrak{M}_n$ to $\mathcal{O}_n$ defined in [12, Section 9]. Then it was shown in [3] that $\Phi$ has many properties. We will use the fact that the maximum number of crossings of a matching $M$ is equal to the maximum height of the oscillating tableau $\Phi(M)$.

We will use the following result of [15]: For any $M \in \mathfrak{M}_n$, we have $\Phi(M^{refl}) = \Phi(M)^{rev}$.

Since the number of palindromic oscillating tableaux is equal to the number of bilaterally symmetric matchings $\mathrm{bsm}_n$, we have, by Proposition 4,

$$\sum_{\mu \vdash (n-2r)} \tilde{f}_n^\mu = \left[\frac{t^n}{n!}\right] \exp(t + t^2), \qquad \sum_{\mu \vdash n} f^\mu = \left[\frac{t^n}{n!}\right] \exp(t + t^2/2), \qquad (6)$$

where the first sum ranges over all nonnegative integers $r$ with $0 \leq r \leq \lfloor n/2 \rfloor$ and partitions $\mu$ of $n - 2r$. The right equation of (6) counts the number of involutions.

There are analogous results if we put restrictions on the height of the tableaux. A $d$-oscillating tableau, also called $d$-symplectic up-down tableau, is an oscillating tableau of a bounded height $d$, by which we mean that the height of every $\mu^i$ is no larger than $d$. Denote by $\tilde{f}_n^\mu(d)$ the number of $d$-oscillating tableaux of shape $\mu$ and length $n$. See [13] for more information.

There is a natural bijection showing that $\tilde{f}_n^\mu(d) = b_n(\bar{\mathbf{0}}, \bar{\mu})$, which has a determinant formula as in (1). The bijection simply takes $(\mu^0, \mu^1, \ldots, \mu^n)$ to the sequence of lattice points $(\bar{\mu}^0, \bar{\mu}^1, \ldots, \bar{\mu}^n)$. Therefore results on oscillating lattice walks can be translated into those on oscillating tableaux. Applying $\gamma$ to $d$-oscillating tableaux of shape $\varnothing$ and length $2n$, and applying Theorem 1, we obtain

$$\sum_{\mu \vdash (n-2r)} (\tilde{f}_n^\mu(d))^2 = b_{2n}(\bar{\mathbf{0}}, \bar{\mathbf{0}}) = \left[\frac{t^{2n}}{(2n)!}\right] \det\left(I_{i-j}(2t) - I_{i+j}(2t)\right)_{1 \leq i,j \leq d}, \qquad (7)$$

an analogy of Theorem 3. This is also the number of $(d+1)$-noncrossing/nonnesting matchings. See [3, Equation (9)] (by setting $k = d + 1$) and references therein.

Theorem 2 actually gives

$$\sum_{\mu \vdash (n-2r)} \tilde{f}_n^\mu(d) = \left[\frac{t^n}{n!}\right] \det(I_{i-j}(2t) + I_{i-j-1}(2t))_{1 \leq i,j \leq d}.$$

This is an analogy of Gessel's determinant formula for involutions. See [5]. See also [12, Sections 4&5].

Let $\mathrm{is}(w)$ be the length of the longest increasing subsequences of $w \in \mathfrak{S}_n$, and let $\mathrm{cr}(M)$ be the maximum crossing number of $M \in \mathfrak{M}_n$. Then we have the following table.

| Classical Objects | Their analogy |
|---|---|
| the general linear group $GL(d)$ | the symplectic group $Sp(2d)$ |
| SYT of bounded height $d$ | oscillating tableaux of bounded height $d$ |
| $\{w \in \mathfrak{S}_n : \mathrm{is}(w) \leq d\}$ | $\{M \in \mathfrak{M}_n : \mathrm{cr}(M) \leq d\}$ |
| $\cdots$ & is an involution | $\cdots$ & is bilaterally symmetric |

Stanley [12] gave a nice survey for the study of increasing and decreasing subsequences of permutations and their variants. One major problem in this area is to

understand the behavior of is($w$). For instance, what is the limiting distribution of is($w$) for permutations? Gessel's determinant formula reduces such problem to analysis, which was solved by Baik, Deift, and Johansson [1] using their techniques. In our table, the limiting distribution formulas of is($w$) for permutations and for involutions, and that of cr($M$) for matchings are known. The distribution for bilaterally symmetric matchings should be obtained in a similar way, but this needs to be checked.

2.3. **Applications.** We first summarize several consequences of Theorem 2.

**Corollary 5.** *The following quantities are equal to*

$$\left[\frac{t^n}{n!}\right] \det(I_{i-j}(2t) + I_{i-j-1}(2t))_{1 \le i,j \le d}.$$

(1) *The number of palindromic Weyl oscillating lattice walks of length $2n$ and starting at $\bar{\mathbf{0}}$.*
(2) *The number of palindromic oscillating tableaux of length $2n$.*
(3) *The number* $\mathrm{bsm}_n(d)$ *of bilaterally symmetric $(d+1)$-noncrossing/nonnesting matchings on $[2n]$.*
(4) *The number of oscillating tableaux of any shape and length $n$.*

We can compute $\mathrm{bsm}_n(d)$ for small $d$. For the case $d = 1$, we have

$$\mathrm{bsm}_{2n}(1) = \binom{2n}{n}, \text{ and } \mathrm{bsm}_{2n+1}(1) = \frac{1}{2}\binom{2n+2}{n+1}. \tag{8}$$

This is a direct consequence of Theorem 2, but we give an alternative proof.

*Proof of* (8). By Corollary 5 part (3), we need to compute noncrossing bilaterally symmetric matchings on $[2n]$. Let $P(t)$ be the generating function $P(t) = \sum_{n \ge 0} \mathrm{bsm}_n(1)t^n$. Consider the possibility of the edge $(1, m)$ in the bilaterally symmetric matching $M$. One sees that if $m > n$, then $m$ must equal $2n$ to avoid a crossing. Thus we have the decomposition of $M$ as in Figure 4, where we use semicircles to indicate noncrossing matchings, and use trapezoid to indicate noncrossing bilaterally symmetric matchings. Therefore, we obtain the functional equation:
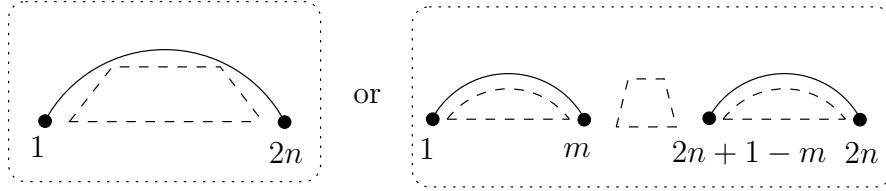


FIGURE 4. Decomposition of noncrossing bilaterally symmetric matchings.

$$P(t) = 1 + tP(t) + t^2 C(t^2)P(t),$$

where $C(t) = \frac{1-\sqrt{1-4t}}{2t}$ is the Catalan generating function, which is known to be the ordinary generating function for noncrossing matchings. Direct algebraic calculation shows (8). $\square$

For the case $d = 2$, we have

$$\mathrm{bsm}_{2n}(2) = \frac{1}{2}\binom{2n+2}{n+1}C_n = \frac{(2n+1)!(2n)!}{(n!(n+1)!)^2}, \tag{9}$$

$$\mathrm{bsm}_{2n+1}(2) = \frac{1}{2}\binom{2n+2}{n+1}C_{n+1} = \frac{(2n+1)!(2n+2)!}{n!(n+1)!^2(n+2)!}. \tag{10}$$

For the case $d = 3$, we obtain

$$\mathrm{bsm}_{2n}(3) = \sum_{s=0}^{n}\frac{2(2s+1)!}{s!^2(s+1)!(s+2)!}\cdot\frac{(2n)!}{(n-s)!(n-s+1)!}, \tag{11}$$

$$\mathrm{bsm}_{2n+1}(3) = \sum_{s=0}^{n}\frac{2(2s+2)!}{s!(s+1)!(s+2)!^2}\cdot\frac{(2n+1)!}{(n-s)!(n-s+1)!}. \tag{12}$$

By a general theory, these sequences are P-recursive, or their generating functions (for any $d$) are D-finite. See, e.g., [11, Chapter 6]. We use the creative telescoping of [9] to find that $\mathrm{bsm}_{2n}(3)$ satisfies a second order P-recursion [11, Chapter 6]:

$$(n+5)(n+4)(n+3)\,\mathrm{bsm}_{2n+4}(3)$$
$$= 4\left(5n^2+30n+43\right)(2n+3)\,\mathrm{bsm}_{2n+2}(3) - 36\,(2n+3)(2n+1)(n+1)\,\mathrm{bsm}_{2n}(3), \tag{13}$$

subject to $\mathrm{bsm}_0(3) = 1, \mathrm{bsm}_2(3) = 3$; and that $\mathrm{bsm}_{2n+1}(3)$ satisfies a $P$-recursion of order 3, which is too lengthy to be given here.

Formulas (9,10) are straightforward by, e.g., the creative telescoping. Formulas (11,12) need some work. We will write $I_i$ for $I_i(2t)$ for short, and use the following facts: $I_i = I_{-i}$, $I_{2i}$ contains only even powers in $t$, and $I_{2i+1}$ contains only odd powers in $t$.

*Proof of (11,12).* By Theorem 2, the exponential generating function is

$$\det\begin{pmatrix} I_1 + I_0 & I_2 + I_1 & I_2 + I_3 \\ I_1 + I_0 & I_1 + I_0 & I_2 + I_1 \\ I_2 + I_1 & I_1 + I_0 & I_1 + I_0 \end{pmatrix}$$
$$= (I_0 - I_2)((I_0{}^2 - I_1{}^2 - I_2{}^2 + I_1 I_3) + (I_0 I_1 + I_0 I_3 - 2I_1 I_2)),$$

where $I_0 - I_2$ contains only even powers in $t$, and in the right factor, we have separated the sum according to the parity of the powers in $t$.

Now it is straightforward, by the creative telescoping, to show that

$$I_0 - I_2 = \sum_{n\geq 0}\frac{t^{2n}}{n!(n+1)!},$$

$$I_0{}^2 - I_1{}^2 - I_2{}^2 + I_1 I_3 = \sum_{n\geq 0}\frac{2(2n+1)!t^{2n}}{n!^2(n+1)!(n+2)!},$$

$$I_0 I_1 + I_0 I_3 - 2I_1 I_2 = \sum_{n\geq 0}\frac{2(2n+2)!t^{2n+1}}{n!(n+1)!(n+2)!^2}.$$

Equations (11,12) then follow.                                                         $\square$

We remark that $\{\mathrm{bsm}_{2n}(2)\}_{n\geq 0}$ gives the sequence A000891 in the Online Encyclopedia of Integer Sequences [10]. One of its interpretation can be stated in

our term: it counts the number of noncrossing partitions of $[2n + 1]$ into $n + 1$ blocks. We also remark that $\{\mathrm{bsm}_{2n}(3)\}_{n \geq 0}$ gives the sequence A064037 in [10]. The only known interpretation is: it counts the number of 3-dimensional oscillating lattice walks of length $2n$, starting and ending at the origin, and staying within the nonnegative octant. Bijections for these objects are desirable.

## 3. Algebraic Description of the Reflection Principle

A classical application of the reflection principle is to the ballot problem, which, in random walks version, asks how many ways there are to walk from the origin to a point $(\lambda_1, \ldots, \lambda_d)_{\geq}$, with each step a positive unit coordinate vector and confined in the region $x_1 \geq x_2 \geq \cdots \geq x_d \geq 0$. The reflection principle of [6, 16] gives a determinant formula, from which the hook-length formula for SYTs can be deduced. Our objective in this section is to give short algebraic derivations of this formula and the formula of Grabiner and Magyar.

In the context of lattice walks, it is convenient to shift the coordinates a little and denote by $W^d = \{\,(x_1, \ldots, x_d) : x_1 > \cdots > x_d > 0, x_i \in \mathbb{Z}\,\}$ the $d$-dimensional *Weyl chamber*. From now on, $\lambda, \mu$ will not denote partitions as in previous sections. Let $\bar{\delta} = (d, d-1, \ldots, 1)$. Then any $\mu \in W^d$ corresponds to a unique partition $\mu - \bar{\delta}$.

### 3.1. **Hook-Length Formula.**

**Theorem 6** (Hook-Length Formula)**.** *The number of standard Young tableaux of shape $\lambda$ is*

$$f^\lambda = (\lambda_1 + \cdots + \lambda_d)! \det \left( \frac{1}{(\lambda_i - i + j)!} \right)_{1 \leq i, j \leq d}. \tag{14}$$

Fixing a starting point $\lambda \in W^d$, we let $f(\lambda; \mu)$ be the number of $W^d$-walks from $\lambda$ to $\mu$, with only positive unit coordinate vector steps. Clearly, the length of such walks, if exist, is $|\mu| - |\lambda|$, where $|\mu| = \mu_1 + \cdots + \mu_d$. Then the number $f^{\mu - \bar{\delta}}$ of SYTs of shape $\mu - \bar{\delta}$ equals $f(\bar{\delta}, \mu)$.

Let $F(x) = F(x_1, \ldots, x_d)$ be the generating function

$$F(x_1, \ldots, x_d) = \sum_{\mu \in W^d} f(\lambda; \mu) x^\mu,$$

where $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_d^{\mu_d}$ records the end points. From known results, $F(x)$ is $D$-finite and does not have a simple expression. But $F(x)$ has a simple rational function *extension*:

$$\bar{F}(x) = \frac{a_\lambda(x)}{1 - (x_1 + x_2 + \cdots + x_d)}, \tag{15}$$

where $a_\lambda(x)$ is the alternant $\det \left( x_i^{\lambda_j} \right)_{1 \leq i, j \leq d}$.

**Proposition 7.** *Let $\bar{F}(x)$ be as above. If we expand*

$$\bar{F}(x) = \frac{a_\lambda(x)}{1 - (x_1 + x_2 + \cdots + x_d)} = \sum_{\eta \in \mathbb{N}^d} \bar{f}(\lambda; \eta) x^\eta,$$

*then $\bar{f}(\lambda; \mu) = f(\lambda; \mu)$ for all $\mu$ in the closure of $W^d$.*

*Proof.* Let $e_i$ be the $i$th unit coordinate vector. Let $\chi(S) = 1$ if the statement $S$ is true and 0 otherwise. Then for $\mu$ in the closure of $W^d$, $f(\lambda; \mu)$ can be uniquely characterized by the following recursion:

(i) If $|\mu| \leq |\lambda|$, then $f(\lambda; \mu) = \chi(\mu = \lambda)$.
(ii) If $\mu_i = \mu_{i+1}$ for $1 \leq i \leq d - 1$, then $f(\lambda; \mu) = 0$.
(iii) If $|\mu| - |\lambda| > 0$, then $f(\lambda; \mu) = \sum_{i=1}^{d} f(\lambda; \mu - e_i)$.

Therefore, it suffices to show that $\bar{f}(\lambda; \mu)$ also satisfies the above three conditions.

Condition (iii) is trivial according to (15); Condition (ii) follows easily from

$$\bar{F}(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_d) = -\bar{F}(x_1, \ldots, x_d);$$

To show condition (i), we notice that the numerator of $\bar{F}(x)$ is homogeneous of degree $|\lambda|$, and the least degree term in the series expansion of $(1 - x_1 - \cdots - x_d)^{-1}$ is 1. This implies that if $|\eta| \leq |\lambda|$, then $\bar{f}(\lambda; \eta)$ equals $(-1)^\pi$ if $\eta = \pi(\lambda)$ for some $\pi \in \mathfrak{S}_d$ and zero otherwise, where $(-1)^\pi$ is the sign of $\pi$ and $\pi(\lambda) = (\lambda_{\pi_1}, \ldots, \lambda_{\pi_d})$. Condition (i) follows since $\pi(\lambda) \in W^d$ only if $\pi$ is the identity.

This completes the proof. $\qquad\square$

By setting $\lambda = \bar{\delta}$, one can derive the hook-length formula, Theorem 6. This completes our first objective of this section.

## 3.2. Grabiner-Magyar Determinant Formula.
The same argument applies to more general situations, such as with a different set of allowing steps. We give one more example to illustrate the idea. Note that the underlying idea is the reflection principle.

Fix a starting point $\lambda \in W^d$ (the most interesting case is $\lambda = \bar{\delta}$). Let $b_n(\lambda; \mu)$ be the number of Weyl oscillating lattice walks of length $n$ from $\lambda$ to $\mu$. Note that we changed the notation here. The $\bar{\lambda}$ is abbreviated by $\lambda$, and similar for $\bar{\mu}$.

**Proposition 8.** *For fixed $\lambda \in W^d$, let*

$$B_\lambda(x; t) = \frac{\det(x_i^{\lambda_j} - x_i^{-\lambda_j})_{1 \leq i,j \leq d}}{1 - t(x_1 + x_1^{-1} + x_2 + x_2^{-1} + \cdots + x_d + x_d^{-1})}. \tag{16}$$

*Then $[x^\mu t^n] B_\lambda(x; t) = b_n(\lambda; \mu)$ for any $\mu$ in the closure of $W^d$ and $n \in \mathbb{N}$.*

*Proof.* Clearly $b_n(\lambda; \mu)$ is uniquely determined by the following recursion:

(i) If $n = 0$ then $b_n(\lambda; \mu) = \chi(\mu = \lambda)$.
(ii) If $\mu_i = \mu_{i+1}$ for $1 \leq i \leq d - 1$, or if $\mu_d = 0$, then $b_n(\lambda; \mu) = 0$.
(iii) If $n \geq 1$, then $b_n(\lambda; \mu) = \sum_{i=1}^{d} b_{n-1}(\lambda; \mu - e_i) + b_{n-1}(\lambda; \mu + e_i)$.

Denote by $\bar{b}_n(\lambda; \eta) = [x^\eta t^n] B_\lambda(x; t)$. It suffices to show that $\bar{b}_n(\lambda; \mu)$ satisfies the same recursion as for $b_n(\lambda; \mu)$ when $\mu$ belongs to the closure of $W^d$. Condition (i) is straightforward; Condition (ii) follows from the identities $B_\lambda(x; t) = - B_\lambda(x; t)|_{x_i = x_{i+1}, x_{i+1} = x_i}$ for $1 \leq i \leq d - 1$, and $B_\lambda(x; t) = - B_\lambda(x; t)|_{x_d = -x_d}$; Condition (iii) follows by writing

$$B_\lambda(x; t)(1 - t(x_1 + x_1^{-1} + x_2 + x_2^{-1} + \cdots + x_d + x_d^{-1})) = \det(x_i^{\lambda_j} - x_i^{-\lambda_j})_{1 \leq i,j \leq d}$$

and then equating coefficients. $\qquad\square$

Now we are ready to complete our second task of this section.

*Proof of Theorem 1.* By Proposition 8, it remains to extract the coefficient of $x^\mu$ in $B_\lambda(x; t)$.

Since the numerator of $B_\lambda(x;t)$ is independent of $t$, it is easy to obtain the exponential generating function

$$\sum_{n\geq 0}\sum_{\eta\in\mathbb{Z}^d}\bar{b}_n(\lambda;\eta)x^\eta\frac{t^n}{n!}=\det(x_i^{\lambda_j}-x_i^{-\lambda_j})_{1\leq i,j\leq d}\exp(t(x_1+x_1^{-1}+\cdots+x_d+x_d^{-1}))$$

$$=\det\left((x_i^{\lambda_j}-x_i^{-\lambda_j})\exp(t(x_i+x_i^{-1}))\right)_{1\leq i,j\leq d}.$$

Now taking the coefficients of $x_1^{\mu_1}\cdots x_d^{\mu_d}$ yields

$$\sum_{n\geq 0}b_n(\lambda;\mu)\frac{t^n}{n!}=\det\left([x_i^{\mu_i-\lambda_j}]\exp(t(x_i+x_i^{-1}))-[x_i^{\mu_i+\lambda_j}]\exp(t(x_i+x_i^{-1}))\right)_{1\leq i,j\leq d},$$

which is equivalent to (1).                                                      □

Note that we use exponential generating functions because $\exp(t(x_1+x_1^{-1}+\cdots+x_d+x_d^{-1}))$ factors nicely enough to be put inside the determinant.

## 4. Two Formulas Relating to Tableaux of Bounded Height

In this section, we will prove Theorems 2 and 3, where the former is a new result and the latter is Gessel's remarkable determinant formula. We will first express our objects as certain constant terms. Then we will play two tricks, the *Stanton-Stembridge trick* and the reverse of the Stanton-Stembridge trick, in evaluating such constant terms.

4.1. **Stanton-Stembridge Trick.** Fix a working ring $\mathcal{K}$ that includes the ring $\mathbb{C}((x_1,\ldots,x_d))$ of formal Laurent series as a subring. For example, in our applications, the working ring is $\mathcal{K}=\mathbb{C}((x_1,\ldots,x_d))[[t]]$. A permutation $\pi\in\mathfrak{S}_d$ acts on elements of $\mathcal{K}$ by permuting the $x$'s, or more precisely

$$\pi\cdot\sum_{i_1,\ldots,i_d\in\mathbb{Z}}a_{i_1,\ldots,i_d}x_1^{i_1}\cdots x_d^{i_d}=\sum_{i_1,\ldots,i_d\in\mathbb{Z}}a_{i_1,\ldots,i_d}x_{\pi_1}^{i_1}\cdots x_{\pi_d}^{i_d}.$$

We say that $\mathcal{K}$ is $\mathfrak{S}_d$-*invariant* if $\pi\cdot\mathcal{K}=\mathcal{K}$ for any $\pi\in\mathfrak{S}_d$. For example, the ring $\mathcal{K}=\mathbb{C}((x_1,\ldots,x_d))[[t]]$ is $\mathfrak{S}_d$-invariant, but the field of iterated Laurent series $\mathbb{C}((x_1))((x_2))$ is not $\mathfrak{S}_2$-invariant (see [14]).

In what follows, we always assume that $\mathcal{K}$ is $\mathfrak{S}_d$-invariant. One can easily check that this condition holds in our application.

**Lemma 9** (Stanton-Stembridge trick). *For any $H(x_1,\ldots,x_d)\in\mathcal{K}$, we have*

$$\operatorname*{CT}_{x_1,\ldots,x_d}H(x_1,\ldots,x_d)=\frac{1}{d!}\operatorname*{CT}_{x_1,\ldots,x_d}\sum_{\pi\in\mathfrak{S}_d}\pi\cdot H(x_1,\ldots,x_d),$$

*where* $\operatorname{CT}_{x_1,\ldots,x_d}$ *means to take the constant term in the $x$'s.*

The lemma obviously holds. The following direct consequence is useful.

**Corollary 10.** *Suppose that $H,U,V\in\mathcal{K}$ and that $U(x)=U(x_1,\ldots,x_d)$ and $V(x)=V(x_1,\ldots,x_d)$ are symmetric and antisymmetric in the $x$'s, respectively.*

*Then*

$$\operatorname*{CT}_{x_1,\ldots,x_d} H(x_1,\ldots,x_d)U(x) = \frac{1}{d!} \operatorname*{CT}_{x_1,\ldots,x_d} U(x) \sum_{\pi \in \mathfrak{S}_d} \pi \cdot H(x_1,\ldots,x_d),$$

$$\operatorname*{CT}_{x_1,\ldots,x_d} H(x_1,\ldots,x_d)V(x) = \frac{1}{d!} \operatorname*{CT}_{x_1,\ldots,x_d} V(x) \sum_{\pi \in \mathfrak{S}_d} (-1)^\pi \pi \cdot H(x_1,\ldots,x_d).$$

We call both the lemma and the corollary the Stanton-Stembridge trick (SS-trick for short). See, e.g., [17, p. 9].

4.2. **Proof of Theorem 2.** First let us write $g_{\bar\delta\mu}(t)$ as a constant term using Theorem 1 and the fact that $I_s(2t) = I_{-s}(2t)$.

$$g_{\bar\delta\mu}(t) = \det\left( \operatorname*{CT}_{x_i} \left[ x_i^{\mu_i - \bar\delta_j} \exp((x_i + x_i^{-1})t) - x_i^{\mu_i + \bar\delta_j} \exp((x_i + x_i^{-1})t) \right] \right)_{1 \le i,j \le d}$$

By factoring out $x_i^{\mu_i} \exp((x_i + x_i^{-1})t)$ from the $i$th row, we obtain

$$g_{\bar\delta\mu}(t) = \operatorname*{CT}_{x} \det\left( x_i^{-\bar\delta_j} - x_i^{\bar\delta_j} \right)_{1 \le i,j \le d} \prod_{i=1}^d x_i^{\mu_i} \exp((x_i + x_i^{-1})t).$$

Therefore, $G(t)$ can be expressed as a constant term in the $x$'s:

$$G(t) = \sum_{\mu \in W^d} g_{\bar\delta\mu}(t) = \operatorname*{CT}_{x} \exp\left( \sum_{i=1}^d (x_i + x_i^{-1})t \right) \det\left( x_i^{-\bar\delta_j} - x_i^{\bar\delta_j} \right) \cdot \sum_{\mu \in W^d} x^\mu.$$

Now we can apply the SS-trick to obtain

$$G(t) = \frac{1}{d!} \operatorname*{CT}_{x} \exp\left( \sum_{i=1}^d \left( x_i + x_i^{-1} \right) t \right) \det\left( x_i^{-\bar\delta_j} - x_i^{\bar\delta_j} \right) \sum_{\mu \in W^d} \sum_{\pi \in \mathfrak{S}_d} (-1)^\pi \pi \cdot x^\mu, \tag{17}$$

where we used the fact that the first factor is symmetric and the second factor is antisymmetric in the $x$'s.

The determinant is well-known to be equal to

$$\det\left( x_i^{-\bar\delta_j} - x_i^{\bar\delta_j} \right)_{1 \le i,j \le d} = \prod_{i=1}^d (1 - x_i^2) \prod_{1 \le i < j \le d} (1 - x_i x_j) a_{\bar\delta}(x^{-1}), \tag{18}$$

where $x^{-1} = (x_1^{-1}, \ldots, x_d^{-1})$, and the alternants are related to the Schur functions as follows:

$$a_\mu(x) := \sum_{\pi \in \mathfrak{S}_d} (-1)^\pi \pi \cdot x_1^{\mu_1} \cdots x_d^{\mu_d} = a_{\bar\delta}(x) s_{\mu - \bar\delta}(x).$$

By the above formula, and the classical identity [11, Equation 7.52] for symmetric functions (by setting $x_k = 0$ for $k > d$), we obtain

$$\sum_{\mu \in W^d} \sum_{\pi \in \mathfrak{S}_d} (-1)^\pi \pi \cdot x_1^{\mu_1} \cdots x_d^{\mu_d} = a_{\bar\delta}(x) \frac{1}{\prod_{i=1}^d (1 - x_i) \prod_{1 \le i < j \le d} (1 - x_i x_j)}. \tag{19}$$

Now substitute (18) and (19) into (17). After a lot of cancelations, we obtain:

$$G(t) = \frac{1}{d!} \operatorname*{CT}_{x} \exp\Big( \sum_{i=1}^{d} \big(x_i + x_i^{-1}\big)\, t\Big) \prod_{i=1}^{d}(1 + x_i) a_{\bar{\delta}}(x^{-1}) \sum_{\pi \in \mathfrak{S}_d} (-1)^\pi \pi \cdot x^{\bar{\delta}}. \quad (20)$$

Now $a_{\bar{\delta}}(x^{-1})$ is antisymmetric. Reversely applying the SS-trick to (20) gives

$$G(t) = \operatorname*{CT}_{x} x_1^d \cdots x_d^1 a_{\bar{\delta}}(x^{-1}) \prod_{i=1}^{d} \exp((x_i + x_i^{-1})t) \prod_{i=1}^{d}(1 + x_i)$$

$$= \operatorname*{CT}_{x} \det\Big( x_i^{j-i}\Big)_{1 \le i,j \le d} \prod_{i=1}^{d} \exp((x_i + x_i^{-1})t) \prod_{i=1}^{d}(1 + x_i)$$

$$= \det\Big( \operatorname*{CT}_{x_i} x_i^{j-i}(1 + x_i) \exp((x_i + x_i^{-1})t)\Big)_{1 \le i,j \le d},$$

which is easily seen to be equivalent to (3).

4.3. **Gessel's Determinant Formula.** The tricks for proving Theorem 3 are similar as in the previous subsection. We remark that previous proofs of this result rely on the powerful tools of symmetric functions. See, e.g., [5, 7].

It follows from the RSK-correspondence that

$$u_d(n) = \sum_{|\alpha|=n} f^\alpha f^\alpha = \sum_{|\mu|=n+|\bar{\delta}|,\, \mu \in W^d} f(\bar{\delta}; \mu)^2,$$

where $\alpha$ ranges over partitions of $n$ of height at most $d$. We will find a generating function of

$$u_d(\lambda; n) := \sum_{|\mu|=n+|\lambda|,\, \mu \in W^d} f(\lambda; \mu)^2.$$

More precisely, we have the following generalized form.

**Theorem 11.** *Let $\lambda \in W^d$ and let $I_s(2t)$ be as in Theorem 1. We have*

$$U_d(\lambda; t) = \sum_{n \ge 0} u_d(\lambda; n) \frac{t^{2n}}{n!^2} = \det(I_{\lambda_i - \lambda_j})_{1 \le i,j \le d}. \quad (21)$$

Note that $f(\lambda; \mu)$ is the number of standard skew Young tableaux of shape $(\mu - \bar{\delta})/(\lambda - \bar{\delta})$. See [11, Equation 7.7.1] (note that there is a change of indices). Therefore $u_d(\lambda; n)$ counts the number of pairs of standard skew Young tableaux of the same shape $(\mu - \bar{\delta})/(\lambda - \bar{\delta})$ with $|\mu| - |\lambda| = n$.

*Proof of Theorem 11.* By Proposition 7,

$$f(\lambda; \mu) = [x^\mu]\, a_\lambda(x) \sum_{k \ge 0}(x_1 + x_2 + \cdots + x_d)^k.$$

Note that when taking the coefficient in $x^\mu$, only the summand with respect to $k = n$ has a contribution, where $n = |\mu| - |\lambda|$ is the length of the lattice walks. It follows that

$$f(\lambda; \mu)\frac{t^n}{n!} = [x^\mu]\, a_\lambda(x) \exp\big((x_1 + x_2 + \cdots + x_d)t\big)$$

$$= \det\Big( [x_i^{\mu_i - \lambda_j}] \exp(tx_i)\Big)_{1 \le i,j \le d}.$$

When written in constant term, we obtain

$$f(\lambda;\mu)\frac{t^n}{n!} = \underset{x}{\mathrm{CT}}\det(x_i^{\lambda_j})_{1\le i,j\le d}\prod_{i=1}^d x_i^{-\mu_i}\exp(tx_i)$$

$$= \underset{x}{\mathrm{CT}}\,a_\lambda(x^{-1})x_1^{\mu_1}\cdots x_d^{\mu_d}\exp(t(x_1^{-1}+\cdots+x_d^{-1})), \tag{22}$$

where the last equality follows by substituting $x_i^{-1}$ for $x_i$.

Now squaring both sides of (22) and summing over all $\mu$, we obtain

$$U_d(\lambda;t) = \sum_{\mu\in W^d} f(\lambda;\mu)^2\frac{t^{2n}}{n!^2}$$

$$= \sum_{\mu\in W^d}\underset{x}{\mathrm{CT}}\,a_\lambda(x^{-1})x^\mu\exp\left(t\sum_{i=1}^d x_i^{-1}\right)\cdot\underset{y}{\mathrm{CT}}\,a_\lambda(y^{-1})y^\mu\exp\left(t\sum_{i=1}^d y_i^{-1}\right)$$

$$= \underset{x,y}{\mathrm{CT}}\,a_\lambda(x^{-1})a_\lambda(y^{-1})\exp\left(t\sum_{i=1}^d x_i^{-1}+t\sum_{i=1}^d y_i^{-1}\right)\sum_{\mu\in W^d}x^\mu y^\mu. \tag{23}$$

We need the following easy formula:

$$\sum_{\pi,\sigma\in\mathfrak{S}_d}(-1)^\pi(\pi\cdot x^\mu)(-1)^\sigma(\sigma\cdot y^\mu) = a_\mu(x)a_\mu(y), \tag{24}$$

and the well-known Cauchy-Binnet formula (see, e.g., [11, p. 397]):

$$\sum_{\mu\in W^d}a_\mu(x)a_\mu(y) = x^{\mathbf{1}}y^{\mathbf{1}}\det\left(\frac{1}{1-x_iy_j}\right)_{1\le i,j\le d}, \tag{25}$$

where $\mathbf{1}$ is the vector of $d$ 1's and $x^{\mathbf{1}} = x_1x_2\cdots x_d$.

Now apply the SS-trick to (23) for the $x$-variables and the $y$-variables separately, and then apply (24). We obtain

$$U_d(\lambda;t) = \frac{1}{d!^2}\underset{x,y}{\mathrm{CT}}\,a_\lambda(x^{-1})a_\lambda(y^{-1})\exp\left(t\sum_{i=1}^d x_i^{-1}+t\sum_{i=1}^d y_i^{-1}\right)\sum_{\mu\in W^d}a_\mu(x)a_\mu(y).$$

Applying (25) gives

$$U_d(\lambda;t) = \frac{1}{d!^2}\underset{x,y}{\mathrm{CT}}\,a_\lambda(x^{-1})a_\lambda(y^{-1})\exp\left(t\sum_{i=1}^d\left(x_i^{-1}+y_i^{-1}\right)\right)x^{\mathbf{1}}y^{\mathbf{1}}\det\left(\frac{1}{1-x_iy_j}\right).$$

Clearly, the last determinant is antisymmetric in the $x$-variables and also in the $y$-variables. Reversely applying the SS-trick for the $x$'s and for the $y$'s, we obtain

$$U_d(\lambda;t) = \underset{x,y}{\mathrm{CT}}\,x^{-\lambda}y^{-\lambda}\exp\left(t\sum_{i=1}^d\left(x_i^{-1}+y_i^{-1}\right)\right)x^{\mathbf{1}}y^{\mathbf{1}}\det\left(\frac{1}{1-x_iy_j}\right)_{1\le i,j\le d}$$

$$= \underset{x,y}{\mathrm{CT}}\det\left(x_i^{1-\lambda_i}y_j^{1-\lambda_j}\exp(tx_i^{-1}+ty_j^{-1})\frac{1}{1-x_iy_j}\right)_{1\le i,j\le d}$$

$$= \det\left(\underset{x_i,y_j}{\mathrm{CT}}\,x_i^{1-\lambda_i}y_j^{1-\lambda_j}\exp(tx_i^{-1}+ty_j^{-1})\frac{1}{1-x_iy_j}\right)_{1\le i,j\le d}. \tag{26}$$

We finally need to evaluate the entries (constant terms) of the above determinant.

$$\underset{x_i,y_j}{\mathrm{CT}}\,\exp(tx_i^{-1}+ty_j^{-1})\frac{x_i^{1-\lambda_i}y_j^{1-\lambda_j}}{1-x_iy_j} = \underset{x_i,y_j}{\mathrm{CT}}\sum_{k,l\geq 0}\frac{t^{k+l}}{k!\,l!}x_i^{-k}y_j^{-l}\sum_{m\geq 0}x_i^{1-\lambda_i+m}y_j^{1-\lambda_j+m}$$

$$= \sum_{m\geq\lambda_i-1,\lambda_j-1}\frac{t^{2-\lambda_i-\lambda_j+2m}}{(1-\lambda_i+m)!(1-\lambda_j+m)!}.$$

By changing the indices $1-\lambda_i+m=n$, we obtain

$$\underset{x_i,y_j}{\mathrm{CT}}\,\exp(tx_i^{-1}+ty_j^{-1})\frac{x_i^{1-\lambda_i}y_j^{1-\lambda_j}}{1-x_iy_j} = I_{\lambda_i-\lambda_j}(2t). \tag{27}$$

This completes the proof.                                                    □

By going over the proof, one can see that the $a_\lambda(y)$ may be replaced with $a_\nu(y)$ without making much difference. This gives the following proposition.

Let $\lambda,\nu\in W^d$ with $|\lambda|\geq|\nu|$. Denote by

$$U_d(\lambda;\nu;t) := \sum_{n\geq 0}u_d(\lambda;\nu;n)\frac{t^{2n+|\lambda|-|\nu|}}{n!(n+|\lambda|-|\nu|)!},$$

where

$$u_d(\lambda;\nu;n) := \sum_{|\mu|=n+|\lambda|,\mu\in W^d}f(\lambda;\mu)f(\nu;\mu).$$

**Proposition 12.** *For $\lambda,\nu\in W^d$ with $|\lambda|\geq|\nu|$, we have*

$$U_d(\lambda;\nu;t) = \det(I_{\lambda_i-\nu_j})_{1\leq i,j\leq d}.$$

Gessel's determinant formula was proved by first deriving a symmetric function identity, and then applying a specialization operator. It is not a surprise that there should be a corresponding symmetric function identity that specializes to Theorem 11, even to Proposition 12. Actually, such a formula was described by Gessel in the same paper [5] in the paragraph just before Theorem 16, and was clearly stated as [4, Theorem 3.5].

Gessel's determinant formula counts permutations of bounded length of longest increasing subsequences. Does Theorem 11 count natural objects?

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