Complexity of Villamayor's algorithm in the monomial case

Rocío Blanco

Abstract

We study monomial ideals, always given by a unique monomial, like a reasonable first step to estimate in general the number of blow ups of Villamayor's algorithm of resolution of singularities. To resolve a monomial ideal $< X_1^{a_1} \cdot \ldots \cdot X_n^{a_n} >$ is interesting due to its equivalence with the particular toric problem $< Z^c - X_1^{a_1} \cdot \ldots \cdot X_n^{a_n} >$.

In the special case, when all the exponents a_i are greater than or equal to the critical value c, we construct the largest branch of the resolution tree which provides an upper bound involving partial sums of Catalan numbers. This case will be called "minimal codimension case". Partial sums of Catalan numbers (starting $1, 2, 5, \ldots$) are $1, 3, 8, 22, \ldots,$ and count, besides this new application, the number of paths starting from the root in all ordered trees with $n+1$ edges. Catalan numbers appear in many combinatorial problems, counting the number of ways to insert n pairs of parenthesis in a word of $n+1$ letters, plane trees with $n+1$ vertices, ..., etc.

In the case of higher codimension, still unresolved, we give an example to state the foremost troubles.

Computation of examples has been helpful in both cases to study the behaviour of the invariant function. Computations have been made in Singular (see [7]) using the *desing* package by G. Bodnár and J. Schicho, see [4].

Introduction

The existence of resolution of singularities in arbitrary dimension over a field of characteristic zero was solved by Hironaka in his famous paper [8]. Later on, different constructive proofs have been given, among others, by Villamayor [12], Bierstone-Milman [1], Encinas-Villamayor [6], Encinas-Hauser [5] and Wodarczyk [13].

This paper is devoted to study the complexity of Villamayor's algorithm of resolution of singularities. This algorithm appears originally in [12] and we will use the presentation given in [6]. In this paper, the authors introduce a class of objects called *basic objects* $B = (W, (J, c), E)$ where W is a regular ambient space over a field k of characteristic zero, $J \subset \mathcal{O}_W$ is a sheaf of ideals, c is an integer and E is a set of smooth hypersurfaces in W having only normal crossings. That is, they consider the ideal J together with a positive integer c, or *critical value* defining the *singular locus* $Sing(J, c) = \{\xi \in X | \; ord_\xi(J) \geq c\}$, where X is the algebraic set defined by J and $ord_\xi(J)$ is the *order* of J in a point ξ.

Let $W \stackrel{\pi}{\leftarrow} W'$ be the blow up with center $\mathcal{Z} \subset Sing(J, c)$, $\pi^{-1}(\mathcal{Z}) = Y'$ is the exceptional divisor. For $\xi \in Sing(J, c)$, $ord_\xi(J) = \theta$, the total transform of J in W' satisfies $J\mathcal{O}_{W'} = I(Y')^\theta \cdot J^\curlyvee$ where $J^\curlyvee$ is the *weak* transform of J, (see [6] for details).

A *transformation* of a basic object $(W, (J, c), E) \leftarrow (W', (J', c), E')$ is defined by blowing up $W \stackrel{\pi}{\leftarrow} W'$ and defining $J' = I(Y')^{\theta - c} \cdot J^\curlyvee$ the *controlled transform* of J.

A sequence of transformations of basic objects

$$(W, (J, c), E) \leftarrow (W^{(1)}, (J^{(1)}, c), E^{(1)}) \leftarrow \cdots \leftarrow (W^{(N)}, (J^{(N)}, c), E^{(N)}) \qquad (1)$$

is a *resolution* of $(W,(J,c),E)$ if $Sing(J^{(N)},c)=\emptyset$.

Remark 0.1. Superscripts $^{(k)}$ in basic objects will denote the k-stage of the resolution process. Subscripts $_i$ will always denote the dimension of the ambient space $W_i^{(k)}$.

Villamayor's algorithm provides a *log-resolution* in characteristic zero. A log-resolution of J is a sequence of blow ups at regular centers as (1) such that each center has normal crossings with the exceptional divisors $E^{(i)}$, and the total transform of J in $W^{(N)}$ is of the form

$$J\mathcal{O}_{W^{(N)}}=I(H_1)^{b_1}\cdot\ldots\cdot I(H_N)^{b_N}$$

with $b_i\in\mathbb{N}$ for all $1\le i\le N$ and $E^{(N)}=\{H_1,\ldots,H_N\}$.

They prove that algorithmic principalization of ideals reduces to algorithmic resolution of basic objects: starting with $c=max\text{–}ord(J)$ the maximal order of J, we obtain a resolution of $(W,(J,c),E)$ as (1), if $max-ord(J^{(N)})=c^{(N)}<c$ but $c^{(N)}>1$ we continue resolving $(W^{(N)},(J^{(N)},c^{(N)}),E^{(N)})$ and so on, until have $max-ord(J^{(\mathcal{N})})=c^{(\mathcal{N})}=1$ what give us a log-resolution of J.

The key of the algorithm is to use induction on the dimension of the ambient space W to construct an invariant function which drops after blowing up.

We shall work with the invariant defined in [6], using the language of mobiles developed in [5]. We remind briefly the main notions. For simplicity, let $W=\mathbb{A}_k^n$ be the ambient space, over a field k of characteristic zero. Let $J\subset\mathcal{O}_W$ be an ideal defining a singular algebraic set $X\subset W$.

The ideal J factors into $J=M\cdot I$, with M the ideal defined by normal crossing divisors, and I some ideal still unresolved. By induction on the dimension of W, we will have this decomposition at every dimension from n to 1, that is $J_i=M_i\cdot I_i$, for $n\ge i\ge 1$, are defined in local flags $W_n\supseteq W_{n-1}\supseteq\cdots\supseteq W_i\supseteq\cdots\supseteq W_1$, where each $J_i,M_i,I_i\in\mathcal{O}_{W_i}$ are in dimension i. There is a critical value c_{i+1} at each dimension i, $(c_{n+1}=c)$, see [5] for details. The process of the algorithm need to resolve these basic objects $(W_i,(J_i,c_{i+1}),E_i)$.

Let E be the exceptional divisor of previous blow ups, and consider $E=\cup_{i=1}^n E_i$ where E_i applies to dimension i. Obviously, we start with $E=\emptyset$.

Once we have expressed the total transform of J as a monomial ideal in terms of the exceptional divisors, to resolve the basic object $(W,(J,c),E)$, we apply Γ function until obtain $Sing(J^{(k)},c)=\emptyset$. The function Γ is the invariant function corresponding to the so-called *monomial case*, following the notation of [6], pages $165-166$.

For any point $\xi\in Sing(J,c)$, the invariant function t will have n coordinates, with lexicographical order, and it will be one of the following three types:

$$\begin{array}{ll}(a) & t(\xi)=(t_n(\xi),t_{n-1}(\xi),\ldots,t_{n-r}(\xi),\ \infty,\ \infty,\ldots,\infty)\\ (b) & t(\xi)=(t_n(\xi),t_{n-1}(\xi),\ldots,t_{n-r}(\xi),\Gamma(\xi),\infty,\ldots,\infty)\\ (c) & t(\xi)=(t_n(\xi),t_{n-1}(\xi),\ldots,t_{n-r}(\xi),\ldots\ldots\ldots,t_1(\xi))\end{array}\quad \text{with } t_i=\left[\frac{\theta_i}{c_{i+1}},m_i\right]$$

where $\theta_i=ord_\xi(I_i)$, m_i is the number of exceptional divisors in E_i, and Γ is the invariant function corresponding to the monomial case.

To resolve the toric hypersurface $\{f=0\}=\{Z^c-X_1^{a_1}\cdot\ldots\cdot X_n^{a_n}=0\}$ we note that its singular locus $Sing(<f>,c)$ is always included in $\{Z=0\}$, so we make induction on the dimension and we reduce to the case where the corresponding ideal J is of the form

$$J=<X_1^{a_1}\cdot\ldots\cdot X_n^{a_n}>\subset\mathcal{O}_W \text{ with } 1\le a_1\le a_2\le\ldots\le a_n,\ \sum_{i=1}^n a_i=d,\ \text{ and } d\ge c, \tag{2}$$

where c is the critical value. If $a_i=0$ for some i, then we may assume $dim(W)<n$.

After blowing up, we always consider the controlled transform of J with respect to c, $J' = I(Y')^{-c} \cdot J^*$ where J^* is the total transform of J and Y' denotes the new exceptional divisor. For the toric problem $J =< Z^c - X_1^{a_1} \cdot \ldots \cdot X_n^{a_n} >$, blowing up the origin we have, in some chart X_i:

$$J^* =< Z^c \cdot X_i^c - X_1^{a_1} \cdots X_i^d \cdots X_n^{a_n} >=< X_i^c \cdot (Z^c - X_1^{a_1} \cdots X_i^{d-c} \cdots X_n^{a_n}) >$$

we can only factorize c times the exceptional divisor.

So we will apply the resolution algorithm to the basic object $(W, (J, c), \emptyset)$ where the ideal $J =< X_1^{a_1} \cdot \ldots \cdot X_n^{a_n} >$, which is already a monomial ideal, but it is not supported by exceptional divisors yet.

1 Monomial case (exceptional monomial)

A *monomial case* is a special kind of monomial ideal, the one given by a unique monomial that can be expressed in terms of the exceptional divisors (once they are known, after several blow ups). This means we have a basic object $(W, (J, c), E)$ where J is one monomial supported by the hypersurfaces in E. We can also call it **exceptional monomial**.

Theorem 1.1. *Let $J \subset \mathcal{O}_W$ be a monomial ideal as in equation (2). Let $E = \{H_1, \ldots, H_n\}$ with $H_i = V(X_i)$ be a normal crossing divisor.*
Then an upper bound for the number of blow ups to resolve $(W, (J, c), E)$ is given by

$$\frac{d - c + gcd(a_1, \ldots, a_n, c)}{gcd(a_1, \ldots, a_n, c)}.$$

Proof. We may assume the greatest common divisor of the exponents a_i and the critical value c is equal to 1, because both the simplified problem and the original problem have the same singular locus. That is, if $gcd(a_1, \ldots, a_n, c) = k$ then $d = k \cdot d_1$, $c = k \cdot c_1$, $a_i = k \cdot b_i$ for all $1 \leq i \leq n$ and $gcd(b_1, \ldots, b_n, c_1) = 1$. The ideal J can be written $J = (J_1)^k$ where $J_1 =< X_1^{b_1} \cdot \ldots \cdot X_n^{b_n} >$ therefore

$$Sing(J, c) = \{\xi \in X \mid ord_\xi((J_1)^k) \geq k \cdot c_1\} = \{\xi \in X \mid ord_\xi(J_1) \geq c_1\} = Sing(J_1, c_1),$$

where X is the algebraic set defined by J. The blowing up center $\mathcal{Z} = \cap_{i=n-(r-1)}^n H_i$ is given by Γ function. For a point $\xi \in \mathbb{A}_k^n$, $\Gamma(\xi) = (-\Gamma_1(\xi), \Gamma_2(\xi), \Gamma_3(\xi))$ where

$$\Gamma_1(\xi) = min\{p \mid \exists\ i_1, \ldots, i_p, a_{i_1}(\xi) + \cdots + a_{i_p}(\xi) \geq c,\ \xi \in H_{i_1} \cap \cdots \cap H_{i_p}\},$$

$$\Gamma_2(\xi) = max\left\{\frac{a_{i_1}(\xi) + \cdots + a_{i_p}(\xi)}{c} \mid p = \Gamma_1(\xi), a_{i_1}(\xi) + \cdots + a_{i_p}(\xi) \geq c,\ \xi \in H_{i_1} \cap \cdots \cap H_{i_p}\right\},$$

$$\Gamma_3(\xi) = max\{(i_1, \ldots, i_p, 0, \ldots, 0) \in \mathbb{Z}^n \mid \Gamma_2(\xi) = \frac{a_{i_1}(\xi) + \cdots + a_{i_p}(\xi)}{c},\ \xi \in H_{i_1} \cap \cdots \cap H_{i_p}\}$$

with lexicographical order in $\mathbb{Z}^n$.

So in any chart X_j, we look to the exponent of X_j after the blow up

$$\left(\sum_{i=n-r+1}^{n} a_i\right) - c < \min_{n-r+1 \leq i \leq n} a_i = a_{n-r+1}$$

because $\sum_{i=n-r+2}^n a_i < c$ by construction of the center $\mathcal{Z}$.

Then the total order of the ideal drops after each blow up by at least one, so in the worst case, we need $d - (c-1)$ blow ups to obtain a total order lower than c. □

Remark 1.2. Note that an exceptional monomial is not resolved yet. If we consider the factorization of J, $J = M \cdot I$ where $J = M$ and $I = 1$. The order of the ideal I does not give any information so we apply Γ function to M until obtain $Sing(J,c) = \emptyset$.

Remark 1.3. This bound is reached only at the following values of c:

$$1,\ a_n + \ldots + a_j + 1 \text{ for } n \geq j \geq 2,\ d.$$

These values are those values of c where the total order of the ideal drops after each blow up exactly by one, and the value $c = d$ in which we finish after only one blow up.

Remark 1.4. If $gcd(a_1, \ldots, a_n, c) = k > 1$, then the bound for an exceptional monomial ideal of order d is $(d-c+k)/k < d-c+1$, so we can use in practice the bound for the case $gcd(a_1, \ldots, a_n, c) = 1$.

2 Case of one monomial

To construct an upper bound for the number of blow ups needed to resolve $(W, (J,c), E = \emptyset)$, J given by a unique monomial, we estimate the number of blow ups to obtain $(W', (J', c), E')$ a transformation of the original basic object with $J' = M'$ (an exceptional monomial) and use theorem 1.1. In order to use theorem 1.1, we need an estimation of the order of M'. This estimation will be valid in general for any stage of the resolution process.

Lemma 2.1. *Let $(W, (J,c), \emptyset)$ be a basic object where $J = M \cdot I$ is a monomial ideal as in equation (2). After N blow ups we have $(W^{(N)}, (J^{(N)}, c), E^{(N)})$. Let $\xi \in \mathbb{A}^n_k$ be a point. Then*

$$ord_\xi(M^{(N)}) \leq (2^N - 1)(d - c)$$

where $ord_\xi(M^{(N)})$ denotes the order at ξ of $M^{(N)}$, the (exceptional) monomial part of $J^{(N)}$.

Proof. It follows by induction over N.

- if $N = 1$, $ord_\xi(M^{(1)}) = d - c$.

  At the beginning, the first blowing up center given by this algorithm is always the origin, so in some chart X_i: $J^{(1)} = M^{(1)} \cdot I^{(1)} =< X_i^{d-c} > \cdot < X_1^{a_1} \cdot \overset{\widehat{i}}{\ldots} \cdot X_n^{a_n} >$ with $E^{(1)} = \{H_i\}$, where $H_i = V(X_i)$.

- We assume the result for $N = m - 1$.

  $$J^{(m-1)} = M^{(m-1)} \cdot I^{(m-1)} =< X_{i_1}^{b_1} \cdots X_{i_s}^{b_s} > \cdot < X_{i_{s+1}}^{a_{i_{s+1}}} \cdots X_{i_n}^{a_{i_n}} > \quad \text{with } \sum_{i=1}^{s} b_i = d'.$$

  By induction hypothesis, after $m - 1$ blow ups, the maximal order d' of the (exceptional) monomial part $M^{(m-1)}$ satisfies

  $$d' \leq (2^{m-1} - 1)(d - c)$$

  For $N = m$, now there are two possibilities:

  1. In the next blowing up center there are only variables appearing in $I^{(m-1)}$.
  2. In the next blowing up center there are variables appearing in $I^{(m-1)}$ and there are also variables appearing in $M^{(m-1)}$.

Case 1:

In the worst case, the blowing up center is $\mathcal{Z}=\cap_{j=s+1}^{n} V(X_{i_j})$. In some chart X_{i_l},

$$J^{(m)} = M^{(m)} \cdot I^{(m)} =< X_{i_1}^{b_1} \cdots X_{i_s}^{b_s} \cdot X_{i_l}^{d-\sum_{j=1}^{s} a_{i_j} - c} > \cdot < X_{i_{s+1}}^{a_{i_{s+1}}} \overset{\widehat{i_l}}{\cdots} X_{i_n}^{a_{i_n}} > .$$

The maximal order of $M^{(m)}$ is

$$\sum_{i=1}^{s} b_i + d - \sum_{j=1}^{s} a_{i_j} - c = d' + d - c - \sum_{j=1}^{s} a_{i_j} \leq d' + d - c$$

by induction hypothesis

$$d' + d - c \leq (2^{m-1} - 1)(d - c) + d - c = 2^{m-1}(d - c) \leq (2^m - 1)(d - c).$$

Case 2: - In some chart X_{i_j}, for $1 \leq j \leq s$

$$J^{(m)} = M^{(m)} \cdot I^{(m)} =< X_{i_1}^{b_1} \overset{\widehat{i_j}}{\cdots} X_{i_s}^{b_s} \cdot X_{i_j}^{\square} > \cdot < X_{i_{s+1}}^{a_{i_{s+1}}} \cdots X_{i_n}^{a_{i_n}} > .$$

- In some chart X_{i_l}, for $s + 1 \leq l \leq n$

$$J^{(m)} = M^{(m)} \cdot I^{(m)} =< X_{i_1}^{b_1} \cdots X_{i_s}^{b_s} \cdot X_{i_l}^{\triangle} > \cdot < X_{i_{s+1}}^{a_{i_{s+1}}} \overset{\widehat{i_l}}{\cdots} X_{i_n}^{a_{i_n}} > .$$

In the worst case, if the center is a point,

$$\square = \triangle = d' + d - \sum_{j=1}^{s} a_{i_j} - c \ .$$

Therefore in both cases the maximal order of $M^{(m)}$ satisfies

$$\leq 2d' + d - c \ \leq \ 2(2^{m-1} - 1)(d - c) + d - c = (2^m - 1)(d - c) \ . \qquad \square$$

Remark 2.2. Due to its general character, this bound is large and far from being optimal.

Remark 2.3. The ideals M_i are supported by normal crossing divisors D_i. Recall that their transformation law after blow up, in the neighbourhood of a point $\xi \in W_i$, is

$$D_i' = \begin{cases} D_i^* + (\theta_i - c_{i+1}) \cdot Y' & \text{if } (t_n'(\xi'),\ldots,t_{i+1}'(\xi'))=(t_n(\xi),\ldots,t_{i+1}(\xi)) \\ \emptyset & \text{in other case} \end{cases} \quad , \ n \geq i \geq 1,$$

$$\left(D_n' = D_n^* + (\theta_n - c) \cdot Y' \text{ always} \right)$$

where D_i^* denotes the pull-back of D_i by the blow up π, Y' denotes the new exceptional divisor, the point $\xi' \in W_i'$ satisfies $\pi(\xi') = \xi$, $\theta_i = ord_\xi(I_i)$ and c_{i+1} is the corresponding critical value.

In order to obtain the ideals J_{i-1}, $n \geq i > 1$, we define the companion ideals P_i and the composition ideals K_i, see [5] for details.
We construct the companion ideals to ensure that $Sing(P_i, \theta_i) \subset Sing(J_i, c_{i+1})$,

$$P_i = \begin{cases} I_i & \text{if } \ \theta_i \geq c_{i+1} \\ I_i + M_i^{\frac{\theta_i}{c_{i+1} - \theta_i}} & \text{if } \ 0 < \theta_i < c_{i+1} \end{cases} \tag{3}$$

where $\xi \in \mathbb{A}^n_k$ is a point, $\theta_i = ord_\xi(I_i)$ and c_{i+1} is the corresponding critical value.

The composition ideal K_i in W_i of an ideal $J_i = M_i \cdot I_i$, where M_i, I_i are ideals in W_i in a point $\xi \in \mathbb{A}^n_k$, with respect to a control c_{i+1} and a normal crossing divisor E_i in $\mathbb{A}^n_k$ is

$$K_i = \begin{cases} P_i \cdot I_{W_i}(E_i \cap W_i) & \text{if } \ I_i \neq 1, \\ 1 & \text{if } \ I_i = 1. \end{cases} \tag{4}$$

The critical value for the following step of induction on the dimension is $c_i = ord_\xi(K_i)$.

The construction of the composition ideal K_i ensures normal crossing with the exceptional divisor E_i.

We say that an ideal K is *bold regular* if $K =< X^a >$, $K \in k[X]$, $a \in \mathbb{N}$.

Finally, we construct the junior ideal J_{i-1}

$$J_{i-1} = \begin{cases} Coeff_V(K_i) & \text{if } \ K_i \text{ is not bold regular or } 1 \\ 1 & \text{otherwise} \end{cases} \tag{5}$$

where V is a hypersurface of maximal contact in W_i (see [5] page 830) and $Coeff_V(K_i)$ is the coefficient ideal of K_i in V (see [5] page 829). The junior ideal J_{i-1} is an ideal in this suitable hypersurface V of dimension $i-1$.

If $\frac{\theta_n}{c} \geq 1$ we are in the first case of equation (3), $\frac{\theta_{n-1}}{c_n} = \frac{\theta_{n-2}}{c_{n-1}} = \ldots = \frac{\theta_j}{c_{j+1}} = 1$ and $t_{j-1} = \ldots = t_1 = \infty$ for $n-1 \geq j \geq 1$, because $D_{n-1} = \ldots = D_1 = \emptyset$ and $P_i = I_i$, and hence J_{i-1} is always given by a unique monomial.

For an ideal $J =< X_1^{a_1} \cdot \ldots \cdot X_n^{a_n} >$ as in equation (2), if $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq c$ then at every stage $\frac{\theta_n}{c} \geq 1$, so we are always in the above situation. The singular locus of (J, c) is always a union of hypersurfaces $\cup_{i=1}^r \{X_i = 0\}$, $1 \leq r \leq n$, so we will call this case the **minimal codimension case**.

If there exists some $a_{i_0} < c$, at a certain stage of the resolution process it may occur $\frac{\theta_n}{c} < 1$. Then we are in the second case of equation (3), the (exceptional) monomial part M_n can appear in some J_j for $n-1 \geq j \geq 1$, and $\frac{\theta_j}{c_{j+1}}$ can be much greater than 1, what increase the number of blow ups. Now its singular locus is a union of intersections of hypersurfaces of the type $\cup_{l_j}(\{X_{l_1} = 0\} \cap \ldots \cap \{X_{l_i} = 0\})$. This is the **higher codimension case**.

3 Bound in the minimal codimension case

Remark 3.1. From now on, we always look to the points where the invariant function is maximal. So the following results about the behaviour of the invariant function always correspond to the points where it reaches its maximal value.

Proposition 3.2. *Let $(W, (J, c), E)$ be a basic object where J is a monomial ideal as in equation (2) with $a_i \geq c$ for all $1 \leq i \leq n$. We can write $J = J_n = M_n \cdot I_n$. Let $\xi \in W$ be a point where $ord_\xi(I_n) = \theta_n$. After each blow up π which drops θ_n, the invariant function in a neighbourhood of ξ is of the form*

$$\left(\left[\frac{d - \sum_{j=1}^s a_{i_j}}{c}, s \right], [1, 0], \ldots, [1, 0] \right) \ \textit{for some} \ \ 1 \leq s \leq n-1.$$

Proof. After blowing up,

$$J'_n = M'_n \cdot I'_n =< X_{i_1}^{b_1} \cdots X_{i_s}^{b_s} > \cdot < X_{i_{s+1}}^{a_{i_{s+1}}} \cdots X_{i_n}^{a_{i_n}} > \ \text{with} \ \ d - \sum_{j=1}^s a_{i_j} = \sum_{j=s+1}^n a_{i_j} \geq c$$

then, $P'_n = I'_n$ and the (exceptional) monomial part does not appear in J'_l for all $n \geq l \geq 1$.

We have $\theta'_n \neq \theta_n$, then $E'_n = Y' + |E|^{\curlyvee}$ and $m_n = s$, we count all the exceptional divisors of the previous steps and the new one. There are no exceptional divisors in lower dimension because $E'_{n-1} = (Y' + |E|^{\curlyvee}) - E'_n = \emptyset$ and, in a similar way, we obtain $E'_l = \emptyset$ for all $n-1 \geq l \geq 1$.

The normal crossing divisors $D'_i = \emptyset$ for all $n-1 \geq i \geq 1$ then the corresponding ideals $M'_{n-1} = \ldots = M'_1 = 1$. In particular, $M'_{n-1} = 1$, hence

$$c'_n = ord_{\xi'}(K'_n) = ord_{\xi'}(Coeff(K'_n)) = ord_{\xi'}(J'_{n-1}) = ord_{\xi'}(I'_{n-1}) = \theta'_{n-1}$$

with $\xi' \in \mathbb{A}^n_k$ such that $\pi(\xi') = \xi$, because $ord(Coeff(K)) = ord(K)$ when K is a monomial ideal, therefore $\frac{\theta'_{n-1}}{c'_n} = 1$. By the same argument we obtain $\frac{\theta'_{n-2}}{c'_{n-1}} = \ldots = \frac{\theta'_1}{c'_2} = 1$. $\square$

Remark 3.3. After each blow up, the exceptional divisors at each dimension are:

$$E'_j = \begin{cases} E_j^{\curlyvee} & \text{if } (t'_n(\xi'),\ldots,t'_{j+1}(\xi'))=(t_n(\xi),\ldots,t_{j+1}(\xi)) \text{ and } \theta'_j=\theta_j \\ (Y'+(E_1\cup\ldots\cup E_n)^{\curlyvee})-(E'_n+\cdots+E'_{j+1}) & \text{in other case} \end{cases}, \quad n > j \geq 1,$$

$(E'_n=E_n^{\curlyvee}$ if $\theta'_n=\theta_n$ or $E'_n=Y'+(E_1\cup\ldots\cup E_n)^{\curlyvee}$ otherwise)

where $E_j^{\curlyvee}$ denotes the strict transform of E_j by the blow up π, Y' denotes the new exceptional divisor, the point $\xi' \in W'_i$ satisfies $\pi(\xi') = \xi$, $\theta'_j = ord_{\xi'}(I'_j)$ and $\theta_j = ord_{\xi}(I_j)$. We denote $|E| = E_1 \cup \ldots \cup E_n$.

Hence, after the first blow up, since $\theta'_n < \theta_n$ we have $E'_n = Y'$ and $E'_{n-1} = \cdots = E'_1 = \emptyset$. After the second blow up, in the chart where $\theta''_n = \theta'_n$ we obtain $E''_n = (E'_n)^{\curlyvee} = \emptyset$, $E''_{n-1} = Y''$, and $E''_{n-2} = \cdots = E''_1 = \emptyset$ and so on. We call this phenomena propagation because every exceptional divisor appears in the invariant function firstly in dimension n, then in dimension $n-1$, $n-2$, and so on, we never lose none.

Definition 3.4. We will call **propagation**, $\mathbf{p(i,j)}$ for $1 \leq i \leq j-1$, $1 \leq j \leq n$, the needed number of blow ups, remaining constant $(t_n, t_{n-1}, \ldots, t_{j+1})$ and θ_j , to eliminate i exceptional divisors in dimension j when there are no exceptional divisors in lower dimensions. That is, passing from the stage

$$([\theta_n, m_n], \ldots, [\theta_{j+1}, m_{j+1}], [\theta_j, i], [1,0], \ldots, [1,0])$$

to the stage

$$([\theta_n, m_n], \ldots, [\theta_{j+1}, m_{j+1}], [\theta_j, 0], [1,0], \ldots, [1,0], \overbrace{\infty, \ldots, \infty}^{i}).$$

Lemma 3.5. Propagation Lemma *Let $(W,(J,c),E)$ be a basic object where J is a monomial ideal as in equation (2) with $a_l \geq c$ for all $1 \leq l \leq n$. Let $p(i,j)$ be the propagation of i exceptional divisors in dimension j in the resolution process of $(W,(J,c),E)$.*
Then, for all $1 \leq j \leq n$,

$$p\ (i,j) = \begin{cases} i + \sum_{k=1}^{i} p\ (k, j-1) & \text{if } \ 0 \leq i \leq j-1 \\ 0 & \text{if } \ i = j \end{cases} \tag{6}$$

Proof.

- If we have i exceptional divisors in dimension i, K_{i+1} is bold regular, $t_i = \infty$ so we do not see dimension i, therefore $p(i,i) = 0$.

  If there are s exceptional divisors at this step of the resolution process, this means that there are $n-s$ variables in I_n. On the other hand, from dimension n until dimension $i+1$ we have $s-i$ excepcional divisors.

When we calculate $J_{n-1}, \dots, J_{i+1}$, we add to the corresponding composition ideal K_j the variables in $I_{W_j}(E_j \cap W_j)$, so in these dimensions we will have $(n-s)+(s-i) = n-i$ variables.

At each step making induction on the dimension, we lose one variable, so in $n-i-1$ steps we obtain that K_{i+1}, that corresponds to the $n-(n-i-1) = i+1$ position, is bold regular. And the variables appearing in these i exceptional divisors do not appear in the next blowing up center.

- By induction on the dimension:
  - If $j = 1$, $p(1,1) = 0$ by the previous argument.
  - If $j = 2$, $p(1,2) = 1$ because when we propagate 1 excepcional divisor from dimension 2 to dimension 1, K_2' is bold regular.

$$\begin{array}{c}([\theta_n, m_n], \dots, [\theta_2, 1], [1, 0]) \\ \downarrow x_i \\ ([\theta_n, m_n], \dots, [\theta_2, 0], \infty)\end{array}$$

    Then $p(1,2) = 1 = 1 + 0 = 1 + p(1,1)$.
  - We assume the result until $j = s-1$. For $j = s$:

$$\begin{array}{c}([\theta_n, m_n], \dots, [\theta_{s+1}, m_{s+1}], [\theta_s, i], [1,0], \dots, [1,0]) \\ \downarrow \\ \\ ([\theta_n, m_n], \dots, [\theta_{s+1}, m_{s+1}], [\theta_s, i-1], [1,1], [1,0], \dots, [1,0]) \\ \left.\begin{array}{c}\downarrow \\ \vdots \\ \downarrow\end{array}\right\} p(1, s-1) \\ ([\theta_n, m_n], \dots, [\theta_{s+1}, m_{s+1}], [\theta_s, i-1], [1,0], \dots, [1,0], \infty) \\ \downarrow \\ ([\theta_n, m_n], \dots, [\theta_{s+1}, m_{s+1}], [\theta_s, i-2], [1,2], [1,0], \dots, [1,0])\end{array}$$

    In the first blow up, we want to remain constant $([\theta_n, m_n] \dots [\theta_{s+1}, m_{s+1}])$ and θ_s, so we look to some suitable chart where $m_s = i$ drops. As m_s drops then $m_{s-1} = i-(i-1) = 1$ and we propagate this exceptional divisor in dimension $s-1$, making $p(1, s-1)$ blow ups. Otherwise, remaining constant $([\theta_n, m_n] \dots [\theta_{s+1}, m_{s+1}])$ and θ_s, the only possibility is to drop m_s again from $i-1$ to $i-2$ looking to a suitable chart, but in this case we would obtain the same invariant function appearing after the propagation. As we want to construct the largest possible sequence of blow ups, we follow the propagation phenomenon as above.

    Blowing up again:

$$\left.\begin{array}{c}\downarrow \\ \vdots \\ \downarrow\end{array}\right\} p(2, s-1)$$

$$\downarrow$$
$$\vdots$$
$$\downarrow$$
$$([\theta_n, m_n], \ldots, [\theta_{s+1}, m_{s+1}], [\theta_s, 1], [1,0], \ldots, [1,0], \overbrace{\infty, \ldots, \infty}^{i-1})$$
$$\downarrow$$
$$([\theta_n, m_n], \ldots, [\theta_{s+1}, m_{s+1}], [\theta_s, 0], [1, i], [1,0], \ldots, [1,0])$$

Then, for $1 \leq i \leq s-1$,

$$p(i,s) = 1 + p(1, s-1) + 1 + p(2, s-1) + \cdots + 1 + p(i, s-1)$$

with $p(l, s-1)$, $1 \leq l \leq i$, defined by the induction hypothesis. $\square$

Remark 3.6. Computation of examples in Singular with *desing* package has been useful to state this behaviour of the exceptional divisors after blowing up. The implementation of this package is based on the results appearing in [3].

Theorem 3.7. *Let $(W, (J,c), \emptyset)$ be a basic object where J is a monomial ideal as in equation (2) with $a_i \geq c$ for all $1 \leq i \leq n$. Then, the invariant function corresponding to $(W, (J,c), \emptyset)$ drops after blowing up in the following form:*

$$\begin{array}{cl}
([\frac{d}{c}, 0], [1,0], \ldots, [1,0]) & \\
\downarrow X_i & 1^{st} \ \textit{blow up} \\
([\frac{d-a_i}{c}, 1], [1,0], \ldots, [1,0]) & \\
\left.\begin{array}{c} \downarrow X_i \\ \vdots \\ \downarrow X_i \end{array}\right\} & p\ (1,n)\ \textit{blow ups} \\
([\frac{d-a_i}{c}, 0], [1,0], \ldots, [1,0], \infty) & \\
\downarrow & \textit{blowing up only variables in } I \\
([\frac{d-a_i-a_j}{c}, 2], [1,0], \ldots, [1,0]) & \\
\left.\begin{array}{c} \downarrow \\ \vdots \\ \downarrow \end{array}\right\} & p\ (2,n)\ \textit{blow ups} \\
([\frac{d-a_i-a_j}{c}, 0], [1,0], \ldots, [1,0], \infty, \infty) & \\
\downarrow & \textit{blowing up only variables in } I \\
\vdots & \vdots \\
\downarrow & \\
([\frac{a_l}{c}, n-1], [1,0], \ldots, [1,0]) & \\
\left.\begin{array}{c} \downarrow \\ \vdots \\ \downarrow \end{array}\right\} & p\ (n-1,n)\ \textit{blow ups} \\
([\frac{a_l}{c}, 0], \infty, \ldots, \infty) &
\end{array}$$

At this stage, $a_l \geq c$ by hypothesis, so the next blowing up center is $\{X_l = 0\}$, and then we obtain an exceptional monomial.

Proof. It follows by propagation lemma and the fact that each time that θ_n drops $E'_n = Y' + |E|^\nu$, and $E'_l = (Y' + |E|^\nu) - (E'_n + \cdots + E'_{l+1}) = \emptyset$ for all $n-1 \geq l \geq 1$. $\square$

Remark 3.8. Following the propagation in the above way provides the largest branch in the resolution tree, because in other case, for example after the first blow up

$$\begin{array}{ccc} & ([\frac{d-a_i}{c}, 1], [1,0], \ldots, [1,0]) & \\ & X_i \swarrow \qquad \searrow X_j & \\ ([\frac{d-a_i}{c}, 0], [1,1], [1,0] \ldots, [1,0]) & & ([\frac{d-a_i-a_j}{c}, 2], [1,0], \ldots, [1,0]) \end{array}$$

looking to some chart X_j with $j \neq i$ we obtain an invariant which will appear later in the resolution process, after the propagation $p(1,n)$.

Corollary 3.9. *Let $(W, (J,c), \emptyset)$ be a basic object where J is a monomial ideal as in equation (2) with $a_i \geq c$ for all $1 \leq i \leq n$. Therefore the needed number of blow ups to transform J into an exceptional monomial is at most*

$$1 + p(1,n) + 1 + p(2,n) + \ldots + 1 + p(n-1,n) + 1 = n + \sum_{j=1}^{n-1} p(j,n). \tag{7}$$

Remark 3.10. In this case we always have $\theta_n \geq c$, so $Sing(J,c) \neq \emptyset$ at every stage of the resolution process. Therefore, in the resolution tree, the branch of theorem 3.7 effectively appears, and it is the largest, hence (7) is exactly the number of blow ups to obtain $J' = M'$.

Proposition 3.11. *Let $(W, (J,c), \emptyset)$ be a basic object where J is a monomial ideal as in equation (2) with $a_i \geq c$ for all $1 \leq i \leq n$. Then the above sum of propagations is a partial sum of Catalan numbers.*

$$n + \sum_{j=1}^{n-1} p(j,n) = \sum_{j=1}^{n} C_j \quad \textit{where} \quad C_j = \left\{ \frac{1}{j+1} \begin{pmatrix} 2j \\ j \end{pmatrix} \right\} \quad \textit{are Catalan numbers.}$$

Proof.

Step 1: Extending p to arbitrary dimension:

$$n + \sum_{j=1}^{n-1} p(j,n) = p(n, n+1).$$

Because of the form of the recurrence equation defining $p(i,j)$ and the fact that $p(n,n) = 0$ by definition, we have

$$p(n, n+1) = n + \sum_{j=1}^{n} p(j,n) = n + \sum_{j=1}^{n-1} p(j,n).$$

Step 2: Solving the recurrence equation defining $p(i,j)$:

(a) We transform the recurrence equation in one defined for every $i, j \geq 0$:
By sending the pair (i,j) to the pair $(i, j-i)$ we extend the recurrence to $i, j \geq 0$, that is we consider

$$\tilde{p}(i,j) = p(i, i+j)$$

then $p(i,j) = \tilde{p}(i, j-i)$. As $p(i,j)$ is defined for $0 \leq i \leq j$ then $\tilde{p}(i,j)$ is defined for $0 \leq i \leq i+j$ for every $i, j \geq 0$.

(b) We obtain an auxiliary recurrence equation:

$$\tilde{p}(i,j) - \tilde{p}(i-1,j+1) = p(i,i+j) - p(i-1,i+j)$$

$$= i + \sum_{k=1}^{i} p(k,i+j-1) - (i-1) - \sum_{k=1}^{i-1} p(k,i+j-1) = p(i,i+j-1) + 1 = \tilde{p}(i,j-1) + 1.$$

Therefore, we have the following recurrence equation involving $\tilde{p}(i,j)$

$$\begin{cases} \tilde{p}(i,j) = 1 + \tilde{p}(i-1,j+1) + \tilde{p}(i,j-1) & \text{for } i,j \geq 1 \\ \tilde{p}(0,j) = \tilde{p}(i,0) = 0 \end{cases}$$

Taking $r(i,j) = p(i,i+j) + 1 = \tilde{p}(i,j) + 1$ we obtain

$$\begin{cases} r(i,j) = r(i-1,j+1) + r(i,j-1) & \text{for } i,j \geq 1 \\ r(0,j) = \tilde{p}(0,j) + 1 = 1,\ r(i,0) = \tilde{p}(i,0) + 1 = 1 \end{cases}$$

(c) Resolving the auxiliary recurrence equation by generating functions:
We define $r_{i,j} := r(i,j)$ and the generating functions

$$R(x,y) = \sum_{i,j\geq 0} r_{i,j} x^i y^j \in \mathbb{C}[[x,y]],\ R_s(x,y) = \sum_{i,j\geq 1} r_{i,j} x^{i-1} y^{j-1} \in \mathbb{C}[[x,y]]$$

by the recurrence equation involving $r(i,j)$

$$R_s(x,y) = \sum_{i,j\geq 1} r_{i-1,j+1} x^{i-1} y^{j-1} + \sum_{i,j\geq 1} r_{i,j-1} x^{i-1} y^{j-1} = \sum_{i\geq 0, j\geq 1} r_{i,j+1} x^i y^{j-1} + \frac{1}{x} \sum_{i\geq 1, j\geq 0} r_{i,j} x^i y^j$$

$= \frac{1}{y^2} \sum_{i\geq 0,j\geq 1} r_{i,j+1} x^i y^{j+1} + \frac{1}{x}\left[\sum_{i\geq 1} r_{i,0} x^i + \sum_{i\geq 1,j\geq 1} r_{i,j} x^i y^j\right]$
$= \frac{1}{y^2} \sum_{i\geq 0,j\geq 2} r_{i,j} x^i y^j + \frac{1}{x}\left[\sum_{i\geq 1} x^i + \sum_{i\geq 1,j\geq 1} r_{i,j} x^i y^j\right]$
$= \frac{1}{y^2}\left[\sum_{i\geq 0,j\geq 1} r_{i,j} x^i y^j - \sum_{i\geq 0} r_{i,1} x^i y\right] + \frac{1}{x}\left[\frac{1}{1-x} - 1 + xyR_s(x,y)\right]$
$= \frac{1}{y^2}\left[\sum_{j\geq 1} r_{0,j} y^j + xyR_s(x,y) - y\sum_{i\geq 0} r_{i,1} x^i\right] + \frac{1}{x}\left[\frac{x}{1-x} + xyR_s(x,y)\right]$
$= \frac{1}{y^2}\left[\frac{y}{1-y} + xyR_s(x,y) - y\sum_{i\geq 0} r_{i,1} x^i\right] + \frac{1}{1-x} + yR_s(x,y)$
$= \frac{1}{y(1-y)} + \frac{x}{y} R_s(x,y) - \frac{1}{y}\sum_{i\geq 0} r_{i,1} x^i + \frac{1}{1-x} + yR_s(x,y).$

Then

$$\left(1 - y - \frac{x}{y}\right) R_s(x,y) = \frac{1}{y(1-y)} + \frac{1}{1-x} - \frac{1}{y}\sum_{i\geq 0} r_{i,1} x^i$$

multiplying the equality by y we have

$$(y - y^2 - x) R_s(x,y) = \frac{1}{1-y} + \frac{y}{1-x} - \sum_{i\geq 0} r_{i,1} x^i$$

$$= \frac{1}{1-y} + \frac{y}{1-x} - r_{0,1} - \sum_{i\geq 1} r_{i,1} x^i = \frac{y}{1-y} + \frac{y}{1-x} - \sum_{i\geq 1} r_{i,1} x^i.$$

Therefore

$$(y - y^2 - x)R_s(x, y) = \frac{y}{1-y} + \frac{y}{1-x} - \sum_{i\geq 1} r_{i,1}x^i$$

so we obtain an equation of the form

$$Q(x, y)R_s(x, y) = K(x, y) - U(x).$$

Now we apply the *kernel method* used in [2], algebraic case 4.3:
If $Q(x, y) = 0$ then $y = \frac{1\pm\sqrt{1-4x}}{2}$. We take the solution passing through the origin, $y = \frac{1-\sqrt{1-4x}}{2}$ and $y = xC(x)$ where $C(x)$ is the generating function of Catalan numbers. On the other hand, $Q(x, y) = 0$ gives $K(x, xC(x)) = U(x)$,

$$K(x, y) = \frac{y}{1-y} + \frac{y}{1-x} = \frac{-y^2 + y - x + 1}{(1-x)(1-y)} - 1$$

so $K(x, xC(x)) = \frac{1}{(1-x)(1-xC(x))} - 1$ and using $\frac{1}{1-xC(x)} = C(x)$ we have

$$U(x) = \frac{C(x)}{1-x} - 1.$$

Making some calculations and using

$$R(x, y) = xyR_s(x, y) + \sum_{j\geq 0} r_{0,j}y^j + \sum_{i\geq 0} r_{i,0}x^i - r_{0,0}$$

we obtain the generating function of $r(i, j)$

$$R(x, y) = \frac{xyC(x) + x - y}{(y^2 - y + x)(1-x)}.$$

Step 3: Obtaining the generating function corresponding to the values $p(n, n + 1)$:
The coefficient of y in $R(x, y)$ is just $\sum_{i\geq 0} r_{i,1}x^i$ then

$$\sum_{i\geq 0} r_{i,1}x^i = \frac{\partial R(x, y)}{\partial y}\Big|_{y=0} = \frac{C(x)}{1-x}$$

is the generating function of the elements in the first column.
If $C(x)$ is the generating function of C_n then the convolution product $C(x) \cdot \frac{1}{1-x}$ is the generating function of $\sum_{k=0}^{n} C_k = S_n$ therefore

$$r_{n,1} = \sum_{k=0}^{n} C_k.$$

As $r(n, 1) = p(n, n + 1) + 1$ then $p(n, n + 1) = r(n, 1) - 1 = \sum_{k=0}^{n} C_k - 1$, as $C_0 = 1$ we have

$$p(n, n + 1) = \sum_{k=1}^{n} C_k$$

where C_k are Catalan numbers.

□

See [11] for more details about Catalan numbers and the web page [10] for further details about their partial sums.

Theorem 3.12. *Let $(W,(J,c),\emptyset)$ be a basic object where J is a monomial ideal as in equation (2) with $a_i \geq c$ for all $1 \leq i \leq n$. Then the needed number of blow ups to resolve $(W,(J,c),\emptyset)$ is at most*

$$\sum_{j=1}^{n} C_j + (2^{\sum_{j=1}^{n} C_j} - 1)(d-c) - c + 1$$

where C_j are Catalan numbers.

Proof. It follows by theorem 1.1, lemma 2.1 and proposition 3.11. □

Example 3.13. In the following table we can see some values of the bound for any monomial ideal J as in equation (2) with $a_i \geq c$ for all $1 \leq i \leq n$.

Table 1: Values of the bound

n	$\sum_{j=1}^{n} C_j$	global bound
1	1	$1 + (d-c) - c + 1$
2	3	$3 + 7(d-c) - c + 1$
3	8	$8 + 255(d-c) - c + 1$
4	22	$22 + 4194303(d-c) - c + 1$

Remark 3.14. Note that, as a consecuence of proposition 3.11, the needed number of blow ups to transform J into an exceptional monomial only depends on n, the dimension of the ambient space.

Corollary 3.15. *Let $J = < Z^c - X_1^{a_1} \cdot \ldots \cdot X_n^{a_n} > \subset k[X_1, \ldots, X_n, Z]$ be a toric ideal with $a_i \geq c$ for all $1 \leq i \leq n$. Then the needed number of blow ups to resolve $(\mathbb{A}_k^{n+1},(J,c),\emptyset)$ is at most*

$$\sum_{j=1}^{n} C_j + (2^{\sum_{j=1}^{n} C_j} - 1)(d-c) - c + 1$$

where C_j are Catalan numbers and $d = \sum_{i=1}^{n} a_i$.

4 Higher codimension case

In the minimal codimension case, the way in which the invariant drops essentially depends on the number of accumulated exceptional divisors. But in this case the first components of the invariant play an important role, they may increase suddenly because the order of the (exceptional) monomial part M_n can appear in some θ_j. We will call this situation *higher codimension case in dimension j*, and after some blow ups, we can obtain a new higher codimension case in other dimension. So, we must estimate the number of blow ups while $\theta_n \geq c$ with a suitable sum of propagations, determine when is going to appear the higher codimension case in dimension 1 and use the known estimation for the order of M_n to estimate the number of blow ups until the following higher codimension case inside this one (if it is possible), afterwards determine when is going to appear the higher codimension case in dimension 2, and so on.

Hence, it has not been possible to obtain a bound for this case in the same way as above due to the complications of the combinatorial problem that perform that we can not know what branch is the largest in the resolution tree (until obtain an exceptional monomial).

Furthermore, in case of giving such bound, the many situations make us expect that this bound would be very huge, even to estimate only the number of blow ups until obtain an exceptional monomial.

Example 4.1. If we consider the basic object $(W,(J,c),\emptyset) = (\mathbb{A}^3_k,(X_1^5X_2^4X_3,4),\emptyset)$, there exists a branch of height 15 until obtain $J' = M'$ or $Sing(J',c) = \emptyset$. So, in dimension 3, we need a bound greater than or equal to 15 for a higher codimension case, in front of the 8 blow ups needed for a minimal codimension case.

In any case, both theorem 1.1 and lemma 2.1 are valid also in the higher codimension case, so the open problem is to find a bound C until obtain an exceptional monomial to construct a global bound of the form

$$C + (2^C - 1)(d - c) - c + 1.$$

For $n = 2$ the higher codimension case appears only in dimension 1 and making some calculations we obtain $C = 3$, that gives the same bound as in the minimal codimension case. This bound can be improved by studying the different branches.

References

[1] E. Bierstone and P. Milman: *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*. Invent. math. **128** (1997), 207 − 302.

[2] M. Bousquet-Mélou and M. Petkovsek: *Linear recurrences with constant coefficients: the multivariate case*. Discrete Mathematics, **225** (2000), 1 − 3, 51 − 75.

[3] G. Bodnár and J. Schicho: *Automated resolution of singularities for hypersurfaces*. Journal of Symbolic Computation, **30**, 4 (2000), 401 − 428.

[4] G. Bodnár and J. Schicho: *"desing"*- A computer program for resolution of singularities. *http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup/*.

[5] S. Encinas and H. Hauser: *Strong resolution of singularities in characteristic zero*. Commentarii Mathematici Helvetici, **77**, 4 (2002), 821 − 845.

[6] S. Encinas and O. Villamayor: *A Course on Constructive Desingularization and Equivariance*. In *Resolution of Singularities, A research textbook in tribute to Oscar Zariski* (Basel, 2000), H. Hauser, J. Lipman, F. Oort, and A. Quirós, Eds. Progress in Math. **181**, Birkhäuser, 147 − 227.

[7] G.-M. Greuel, G. Pfister, and H. Schönemann. Singular 2.0.5. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern (2003). *http://www.singular.uni-kl.de*.

[8] H. Hironaka: *Resolution of singularities of an algebraic variety over a field of characteristic zero*. Annals of Mathematics, **79** (1964), 109 − 362.

[9] J. Kollár: *Lectures on Resolution of Singularities*. Annals of Mathematics Studies, **166** (Princeton and Oxford, 2007), Princeton University Press.

[10] N. J. A. Sloane: Sequences number A014138 and A000108 from The On-Line Encyclopedia of Integer Sequences, *http://www.research.att.com/∼njas/sequences/index.html*.

[11] R. P. Stanley: *Enumerative Combinatorics*. Vol 2. Cambridge Studies in Avanced Mathematics, **62** (Cambridge, 1999), Cambridge University Press.

[12] O. Villamayor: *Constructiveness of Hironaka's resolution*. Annales Scientifiques École Normale Supérieure, 4^e serie, **22** (1989), $1-32$.

[13] J. Wodarczyk: *Simple Hironaka resolution in characteristic zero*. Journal of the Americal Mathematical Society, **18** (2005), $779-822$.

Departamento de Matemática Aplicada
E.T.S. Arquitectura, Universidad de Valladolid
Avda. de Salamanca s/n. 47014 Valladolid Spain
rblanco@modulor.arq.uva.es