# Simple formulas for lattice paths avoiding certain periodic staircase boundaries

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**Abstract**

There is a strikingly simple classical formula for the number of lattice paths avoiding the line $x = ky$ when $k$ is a positive integer. We show that the natural generalization of this simple formula continues to hold when the line $x = ky$ is replaced by certain periodic staircase boundaries—but only under special conditions. The simple formula fails in general, and it remains an open question to what extent our results can be further generalized.

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## 1 Background and main results

Throughout this paper, a *lattice path* will mean a lattice path in the plane whose only allowable steps are north $(0, 1)$ and east $(1, 0)$.

It is a classical theorem [1][2] that if $k$ is a positive integer, then the number of lattice paths from $(0, 0)$ to $(a + 1, b)$ (where $a \geq kb$) that avoid touching or crossing the line $x = ky$ except at $(0, 0)$ is given by the formula

$$\binom{a+b}{b} - k\binom{a+b}{b-1}. \tag{1}$$

In fact, more is true: There are

$$\binom{a}{c-1}\binom{b}{c-1} - k\binom{a-1}{c-2}\binom{b+1}{c} \tag{2}$$

such paths with $c-1$ northwest corners. (There is a similar-looking formula for paths with a given number of southeast corners.) This stronger result appears explicitly in [4] and implicitly even earlier, but our favorite proofs of all these facts are the bijective proofs of Goulden and Serrano [3].

It is natural to ask if there are similar simple formulas for lattice paths from $(0,0)$ to $(a,b)$ that avoid the line $x = ky$, if $k$ is allowed to be an arbitrary positive rational number. While one can write down a determinantal formula (indeed, a determinantal formula exists for an arbitrarily shaped boundary), nothing as simple as (1) is known, and empirical investigation does not suggest any obvious conjecture.

Our first main result is that for certain periodic staircase boundaries (instead of straight-line boundaries), there *are* simple enumerative formulas that generalize (1) and (2), at least for certain special starting and ending points.

**Definition 1** *Given positive integers $s$ and $t$, let $A_{s,t}$ be the infinite staircase path that starts at $(0,t)$, then takes $s$ steps east, $t$ steps north, $s$ steps east, $t$ steps north, and so on.*

**Definition 2** *Given a set $S$ of (finite) lattice paths, take each path $\pi \in S$, and augment it by prepending a north step to the beginning of $\pi$ and appending a north step to the end of $\pi$. Let $S^+$ denote the resulting set of lattice paths.*

**Theorem 3** *Let $s$, $t$, $n$, and $c$ be positive integers.*

*(1) Let $S_1$ be the set of lattice paths from $(0,0)$ to $(sn+1,tn)$ that avoid $A_{s,t}$. There are*

$$t\binom{sn}{c-1}\binom{tn}{c-1} - s\binom{sn-1}{c-2}\binom{tn+1}{c} \tag{3}$$

*paths in $S_1^+$ with $c$ northwest corners (equivalently, $c$ southeast corners).*

*(2) Let $S_2$ be the set of lattice paths from $(1,0)$ to $(sn,tn-1)$ that avoid $A_{s,t}$. There are*

$$t\binom{sn-1}{c-1}\binom{tn-1}{c-1} - s\binom{sn-2}{c-2}\binom{tn}{c} \tag{4}$$

*paths in $S_2^+$ with $c$ northwest corners (equivalently, $c$ southeast corners).*

The equivalence between counting northwest and southeast corners follows because in a lattice path that starts with a north step and ends with a north step, the first corner must be a northwest corner and the last corner must be a southeast corner, and northwest and southeast corners must alternate. Also, since $|S| = |S^+|$ for any $S$, summing over all $c$ and applying Vandermonde convolution immediately yields the following corollary.

**Corollary 4** *Let $s$, $t$, and $n$ be positive integers. Then*

$$|S_1| = t\binom{sn+tn}{tn} - s\binom{sn+tn}{tn-1} \tag{5}$$

*and*

$$|S_2| = t\binom{sn+tn-2}{tn-1} - s\binom{sn+tn-2}{tn-2}. \tag{6}$$

Note that avoiding $A_{k,1}$ is the same as avoiding $x = ky$ except at $(0,0)$, so our results generalize (1) and (2) in one direction, by allowing arbitrary $s$ and $t$, but are simultaneously more special in another direction, since only certain special endpoints are allowed. More precisely, note that if we set $a = kn$ and $b = n$ in (1) and (2), then we get the same answers as if we set $s = k$ and $t = 1$ in (5) and (3). (When $t = 1$, the map $S \mapsto S^+$ simply adds a northwest corner to every path.)

Our proof of Theorem 3 is similar to Goulden and Serrano's in several ways but differs in one crucial way. Like Goulden and Serrano, we interpret (3) and (4) as counting *all* paths of a certain type, minus the *bad* paths. Another similarity is the idea of breaking the bad path into two halves $\rho$ and $\sigma$ at the first "bad point" so as to manipulate $\rho$ and $\sigma$ into something that is easier to count. The crucial difference is that Goulden and Serrano *rotate* $\rho$, whereas we *interchange* $\rho$ *and* $\sigma$. Therefore our bijection does not specialize to Goulden and Serrano's rotation principle nor to André's reflection principle.

We also give a second proof of Corollary 4, which is based on a well-known argument of Raney [6] regarding cyclic shifts of integer sequences.

It is frustrating that Theorem 3 applies only to special endpoints. Can anything be said about other endpoints? We do not have a satisfactory answer to this question, but our second main result is a tantalizing hint that more general theorems lie waiting to be found. It is best stated in the language of binary strings; we draw the connection to lattice paths afterwards.

**Theorem 5** *For $n \geq 1$, $s \geq 0$, and $0 \leq r \leq 2n$, let $a(n,s,r)$ be the number of binary sequences of length $(s+2)n+1$ such that for all $j$, the $j$th occurrence of 10 (if it exists) appears in positions $(s+2)j+1$ and $(s+2)j+2$ or later, and such that the total number of occurrences of 10 and 01 is at most $r$. Then*

$$a(n,s,r) = 2\binom{(s+2)n-1}{r} - (s-2)\sum_{i=0}^{r-1}\binom{(s+2)n-1}{i}. \tag{7}$$

Our proof of Theorem 5 is again an application of Raney's argument, combined with a straightforward induction on $n$.

To convert Theorem 5 into lattice-path language, let $\beta = (b_1, b_2, \ldots, b_{(s+2)n+1})$ be a binary sequence, let $b_0 = 0$, and define $\Delta\beta$ by $(\Delta\beta)_i = |b_i - b_{i-1}|$, for $i \geq 1$. If we convert $\Delta\beta$ into a lattice path by turning 0's into east steps and 1's into north steps, then it is easily checked that the binary sequences in Theorem 5 turn into lattice paths avoiding $B_s$, as defined below.

**Definition 6** *For $s \geq 0$, define $B_s$ to be the staircase path that starts at $(0, 2)$, then takes $s + 1$ steps east, 2 steps north, $s$ steps east, 2 steps north, $s$ steps east, and so on, always alternating between 2 steps north and $s$ steps east except for the first segment of $s + 1$ steps east.*

For example, $B_4$ is the dashed line in the lower picture in either Figure 2 or Figure 3 below. Curiously, we have not been able to generalize Theorem 5 to more general staircase boundaries, or to refine the count according to northwest or southeast corners. But for the special case when $s = 2k$ is even, we have a second, purely bijective proof of the following corollary of Theorem 5.

**Corollary 7** *For all $n \geq 1$ and $k \geq 0$, the number of lattice paths of length $2(k+1)n+1$ that start at $(0, 0)$ and that avoid touching or crossing $B_{2k}$ equals the number of lattice paths of length $2(k + 1)n + 1$ that start at $(0, 0)$ and that avoid touching or crossing the line $x = ky$ except at $(0, 0)$. This number has the explicit formula*

$$\binom{2(k+1)n}{2n} - (k-1)\sum_{i=0}^{2n-1}\binom{2(k+1)n}{i}. \tag{8}$$

Formula (8) is of course just obtained by summing over the appropriate instances of (1). These numbers also appear as A107027 in Sloane's Online Encyclopedia of Integer Sequences. This cries out for a combinatorial interpretation of each summand as counting lattice paths avoiding $B_{2k}$ but with varying endpoints. Unfortunately, we do not know how to make this idea work.

Note that the case $s = 2$ of Theorem 5 is particularly simple:

**Corollary 8** *For $n \geq 1$, there are $\binom{4n}{2n}$ binary sequences of length $4n+1$ with the property that for all $j$, the $j$th occurrence of 10 appears in positions $4j+1$ and $4j + 2$ or later (if it exists at all).*

We suspect that we have not yet found the "proof from the Book" of Corollary 8, and encourage the reader to find it.

The outstanding open question is whether our results generalize further. We should mention two papers [5] and [7] that consider staircase boundaries similar to $A_{s,t}$ and that prove results related to Corollary 4. Although our results do not seem to imply or be implied by their results, perhaps it would be fruitful to investigate the precise relationships among them.

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## 2 Proofs of Theorem 3 and Corollary 4

**PROOF.** (of Theorem 3)

We prove part (1) first. It will be convenient to first prove formula (5) bijectively, and then track corner counts through the bijection.

As we hinted above, we interpret (5) as counting the set $T$ of *all* paths of a certain type, minus the set of *bad* paths. For $0 \le i \le t-1$, let $T_i$ be the set of all lattice paths from $(1, i)$ to $(sn+1, tn+i)$, and let $T = \bigcup_i T_i$. Then

$$|T_i| = \binom{sn+tn}{tn} \qquad \text{and} \qquad |T| = t\binom{sn+tn}{tn}. \tag{9}$$

We regard $S_1$ as a subset of $T$ as follows. Given any $\pi \in S_1$, find the smallest $i$ such that $(1, i) \in \pi$; such an $i$ must exist. Then there exists a unique $\pi' \in T_i$ that agrees exactly with the remainder of $\pi$, provided that we append $i$ north steps to the end of $\pi$. Identifying $\pi$ with $\pi'$ embeds $S_1$ in $T$. It remains to show that the number of bad paths—i.e., the paths in $T \backslash S_1$—is $s\binom{sn+tn}{tn-1}$.

We partition the set $T \backslash S_1$ into $s$ disjoint sets $U_1, \dots, U_s$ as follows. By definition, every path in $T \backslash S_1$ must hit a bad point, i.e., a point on the boundary $A_{s,t}$. For $1 \le j \le s$, we let $U_j$ be the set of all paths in $T \backslash S_1$ whose first bad point has an $x$-coordinate that is congruent to $j$ modulo $s$. To prove formula (5), it suffices to show that $|U_j| = \binom{sn+tn}{tn-1}$, independent of $j$.

Fix any $j$. Given $\pi \in U_j$, observe that the step that terminates in the first bad point of $\pi$ must be a north step. Let $\rho$ be the portion of $\pi$ prior to this fatal north step, and let $\sigma$ be the portion of $\pi$ after the bad point. Thus $\pi = (\rho, \text{north}, \sigma)$. Now comes the crucial part of the proof, where we interchange $\rho$ and $\sigma$. More precisely, let $\pi'$ be the lattice path that starts at $(j-1, t)$ and takes steps $(\sigma, \text{east}, \rho)$. See Figure 1 for an example.

We claim that $\pi \mapsto \pi'$ bijects $U_j$ onto the set $U'_j$ of all paths from $(j-1, t)$ to $(sn+j, tn+t-1)$. First note that since $\pi$ and $\pi'$ have the same total number of north steps and the same total number of east steps except that one north step of $\pi$ has been changed into an east step of $\pi'$, it follows that $\pi'$ does in fact terminate at $(sn+j, tn+t-1)$. Now, given any path $\pi' \in U'_j$, let $\sigma$ be

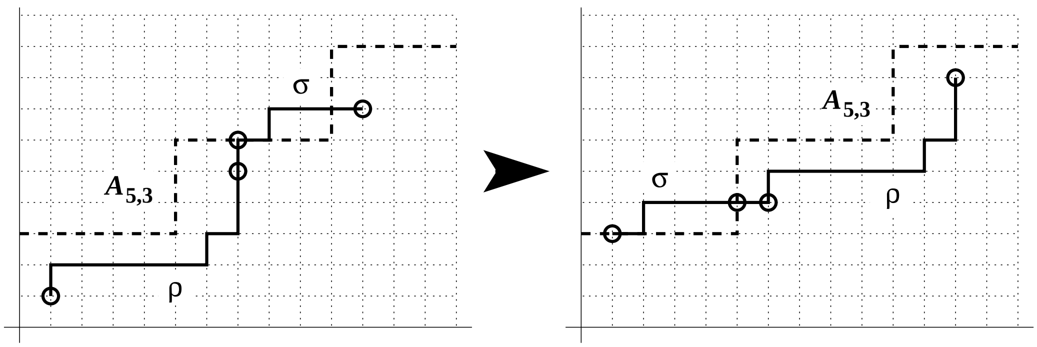


Fig. 1. Example of $\pi \mapsto \pi'$ with $s = 5$, $t = 3$, $n = 2$, and $\pi \in T_1 \cap U_2$

the initial segment of $\pi'$ up to the last point of $\pi'$ that lies on the boundary $A_{s,t}$. The next step after that must be an east step; let $\rho$ be the remainder of $\pi'$ after that. It is straightforward to check that this allows us to construct a unique preimage $\pi$ of $\pi'$. This proves formula (5).

Now we prove the stronger formula (3), for northwest corners. The initial north step of each lattice path in $S_1^+$ forces there to be a northwest corner with $x$-coordinate zero, whereas the final north step does not affect the northwest corner count. Therefore if we embed $S_1$ in $T$ as above, we really want to count lattice paths with $c-1$ northwest corners (rather than $c$ northwest corners). There are $t\binom{sn}{c-1}\binom{tn}{c-1}$ paths in $T$ with $c-1$ northwest corners, because we can pick the $x$-coordinates and $y$-coordinates of the corners independently. It therefore suffices to show that for all $j$, there are $\binom{sn-1}{c-2}\binom{tn+1}{c}$ paths in $U_j$ with $c-1$ northwest corners.

If $\alpha$ is a binary string, let $|\alpha|$ denote its length, and let $w(\alpha)$ denote its *weight,* i.e., the number of 1's in $\alpha$. Let $U_j''$ be the set of ordered pairs $(\alpha, \beta)$ of binary strings such that $|\alpha| = sn-1$, $|\beta| = tn+1$, and $w(\beta) = w(\alpha)+2$. It suffices to describe a bijection from $U_j'$ to $U_j''$ such that the composite map $\pi \mapsto \pi' \mapsto (\alpha, \beta)$ sends paths with $c-1$ northwest corners to pairs $(\alpha, \beta)$ with $w(\beta) = c$.

Before describing this bijection, we make two observations. Let $\pi$, $\rho$, $\sigma$, and $\pi'$ be as above. The first observation is that, because of the position of the endpoint of $\pi$ relative to the boundary $A_{s,t}$, $\sigma$ always has at least one east step. The second observation is that we lose a northwest corner when passing from $\pi$ to $\pi'$ iff $\sigma$ starts with an east step, and we gain a northwest corner as we pass from $\pi$ to $\pi'$ iff $\sigma$ ends with a north step. (Note that we can both gain a corner and lose a corner, leaving the total corner count unchanged.) So to track corners properly, we must watch the first and last steps of $\sigma$.

Now for the bijection. Given $\pi' \in U_j'$, construct $\alpha$ by first writing down a binary string of length $sn$ whose $i$th digit $(1 \le i \le sn)$ is 1 iff $j+i-1$ is the $x$-coordinate of a northwest corner of $\pi'$, and then deleting the digit corresponding to the point where $\pi'$ intersects $A_{s,t}$ for the last time. This digit must exist, because $\sigma$ has at least one east step. For example, in Figure 1, we first write down 1000100001, and then delete the 4th digit to obtain $\alpha = 100100001$.

The first $tn - 1$ digits of $\beta$ are obtained by writing down the binary string of length $tn - 1$ whose $i$th digit ($1 \le i \le tn - 1$) is 1 iff $t + i$ is the $y$-coordinate of a northwest corner of $\pi'$. The next digit of $\beta$ is 1 iff $\sigma$ does not start with a north step, and the last digit of $\beta$ is the complement of the deleted digit of $\alpha$. For example, in Figure 1, $\beta = 1110011$.

To see that $w(\beta) = w(\alpha) + 2$, first pair off the 1's in $\alpha$ and $\beta$ arising from northwest corners that they both "see," and then note that $\beta$ will have two extra 1's corresponding to the columns in which the first and last vertices of $\sigma$ appear: Either $\beta$ sees a northwest corner in that column (and $\alpha$ of course does not see it), or there is no such corner, in which case the appropriate trailing bit of $\beta$ will be set. Either way, $w(\beta) = w(\alpha) + 2$.

Similarly, as we pass from $\pi$ to $\pi'$ to $\beta$, a corner that is lost from $\pi$ to $\pi'$ is "caught" by the penultimate bit of $\beta$, and $\beta$ will gain an extra 1 either by catching a gained corner or, if no corner is gained, by setting its last bit. Thus $w(\beta)$ is one more than the number of northwest corners of $\pi$. Equivalently, $w(\alpha)$ is one *less* than the number of northwest corners of $\pi$.

It remains to show that $\pi' \mapsto (\alpha, \beta)$ is a bijection. Since $|U'_j| = |U''_j|$, it suffices to show that $\pi'$ can be reconstructed from its image $(\alpha, \beta)$. To reconstruct $\pi'$ it suffices to reconstruct the northwest corners. The penultimate digit of $\beta$ is 0 iff $\pi'$ has a northwest corner with $x$-coordinate $j - 1$, so we need only reconstruct the deleted digit of $\alpha$. The *value* of the deleted digit is the complement of the last digit of $\beta$, so we need only reconstruct its *position*. To do this, take $(\alpha, \beta)$ and begin constructing $\pi'$ from the end backwards without regard to the deleted digit. At some point, the partially reconstructed path will touch or cross the boundary $A_{s,t}$. It is easy to check that the first such contact point with $A_{s,t}$ yields the position of the deleted digit of $\alpha$.

This completes the proof of part (1). The proof of part (2) is very similar, so we focus only on the details that differ. For $0 \le i \le t - 1$, let $T_i$ be the set of all lattice paths from $(1, i)$ to $(sn, tn + i - 1)$, and let $T = \bigcup_i T_i$. Then $T$ is our set of *all* paths. Note that $S_2$ is already naturally a subset of $T$—in fact, $S_2 \subset T_0$—so we do not have to embed $S_2$ in $T$. The definition of the sets $U_j$ is exactly analogous. However, $\pi'$ now starts at $(j, t+1)$ rather than at $(j-1, t)$, and ends at $(sn + j, tn + t - 1)$. The proof of (6) now goes through as before.

To do the corner count, we need to define the map $\pi' \mapsto (\alpha, \beta)$ in the case that $\sigma$ is empty or vertical, i.e., has no east steps. In this case, we always delete the *first* digit of $\alpha$. The definition of $\beta$ is the same as before. The arguments that $w(\beta) = w(\alpha) + 2$ and that $\pi' \mapsto (\alpha, \beta)$ is a bijection still work.

However, $w(\alpha)$ is no longer always one less than the number of northwest corners of $\pi$. Let $V$ denote the set of paths in $\bigcup_j U_j$ for which $\sigma$ is vertical or empty and $\rho$ starts with a horizontal step. Then it is straightforward to check that for $\pi \in V$, $w(\alpha)$ is *equal* to the number of northwest corners of $\pi$. So if we let $X^c$ denote the members of $X$ with $c$ northwest corners, then pulling back $U_j''$ to $U_j$ shows that formula (4) is the cardinality of the set

$$\left(T^{c-1} \backslash \bigcup_{j=1}^{s} U_j^{c-1}\right) \cup (V^{c-1} \backslash V^{c-2}) = S_2^{c-1} \cup (V^{c-1} \backslash V^{c-2}). \tag{10}$$

On the other hand, if we let $N_2$ denote the subset of $S_2$ consisting of paths that start with a north step, and observe that prepending a north step to $\pi \in S_2$ adds a northwest corner to $\pi$ iff $\pi$ starts with an east step, then we see that $(S_2^+)^c$ is equinumerous with

$$S_2^{c-1} \cup (N_2^c \backslash N_2^{c-1}). \tag{11}$$

Thus to show that (10) and (11) are equinumerous, it suffices to show that $N_2^c = V^{c-1}$ for any $c$. But this bijection is easily described: Given a path in $N_2$, simply move all the initial north steps to the end; this creates a path in $V$ with one fewer northwest corner. This completes the proof.

**PROOF.** (of Corollary 4)

Of course this follows from Theorem 3, but we have another proof. The formula in equation (5) can be rewritten as $\frac{1}{n}\binom{sn+tn}{sn+1}$. Consider the set $S_1'$ of all paths starting at the origin that end with a north step and that have a total of $sn+1$ east steps and a total of $tn$ north steps. Clearly $|S_1'| = \binom{sn+tn}{sn+1}$ and $S_1 \subset S_1'$. We need to show that $S_1$ comprises precisely $1/n$ of the paths in $S_1'$.

Decompose any path $\pi \in S_1'$ into $n$ consecutive subpaths $\pi_1, \pi_2, \ldots, \pi_n$, where each $\pi_j$ contains exactly $t$ north steps and ends in a north step. Our desired result follows immediately from the following key claim: For *any* $\pi \in S_1'$, there is exactly one "cyclic shift" of $\pi$ that lies in $S_1$, where by a cyclic shift we mean one of the $n$ paths of the form

$$\pi_j, \pi_{j+1}, \ldots, \pi_n, \pi_1, \pi_2, \ldots, \pi_{j-1}$$

obtained from $\pi$ by concatenating the subpaths in a cyclically permuted order.

To see the key claim, one first readily verifies that $\pi \in S_1$ iff for all $i > 0$, the total length of the first $i$ subpaths $\pi_1, \ldots, \pi_i$ is at least $(s+t)i + 1$. Now we apply an argument patterned after a classic proof of Raney [6]. For all

$j \geq 1$, let $\ell_j$ be the length of $\pi_{(j \bmod n)}$. Consider the graph in the $xy$ plane with straight-line segments between vertices $P_j$ and $P_{j+1}$, where

$$P_j = \left( j, \sum_{i=1}^{j} \ell_i \right).$$

The "average" slope of this graph is $(sn+tn+1)/n = s+t+\frac{1}{n}$. The line of the form $y = (s+t+\frac{1}{n})x+C$ that is "tangent" to this graph from below intersects the graph exactly once every $n$ points, because the graph has period $n$ and the coefficient of $x$ is an integer plus $1/n$. The points of intersection have the form $P_j$, $P_{j+n}$, $P_{j+2n}$, etc., and the value of $j$ here yields the unique cyclic shift having the desired property. This proves the claim.

The proof of equation (6) is similar. Define $S_2'$ to be the set of all paths from the origin that have a total of $sn-1$ east steps and a total of $tn-1$ north steps. Decompose any $\pi \in S_2'$ as follows:

$$\pi = \pi_1, \text{north}, \pi_2, \text{north}, \ldots, \text{north}, \pi_n$$

where each $\pi_j$ has $t-1$ north steps. Then $\pi \in S_2$ iff for all $0 \leq i < n$ we have

$$|\pi_1| + |\pi_2| + \cdots + |\pi_i| \geq i(s+t-1).$$

Exactly one "cyclic shift" of $\pi$ has the equivalent property that for all $i$,

$$|\pi_1| + |\pi_2| + \cdots + |\pi_i| \geq (i/n)(sn+tn-n-1).$$

Thus there are $\frac{1}{n}\binom{sn+tn-2}{tn-1}$ paths in $S_2$, which is equivalent to equation (6).

## 3 Proofs of Theorem 5 and Corollary 7

**PROOF.** (of Theorem 5)

It is easily verified that $a(1,s,0) = 2$ and $a(1,s,1) = a(1,s,2) = s+4$. If $n \geq 2$ and $r \leq 2n-2$, then we claim that the following recursion holds:

$$a(n,s,r) = \sum_{d=0}^{s+2} \binom{s+2}{d} a(n-1,s,r-d). \qquad (12)$$

The reason is that an admissible binary string of order $n-1$ can be extended by any sequence of $s+2$ bits without danger of causing inadmissibility, provided that the resulting string changes from 1 to 0 or vice versa at most $2n-2$ times. The parameter $d$ counts the number of changes introduced by the last

$s+2$ bits, and the binomial coefficient counts the number of ways to position the $d$ changes.

By Vandermonde convolution, the recurrence (12) almost gives us a proof by induction on $n$, except that we need to handle the cases $r = 2n-1$ and $r = 2n$. Note that no string of order $n$ can have more than $2n-1$ changes, and that equation (7) takes the same value for $r = 2n-1$ and $r = 2n$. So to complete the proof of Theorem 5, it is enough to show that

$$a(n, s, 2n-1) - a(n, s, 2n-2) = 2\binom{(s+2)n-1}{2n-1} - s\binom{(s+2)n-1}{2n-2},$$

which can be rewritten as $\frac{1}{n}\binom{(s+2)n}{2n-1}$. The left-hand side counts the admissible strings with exactly $2n-1$ changes, and we use the proof technique of Raney as before. Any such string $\sigma$ must start with 0; we decompose it into substrings $\sigma_1, \sigma_2, \ldots, \sigma_n$, where each $\sigma_j$ consists of $a_j$ zeroes followed by $b_j$ ones, and $a_j, b_j > 0$. The condition for admissibility can now be expressed as

$$|\sigma_1| + |\sigma_2| + \cdots + |\sigma_i| \geq i(s+2) + 1$$

for all $0 \leq i < n$. Exactly one cyclic shift of $\sigma$ has the equivalent property that for all $i$,

$$|\sigma_1| + |\sigma_2| + \cdots + |\sigma_i| \geq (i/n)((s+2)n + 1).$$

Thus the number of admissible strings with exactly $2n-1$ changes is equal to $1/n$ times the number of ways to partition $(s+2)n+1$ into $2n$ positive integers, corresponding to the numbers $a_j$, $b_j$. This is well known to be $\binom{(s+2)n}{2n-1}$, and this completes the proof.

**PROOF.** (of Corollary 7)

We can deduce this easily from Theorem 5 just by showing that equation (7) reduces to equation (8) when $r = 2n$ and $s = 2k$. We have

$$a(n, 2k, 2n) = 2\binom{2(k+1)n-1}{2n} - 2(k-1)\sum_{i=0}^{2n-1}\binom{2(k+1)n-1}{i};$$

breaking up the sum, the right-hand side becomes

$$\begin{aligned}2&\binom{2(k+1)n-1}{2n} - (k-1)\binom{2(k+1)n-1}{2n-1} \\ &- (k-1)\sum_{i=0}^{2n-1}\left[\binom{2(k+1)n-1}{i} + \binom{2(k+1)n-1}{i-1}\right],\end{aligned}$$

or

$$\left[2\frac{2kn}{2(k+1)n} - (k-1)\frac{2n}{2(k+1)n}\right]\binom{2(k+1)n}{2n} - (k-1)\sum_{i=0}^{2n-1}\binom{2(k+1)n}{i},$$

which then collapses to formula (8).

However, we also give a direct bijective proof. If $k = 0$ then formula (8) simplifies to $4^n$, the boundary conditions are nearly vacuous, and the result is easy to prove. So fix $k \geq 1$ and $n \geq 1$.

Our bijection is actually between two sets of lattice paths that are slightly different from those mentioned in the corollary.

Let $\mathcal{F}_k(n)$ be the set of lattice paths of length $2(k+1)n$ (note the shorter length) that start at $(0,0)$ and avoid the line $x = ky$ except at $(0,0)$.

Let $\mathcal{G}_k(n)$ be the set of lattice paths of length $2(k+1)n+1$ that start at $(0,0)$ and avoid $B_{2k}$, and that touch the line $x = ky+1$ at least once for $y > 0$.

To see that a bijection between $\mathcal{F}_k(n)$ and $\mathcal{G}_k(n)$ implies the corollary, we make two observations. First, because $2(k+1)n$ is a multiple of $k+1$, every lattice path of length $2(k+1)n$ that avoids $x = ky$ can be extended by either an east step or a north step without hitting the line $x = ky$; therefore $|\mathcal{F}_k(n)|$ is exactly half the number of lattice paths of length $2(k+1)n+1$ that avoid $x = ky$. Second, the paths excluded by the final condition on $\mathcal{G}_k(n)$ are precisely those that avoid the line $x = ky + 1$ after $(1,0)$, and therefore are in bijection with $\mathcal{F}_k(n)$—simply prepend an east step to each path in $\mathcal{F}_k(n)$.

The rest of the proof is devoted to describing a bijection $\varphi : \mathcal{F}_k(n) \to \mathcal{G}_k(n)$.

We define a procedure called *trisection* that we need in our construction of $\varphi$. Define the *potential* of a point $(x,y)$ to be $x - ky$. Let $P$ be a lattice path, not necessarily starting at $(0,0)$, but with the property that the potential difference of $P$—i.e., the potential of the last point of $P$ minus the potential of the first point of $P$—is at least $k+1$. To trisect $P$, first look at the last $k+1$ steps of $P$. If all of these steps are east steps, then the trisection procedure fails. Otherwise, let $b$ be the segment of $P$ consisting of the last north step of $P$ along with all the east steps after that. Let $l$ be the length of $b$. Find the *last* lattice point $p \in P$ such that the initial segment $a$ of $P$ comprising everything up to $p$ has potential difference exactly $k+1-l$. Such a point $p$ must exist and occur prior to $b$. Let $P'$ be the segment of $P$ between $a$ and $b$. The decomposition $P = (a, P', b)$ is the trisection of $P$. Note that the potential difference of $P'$ is the same as that of $P$.

We are now ready to describe $\varphi$. Given $P \in \mathcal{F}_k(n)$, the construction of $\varphi(P)$ has two phases. In Phase 1, we decompose $P$ into segments; in Phase 2, we build $\varphi(P)$ using the segments constructed in Phase 1.

Observe that the potential difference of any $P \in \mathcal{F}_k(n)$ is at least $k+1$. We begin Phase 1 by trying to trisect $P$ into $(a_1, P', b_1)$. If this fails, we proceed to Phase 2. Otherwise, if the *height* of $a_1$ (i.e., the $y$-coordinate of the last point of $a_1$ minus the $y$-coordinate of the first point of $a_1$) is even, then we proceed to Phase 2. Otherwise, we try to trisect $P'$ into $(a_2, P'', b_2)$, proceeding to Phase 2 if the trisection fails or if $a_2$ has even height. If we still do not reach Phase 2, then we try to trisect $P''$, and so on.

Eventually we must reach Phase 2, with a decomposition

$$P = (a_1, a_2, \ldots, a_{m-1}, a_m, Q, b_m, b_{m-1}, \ldots, b_2, b_1),$$

for some $m$, where $Q$ denotes whatever remains in the middle. If we reach Phase 2 because the height of $a_m$ is even, then we set

$$\varphi(P) = (\text{east}, a_m, b_m, a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_{m-1}, b_{m-1}, Q),$$

where the "east" means that we begin $\varphi(P)$ with an east step. For an example with $k = 2$ and $n = 7$, see Figure 2.

The other way to reach Phase 2 is for the last $k+1$ steps of $Q$ to all be east steps. Decompose $Q = (Q', b_{m+1})$ where $b_{m+1}$ comprises those final $k+1$ east steps. Then set

$$\varphi(P) = (\text{north}, b_{m+1}, a_1, b_1, a_2, b_2, \ldots, a_m, b_m, Q').$$

For an example, again with $k = 2$ and $n = 7$, see Figure 3.

It is readily checked that $\varphi(P) \in \mathcal{G}_k(n)$; the key is that each $(a_i, b_i)$ pair takes us from one point on the line $x = ky + 1$ to the next. The way $a_i$ and $b_i$ are constructed ensures that $\varphi(P)$ avoids $B_{2k}$.

To invert $\varphi$, suppose we are given $P \in \mathcal{G}_k(n)$. Whether Figure 2 or Figure 3 applies depends on whether the first step of $P$ is north or east. Mark all the points of $P$ that lie on the line $x = ky + 1$; there must be at least one, since $P \in \mathcal{G}_k(n)$. Each such point is a terminal point of a $b_i$. By backing up from such a point until we find a north step, we can reconstruct the $b_i$, and therefore also the $a_i$ and $Q$. Hence $\varphi$ is easily reversed. We leave the straightforward verification of the details to the reader.

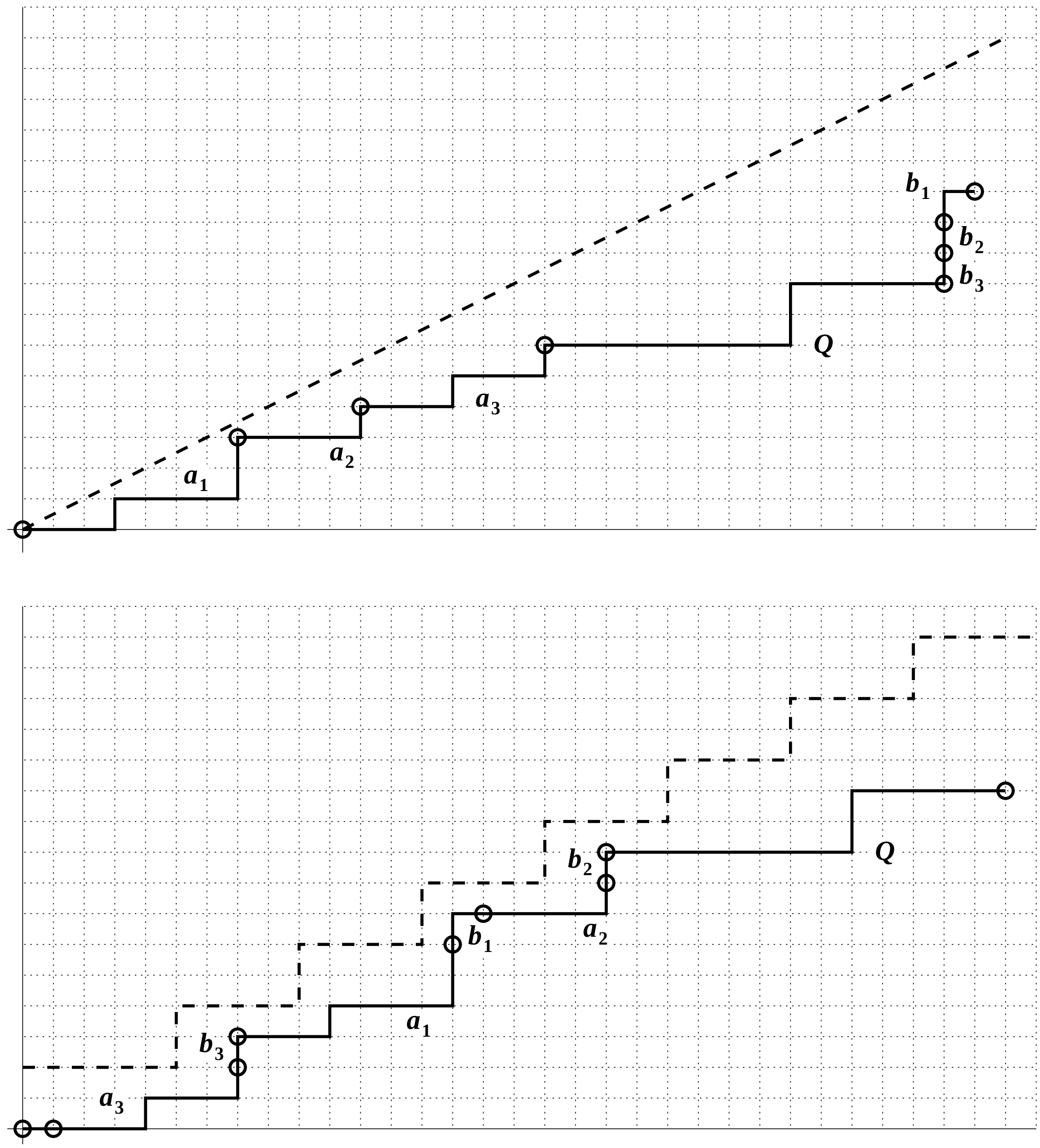


Fig. 2. Example of $P \mapsto \varphi(P)$ when $a_m$ has even height

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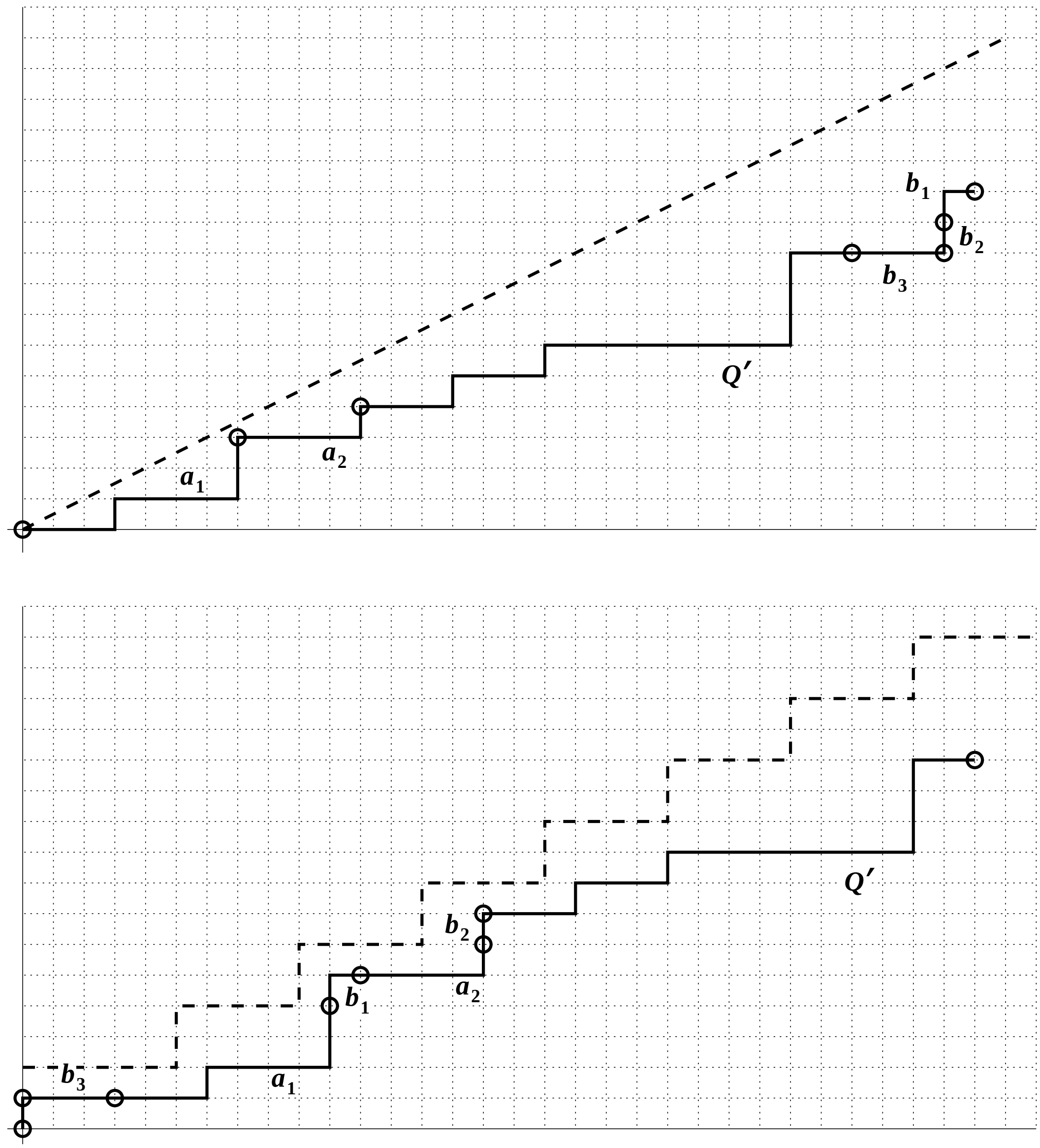

Fig. 3. Example of $P \mapsto \varphi(P)$ when $b_{m+1}$ comprises $k+1$ east steps

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