ABSTRACT FACTORIAL FUNCTIONS AND THEIR APPLICATIONS

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Abstract. We define the notion of an abstract factorial function on the set of natural numbers and show that, given any subset of $\mathbb{Z}$, we can associate to it another set with which, if non-trivial, we can define one or more (generally independent) abstract factorial functions. These associated sets are studied and arithmetic relations are revealed. In particular, we show that one of the associated sets of the set of primes is a subset of a class of numbers that also contains the highly composite numbers of Ramanujan. Furthermore, we show that for any given abstract factorial function the series of reciprocals of its factorials always converges to an irrational number.

1. Introduction

We consider the problem of defining an abstract factorial function $!_a : \mathbb{N} \to \mathbb{Q}$ and show how these can be constructed using arbitrary sets X of integers in $\mathbb{Q}$. Among the various questions considered here we find partial answers to these: What do these abstract factorials look like? How does one construct possibly infinite families of such generalized factorials simply? Finally, are there irrationality-type results for numbers defined by series of the reciprocals of such abstract factorials?

The definition of an abstract factorial requires an implicit definition of an abstract/generalized binomial coefficient as well. Our definition appears to differ from all preceding definitions of this concept in that all that is required is the divisibility of $n!_a$ (as odd as it may be) by the quantities $k!_a(n-k)!_a$ for every $0 \leq k \leq n$. The form of the factorial function itself need not be of a "falling" chain type as has been usually assumed. In [6] Knuth and Wilf define a generalized binomial coefficient by first beginning with a positive integer sequence $\mathcal{C} = \{C_n\}$ and then defining the binomial coefficient $\binom{n+m}{m}_{\mathcal{C}} = C_{m+n}C_{m+n-1}\dots C_{m+1}/C_nC_{n-1}\dots C_1$ that is reminiscent of the form of the usual expression for the binomial coefficient *after* we have canceled out the $m!$ from both numerator and denominator. There is an advantage to leaving things as they are in this respect as we see below (i.e., no cancelations) as this allows for much greater generality especially when it concerns the notion of an associated set and irrationality questions. In addition, the C_j above ought to satisfy certain recurrence relations [6] in order for said binomial coefficients to be integers, a condition that we do not require *per se*.

Date: Oct. 31, 2007- Dedicated to the memory of my teacher, Hans A. Heilbronn.

Besides clever guesswork, the construction of abstract factorial functions is most easily accomplished by starting from a given integer sequence (or set of integers, X, always assumed non-empty and one in which multiplicities are counted along with its elements). Regardless of the form of X we create a new (generally not unique) set called an "associated set" of X with which one can produce many abstract factorial functions (generally infinitely many linearly independent ones if X is infinite). One can think of an associated set as exhibiting a *generalized binomial sequence* $\{b_n\}$ that is an integer sequence with the property that for every $n \in \mathbb{N}$, $b_k\, b_{n-k} | b_n$ for every $k = 0, 1, 2, \ldots, n$. Such associated sets arise naturally from a construction based on the original set X, a construction which leads to a class of numbers that can be characterized (Theorem 4.1).

For instance, in Example 4.2 we show that one of the associated sets of the sequence $\{1, q, q, q, \ldots\}$ where $q \in \mathbb{Z}^+$, $q \geq 2$, is given by the set $\{B_n\}$ where

$$B_n = q^{d(1)+d(2)+\cdots+d(n)}$$

where $d(n)$ is the usual divisor function. In the same spirit we show in Example 4.5 that for $q \in \mathbb{Z}^+$, $q \geq 2$, the set $\{q^n : n \in \mathbb{N}\}$ has an associated set $\{B_n\}$ where

$$B_n = q^{\sigma(1)+\sigma(2)+\cdots+\sigma(n)}$$

where now $\sigma(n)$ is the sum of the divisors of n, a result that can be extended to the case of integer sequences of the form $\{q^{n^k}\}$ for given $k \geq 1$. Standard arithmetic functions abound in this context as can be gathered by considering the arithmetic progression $a\mathbb{Z}^+$ of a fixed multiple of the positive integers. Here, one of the associated sets is given by numbers of the form

$$B_n = a^{d(1)+d(2)+\cdots+d(n)} \prod_{i=1}^{n} i^{\lfloor n/i \rfloor},$$

where $d(n)$ is the divisor function and the product on the right is the arithmetic function that defines the cumulative product of all the divisors of the integers $1, 2, \ldots, n$ (see Remark 9).

Such associated sets can have many fascinating arithmetic properties. Recall that in 1915 Ramanujan [8] introduced the *highly composite numbers* (hcn), that is numbers whose number of divisors exceeds the number of divisors of all previous numbers, or equivalently, numbers n such that $d(m) < d(n)$ for all $m < n$, where $d(n)$ is the divisor function. In a later paper [9] Ramanujan goes on to study the asymptotic distribution of the function $Q(x)$ that counts the totality of numbers less than or equal to x of the form

$$q \equiv p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n} \tag{1}$$

where p_i is the i^{th}-prime, and $a_1 \geq a_2 \geq \ldots \geq a_n$ are positive integers. His motivation was that this class q in fact contains all the highly composite numbers and that he hoped to obtain similar results for $H(x)$, the number of hcn less than or equal to x, but considered this a problem of "extreme difficulty" [[9],p.119]. We show that one of the *associated sets* of the set of ordinary prime numbers is a subset of the class q of numbers of the form (1) considered by Ramanujan in [9]. This special associated set is yet another one of the few naturally occuring sets of integers that is a subset of the class of numbers defined by q.

After defining the notion of an abstract factorial function in the first part (Definition 2.1), we study the ratios of such consecutive generalized factorials showing that strings of three or more consecutive equal factorials cannot occur (Lemma 2.1). We then present (Lemma 2.3) sharp upper bounds on the growth of the quantity $k!_a/(k+1)!_a \equiv 1/b_k$ by exhibiting a dichotomy: For any abstract factorial function, either equations (3) or (4) below hold with cases of equality exhibited by specific examples (Example 2.1 in the former case, and use of the Bhargava factorials for the set of primes [2] in the latter case). The recent work of Bhargava [1],[2],[3] dealing with the factorials of an arbitrary set should be consulted in this connection and we will review the basic definitions herewith, as these factorials are particular cases of the abstract factorial functions defined here (Proposition 2.2).

We present a global irrationality result for numbers defined by series of reciprocals of abstract factorials akin to the representation of e using ordinary factorials. Indeed, we show that given any abstract factorial function whatsoever, the sum of the reciprocals of its generalized factorials is always irrational (Theorem 3.2). As an illustration we obtain, for instance, that the series of reciprocals of the k^{th}-powers ($k \geq 1$) of the cumulative products of all the divisors of the integers from 1 to n, i.e.,

$$\sum_{n=1}^{\infty} 1/\prod_{i=1}^{n} i^{k\,\lfloor n/i \rfloor},$$

is irrational (Example 4.4 and Remark 9).

After proceeding to the calculation of the natural associated set of the set of primes, we note that the first six numbers of this set are actually highly composite numbers and, in fact, we prove that these are the only ones (Proposition 5.1). In addition, we show that there exists an infinite set of hcn, $\{h_n\}$, such that the series

$$\sum_{n=0}^{\infty} 1/h_1^{\lfloor n/1 \rfloor} h_2^{\lfloor n/2 \rfloor} h_3^{\lfloor n/3 \rfloor} \cdots h_n^{\lfloor n/n \rfloor}$$

has an irrational sum (Proposition 5.4). Furthermore, we show that any associated set of the set of primes contains only finitely many hcn (Proposition 5.2).

An application of the theory developed here allows us to derive that the series of reciprocals of all the members of this associated set is an irrational number; that is the sum of the series

$$\sum_{n=0}^{\infty} 1/p_1^{\lfloor n/1 \rfloor} p_2^{\lfloor n/2 \rfloor} p_3^{\lfloor n/3 \rfloor} \cdots p_n^{\lfloor n/n \rfloor}$$

where p_i is the i^{th}-prime is irrational.

2. Preliminaries

In the sequel the symbols X, I will always stand for non-empty subsets of $\mathbb{Z}$; either may be finite or infinite, as the case may be, and their elements are not necessarily distinct (e.g., thus the set $X = \{1, q, q, q, \ldots\}$ is considered an infinite set). In the event that these sets have repeating elements it helps to think of them as integer sequences, as arbitrary as desired.

Definition 2.1. *An abstract (or generalized) factorial function is a function* $!_{a}$: $\mathbb{N} \to \mathbb{Z}^+$ *that satisfies the following conditions:*

(1) $0!_{a} = 1$,
(2) *For every non-negative integers* n, k, $0 \le k \le n$ *the generalized binomial coefficients*
$$\binom{n}{k}_{a} := \frac{n!_{a}}{k!_{a}\,(n-k)!_{a}} \in \mathbb{Z}^+,$$
(3) *For every positive integer* n, $n!$ *divides* $n!_{a}$.

Remark 1. Since, by hypothesis (2) above, $\binom{n+1}{n}_{a} \in \mathbb{Z}^+$ for every $n \in \mathbb{N}$ the sequence of factorials $n!_{a}$ is non-decreasing.

Thus, the ordinary factorial function $n! := 1 \cdot 2 \cdot 3 \cdots \cdot n$ is an abstract factorial function as is the function defined by setting $n!_{a} := 2^{n(n+1)/2} n!$. The latter is found by considering the set $X = \{n\, 2^n : n \in \mathbb{Z}^+\}$, and multiplying its first n-terms together. Bhargava's factorial function [1], [2] defined on arbitrary sets X is also an abstract factorial function (see Proposition 2.2).

One of the curiosities of abstract factorial functions is the possible existence of equal consecutive factorials. Thus, in order to proceed we need to understand their role and their connection to the rest of the theory.

Definition 2.2. *Let* $!_{a}$ *be an abstract factorial function. By a pair of equal consecutive factorials we mean a pair of consecutive abstract factorials such that, for some* $k \ge 2$, $k!_{a} = (k+1)!_{a}$.

Remark 2. Of course, for a given abstract factorial function, Definition 2.1 does not generally preclude the existence of such equal consecutive factorials as we do not tacitly assume that the factorials form a strictly monotone sequence (cf., Example 2.1 below).

In order to begin our study of such equal abstract factorials we consider the properties of ratios of *nearby* factorials. We adopt the following notation for ease of exposition: For a given integer k and for a given abstract factorial function $!_{a}$, we write
$$b_k = \frac{(k+1)!_{a}}{k!_{a}}. \tag{2}$$
Since generalized binomial coefficients are integers by Definition 2.1, b_k is an integer for every $k = 0, 1, 2, \ldots$. The next result shows that strings of three or more equal consecutive abstract factorials cannot occur.

Lemma 2.1. *There is no abstract factorial function* $!_{a}$ *with three equal consecutive factorials.*

Now that we know that equal consecutive factorials must be isolated and occur in pairs if they exist at all, we explore the relation of the factorials just preceding the pair to the pair itself.

Lemma 2.2. *Let* $!_a$ *be given and let* $2!_a \neq 2$. *If* $b_k = 1$ *for some* $k \geq 2$, *then* $b_{k-1} \geq 3$.

The next result gives a limit to the asymptotics of sequences of ratios of abstract factorials defined by the reciprocals of the b_k. These ratios do not necessarily tend to zero as one may expect (as in the case of the ordinary factorial), but may have subsequences approaching non-zero limits!

Lemma 2.3. *For any given abstract factorial function* $!_a$, *either*

$$\limsup_{k\to\infty} \frac{1}{b_k} = 1, \tag{3}$$

or

$$\limsup_{k\to\infty} \frac{1}{b_k} \leq 1/2, \tag{4}$$

the upper bound in (4) *being sharp.*

Definition 2.3. An abstract factorial function $!_a$ whose factorials satisfy (3) will be called **exceptional.**

Note: Using the generalized binomial coefficients $\binom{n+1}{n}_a$ it is easy to see that a necessary condition for the existence of such exceptional factorial functions is that $1!_a = 1$. The question of their existence comes next.

Proposition 2.1. *There is an abstract factorial function satisfying* (3).

To see this define a function $!_a : \mathbb{N} \to \mathbb{Z}^+$ by initially setting $0!_a = 1$, $1!_a = 1$. The exceptional abstract factorial is now defined inductively.

$$n!_a = \begin{cases} (n+1)!_a = n!\,(n+1)!\prod_{j=1}^{n-1}(n-j)!_a{}^2, & \text{if } n = 2, 5, 8, 11, \ldots, \\ n!\prod_{j=1}^{n-1}(n-j)!_a{}^2, & \text{if } n = 4, 7, 10, 13, \ldots. \end{cases}$$

To see that this is an abstract factorial function we must show that the generalized binomial coefficients $\binom{n}{k}_a$ are positive integers for $0 \leq k \leq n$ as the other two conditions in Definition 2.1 are clear by construction. Putting aside the trivial cases where $k = 0, k = n$ we may assume that $1 \leq k \leq n-1$.

To see that $\binom{n}{k}_a \in \mathbb{Z}^+$ for $k = 1, 2, \ldots, n-1$ we note that, by construction, the expression for $n!_a$ necessarily contains two copies of each of the terms $k!_a$ and $(n-k)!_a$ for each such k whenever $2k \neq n$. It follows that the stated binomial coefficients are integers whenever $2k \neq n$. On the other hand, if $2k = n$ the two copies of $k!_a$ in the denominator are canceled by two of the respective four copies in the numerator (since now $(n-k)!_a = k!_a$). Observe that (3) holds by construction.

Remark 3. The construction in Proposition 2.1 may be generalized simply by varying the exponent outside the finite product from 2 to any arbitrary integer greater than two. There then results an infinite family of such exceptional factorials. The quantity defined by $\prod_{j=1}^{n-1}(n-j)!_a$, may be thought of as an abstract generalization of the *super factorial* (see [[11], A000178]).

Example 2.1. *The first few terms of the exceptional factorial defined in Proposition 2.1 are given by* $1!_a = 1$*,* $2!_a = 3!_a = 12$*,* $4!_a = 497664$*,* $5!_a = 6!_a = 443722221348087398400$*, etc.*

Since the preceding results are valid for abstract factorial functions they include, in particular, the recent factorial function considered by Bhargava [1], [2] and we summarize its construction for completeness. Let $X \subseteq \mathbf{Z}$ be a finite or infinite set of integers. Following Bhargava [1],[2],[3] we define the notion of a p-ordering of X (it is defined more generally for subsets of Dedekind rings) and use it to define the generalized factorials of the set X inductively. By definition $0!_X = 1$. For p a prime, we fix an element $a_0 \in X$ and, for $k \geq 1$, we select a_k such that the highest power of p dividing $\prod_{i=0}^{k-1}(a_k - a_i)$ is minimized. The resulting sequence of a_i is called a p-ordering of X. As one can gather from the definition, such p-orderings are not unique, as one can vary a_0. Associated with such a p-ordering of X we define an associated p-sequence $\{\nu_k(X,p)\}_{k=1}^{\infty}$ by

$$\nu_k(X,p) = w_p(\prod_{i=0}^{k-1}(a_k - a_i)),$$

where $w_p(a)$ is, by definition, the highest power of p dividing a (e.g., $w_3(162) = 81$). Bhargava [1], [2] shows that although the p-ordering is not unique the associated p-sequence is independent of the p-ordering being used. Since this quantity is an invariant, this led Bhargava to define generalized factorials of X by setting

$$k!_X = \prod_p \nu_k(X,p), \tag{5}$$

where the (necessarily finite) product extends over all primes p.

Proposition 2.2. *Bhargava's factorial function* (5) *is an abstract factorial function.*

It follows from these definitions that for $X = \mathbf{Z}$ the notion of the generalized factorial considered in [1] etc. is identical to the ordinary factorial and we write $n!_{\mathbb{Z}+} := n!$ as usual.

As we mentioned above, the question of the possible existence of equal consecutive (Bhargava) factorials is of interest. We show herewith that although this appears to be a remote possibility for the ring of rational integers, such examples do exist even in this scenario.

Example 2.2. There exist sets X with consecutive Bhargava factorials, $!_X$. Perhaps the easiest example of such an occurrence lies in the set of generalized factorials of the set of cubes of the integers, $X = \{n^3 : n \in \mathbb{N}\}$, where one can show directly that $3!_X = 4!_X (= 504)$. Actually, the first occurrence of this is for the finite subset $\{0, 1, 8, 27, 64, 125, 216, 343\}$.

Another such set of equal Bhargava factorials is given by the finite set of Fibonacci numbers $X = \{F_2, F_3, \ldots, F_{18}\}$, where one can show directly that $7!_a = 8!_a (= 443520)$. We point out that the calculation of Bhargava factorials for finite sets is greatly simplified through the use of Crabbe's algorithm [4].

In what follows we will use the phrases "abstract factorials" and "generalized factorials" interchangeably. Inspired by the factorial representation of the base of the natural logarithms, one of the basic objects of study here is the series defined by the sum of the reciprocals of the abstract factorials in question.

Definition 2.4. *Let $!_a$ be a given abstract factorial function. The constant e_a is defined by the series of reciprocals of its generalized factorials, i.e.,*

$$e_a \equiv \sum_{r=0}^{\infty} \frac{1}{r!_a}. \tag{6}$$

Note that the series appearing in (6) converges on account of Definition 2.1(3) and $1 < e_a \leq e$. Thus, the usual factorial function gives that $e_a = e$, Euler's number ($\approx 2.718\ldots$). The generalized factorial function defined by $n!_a := 2^{n(n+1)/2} n!$ has an $e_a \approx 1.56514$.

3. An irrationality result

We now state a few lemmas leading to an irrationality result for sums of reciprocals of such abstract factorials.

Lemma 3.1. *Let $!_a$ be an abstract factorial function whose factorials satisfy* (4)*. Then e_a is an irrational number.*

Remark 4. Although condition (3) in Definition 2.1 (i.e., $n!|n!_a$) of an abstract factorial function appears to be very stringent, one cannot do without something like it; that is Lemma 3.1 above is false for "factorials functions" without this or some other similar property. For example, for $q > 1$ an integer, define the "factorial" $n!_a = q^n$. It satisfies properties (1) and (2) of Definition 2.1 but not (3). In this case it is easy to see that even though our "factorial" satisfies equation (4), e_a so defined is rational.

Corollary 3.1. *Let X be the set of prime numbers and $!_a$ the Bhargava factorial function of this set given by* [[2],p.793]

$$n!_a = \prod_p p^{\sum_{m=0}^{\infty} \left[\frac{n-1}{p^m(p-1)}\right]},$$

where the (finite) product extends over all primes. Then $e_a \approx 2.562760934$, is irrational.

The previous result holds because the Bhargava factorials of the set of primes satisfy (4) with equality (as is not difficult to show). The next lemma covers the logical alternative exhibited by equation (3) in Lemma 2.3.

Lemma 3.2. *Let $!_a$ be an abstract factorial function whose generalized factorials satisfy* (3)*. Then e_a is irrational.*

We summarize the previous two lemmas in the following theorem.

Theorem 3.2. *For any abstract factorial function* $!_a$ *the number* e_a *is irrational.*

Remark 5. This, therefore, is one possible setting for an extension of Lambert's classic theorem on the irrationality of e, showing that its irrationality appears to be due more to the structure of the factorial function in question than the underlying theory about the base of the natural logarithms. As a direct application, we note that since the Bhargava factorials of an arithmetic progression $X = \{an+b : n \in \mathbb{N}\}$, $a > 0$ are given by $n!_a = a^n\, n!$ [[2], Example 17], we can immediately deduce the classic irrationality of the number $e^{1/a}$, for any integer $a > 0$. Nontrivial examples appear below.

Example 3.3. Define a factorial function by $n!_a := (2n+1)!/2^n$, $n = 0, 1, 2 \dots$. Then this is an abstract factorial function. An immediate application of Theorem 3.2 gives that

$$\sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!} = \frac{\sinh \sqrt{2}}{\sqrt{2}} \notin \mathbb{Q}.$$

More generally, for $b \in \mathbb{Z}^+$ we get

$$\sum_{n=0}^{\infty} \frac{b^n}{(bn+1)!} \notin \mathbb{Q}.$$

Using $b = 4$ for example we can derive the irrationality of $\sqrt{2}(\sinh \sqrt{2} + \sin \sqrt{2})$.

Example 3.4. Let F_n denote the classical Fibonacci numbers defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $F_0 = F_1 = 1$. The "Fibonacci factorials" [[11], id:A003266], denoted by $FF(n)$ are defined by

$$FF(n) = \prod_{k=1}^{n} F_k.$$

Define an abstract factorial function by setting $FF(0) := 1$ and

$$n!_a := n!FF(n), \quad n = 1, 2, \dots$$

In this case, the generalized binomial coefficients involve the *Fibonomial coefficients* $(= FF(n)/FF(k)FF(n-k))$ so that

$$\binom{n}{k}_a = \binom{n}{k}\binom{n}{k}_F$$

where the Fibonomial coefficients [[11], id:A010048], [6] on the right are integers for $k = 0, 1, , \dots, n$. Once again, an application of Theorem 3.2 yields

$$\sum_{n=0}^{\infty} \frac{1}{n!FF(n)} \approx 1.5905614 \notin \mathbb{Q}.$$

Actually, by redefining the abstract factorial appropriately we see that a stronger result holds namely,

$$\sum_{n=0}^{\infty} \frac{1}{n!(FF(n))^k} \notin \mathbb{Q}$$

for every $k \in \mathbb{Z}^+$.

Example 3.5. The (exceptional) abstract factorial of Example 2.1 gives the series of reciprocals of generalized factorials:

$$e_{_a} = 1 + 1 + \frac{1}{12} + \frac{1}{12} + \frac{1}{497664} + \frac{1}{443722221348087398400} + \frac{1}{443722221348087398400} + \cdots$$
$$\approx 2.166668676 \notin \mathbb{Q}.$$

4. Associated sets and their properties

Besides creating factorial functions using clever constructions, the easiest way to generate them is by considering integer sequences. As we referred to earlier it is shown in [1] that on every subset $X \subseteq \mathbb{Z}$ one can define an abstract factorial function. For example, if $k = 2$, $a \in \mathbb{Z}^+$, then the integer sequence $\{a\, n^k\}_{n=1}^{\infty}$ is a p-sequence for all primes p simultaneously. Its Bhargava factorials are then a simple matter [[2], Lemma 16 and Example 19], however, this is not the case if $k = 3$, and this for any $a > 0$.

Now, there are cases where the product of the first n-terms $b_1\, b_2 \cdots b_n$ of the given set X can be used to define an abstract factorial function as well, one which may (resp. may not) agree with the Bhargava factorial function of X (e.g., $X = \mathbb{Z}^+$, resp. $X = \{n\, 2^n \,:\, n \in \mathbb{Z}^+\}$).

However, if our set X fails to have any special property of the type just discussed we show that there is still another method to define an abstract factorial function using a set "associated" to X in such a way that if X is infinite (recall that we count multiplicities here), one can define infinitely many generally linearly independent abstract factorial functions suggested by this associated set. The construction of this special set is next.

4.1. **The construction of an associated set.** Given $I = \{b_1, b_2, \ldots\}$, $I \subset \mathbb{Z}$, with or without repetitions, we can affiliate to it another set $X_I = \{B_0, B_1, \ldots, B_n, \ldots\}$ of non-negative integers, termed an *associated set of* I. In this case I is called a *primitive* set. The elements of this set $X_I \equiv X$ are defined as follows: $B_0 = 1$ by definition, B_1 is (the absolute value of) an arbitrary but fixed element of I, say, $B_1 = |b_1|$ (so that the resulting associated set X generally depends on the choice of b_1). Next, B_2 is the smallest (positive) number of the form ${b_1}^{\alpha_1}\, {b_2}^{\alpha_2}$ (where the $\alpha_i > 0$) such that ${B_1}^2 | B_2$. Hence $B_2 = |{b_1}^2\, b_2|$. Next, B_3 is defined as the smallest (positive) number of the form ${b_1}^{\alpha_1}\, {b_2}^{\alpha_2} {b_3}^{\alpha_3}$ such that $B_1 B_2 | B_3$. Thus, $B_3 = |{b_1}^3\, b_2\, b_3|$. Now, B_4 is defined as that smallest (positive) number of the form $\prod_{k=1}^{4} {b_k}^{\alpha_k}$ such that $B_1 B_3 | B_4$ and ${B_2}^2 | B_4$. This calculation gives us $B_4 = |{b_1}^4\, {b_2}^2\, b_3 b_4|$. In general, we build up the elements B_i, $i = 2, 3, \ldots, n-1$, inductively as per the preceding construction and define the element B_n as that smallest (positive) number of the form $|\prod_{k=1}^{n} {b_k}^{\alpha_k}|$ such that $B_i B_j | B_n$ for every i, j, $0 \le i \le j \le n$, and $i + j = n$.

Of course, such an associated set may be finite (if X is finite) or infinite, and trivial (e.g., if $b_1 = 0$ or some other $b_k = 0$) or non-trivial, see below. So, for instance, if none of the b_i vanish this construction guarantees that if we define $n!_a = n!\, B_1 B_2 \cdots B_n$ then all the generalized binomial coefficients $\binom{n}{k}_a$ are integers for $k \leq n$, (see Definition 2.1(2)) so that the function just defined is indeed an abstract factorial function. Observe that $n!_a = n!\, B_n$, $n!_a = n!\, B_n B_{n+1}/B_1$, $n!_a = n!\, B_n B_{n+1} B_{n+2}/B_1 B_2$ etc. all define abstract factorial functions, and all are suggested by considering the structure of our associated set X_I. From now on, *we will always assume that any/all associated sets are non-trivial.*

The basic properties of any one of the associated sets of a set of integers, all of which follow from the construction, can be summarized as follows.

Remark 6. Let $I = \{b_i\} \subset \mathbb{Z}$ be any infinite subset of the integers. For any fixed $b_m \in I$, the associated set $X_{b_m} = \{B_1, B_2, \ldots, B_n, \ldots\}$ exists and for every $n > 1$ and for every i, j, $0 \leq i \leq j \leq n$ and $i + j \leq n$, we have $B_i B_j | B_n$. In addition, if the elements of I are all positive, then the B_i are monotone.

The above construction of an associated set leads to very specific sets of integers, sets whose elements we characterize next (as we show below, it is helpful to think of these B_n as *generalized factorials*)

Theorem 4.1. Given $I = \{b_i\} \subset \mathbb{Z}$, the terms

$$B_n = {b_1}^{\lfloor n \rfloor} {b_2}^{\lfloor n/2 \rfloor} {b_3}^{\lfloor n/3 \rfloor} \cdots {b_n}^{\lfloor n/n \rfloor} \tag{7}$$

characterize its associated set X_{b_1} (where we leave out the absolute values around the b's in (7) by convention. As usual $\lfloor x \rfloor$ is the greatest integer not exceeding x).

Before proceeding with some applications we require a few basic lemmas.

Lemma 4.1. *For $k \geq 0$ an integer, let $\sigma_k(n)$ denote the sum of the k-th powers of the divisors of n, (where, $\sigma_0(n) = d(n)$ and $d(n)$ is the number of divisors of n, including 1 and n). Then*

$$\sum_{i=1}^{n} \sigma_k(i) = \sum_{i=1}^{n} i^k \lfloor n/i \rfloor. \tag{8}$$

Note: The left-side of (8) is often called "the average order of the (arithmetical) function..." (although it should be divided by n also), see Hardy and Wright [[5], Section 18.2] where this notion is used in connection with the determination of the asymptotics of various arithmetical functions.

Lemma 4.2. *Let $\alpha(n)$ denote the cumulative product of all the divisors of the numbers $1, 2, \ldots, n$ (including the numbers 1 and n). Then*

$$\alpha(n) = \prod_{i=1}^{n} i^{\lfloor n/i \rfloor}. \tag{9}$$

Remark 7. It is also known that

$$\alpha(n) = \prod_{k=1}^{n} \lfloor \frac{n}{k} \rfloor !$$

(see [11], id.A092143, Formula).

We now move on to examples where we describe explicitly some of the associated sets of various basic integer sequences.

Example 4.2. We find the associated set X_I of the set I of essentially constant integers: $I = \{1, q, q, q, q, \ldots\}$ as per the construction where $q \geq 2$ is given. Choosing $B_1 = q$ in the construction gives the associated set

$$X_I = \{1, q, q^3, q^5, q^8, q^{10}, q^{14}, q^{16}, q^{20}, q^{23}, q^{27}, q^{29}, q^{35}, \ldots\} \tag{10}$$

a set whose $n-$th term $B_n = q^{a(n)}$, where $a(n) = \sum_{k=1}^{n} d(k)$ (by Theorem 4.1 and Lemma 4.1) and $d(k)$ is, as before, the number of divisors of k. Note that by the very nature of the set itself $n!$ does not, in general, divide B_n for every n. In other words, $n!_a = B_n$ does not necessarily define an abstract factorial function. On the other hand, the function defined by $n!_a = n!\, B_n$ does define an abstract factorial function. Here we see that equal consecutive factorials cannot occur by construction so in particular, by Lemma 3.1, the number e_a defined by the sum of the reciprocals of these generalized factorials is irrational.

Remark 8. Note that the Bhargava factorials of I are mostly zero here while those few that one can calculate by hand for the associated set X_I indicate that the factorials are not of the form of the B_n above. Thus it appears that, generally speaking, the factorials defined here are distinct from Bhargava's.

Definition 4.1. Let I be an infinite subset of $\mathbb{Z}$ with corresponding associated set $X_I = \{B_n\}$. If $n!|B_n$ for every n, we say that this associated set X_I is a **self-factorial set**.

The motivation for this terminology is that the function defined by setting $n!_a = B_n$ is an abstract factorial function. In other words, a self-factorial set may be thought of as an infinite integer sequence of consecutive generalized factorials (identical to the set itself). The next result is very useful when one wishes to iterate the construction of an associated set *ad infinitum* (i.e., when finding the associated set of an associated set, etc.).

Lemma 4.3. *If $I = \{b_n\}$ is a set with $n!|b_n$ for every n, then its associated set X_{b_1} is a self-factorial set. The same idea may be used to prove that*

Corollary 4.3. *The associated set X_{B_1} of the self-factorial set $X = \{B_n\}$ is a self-factorial set.*

Next, we show that set I of positive integers also has an associated set with interesting properties.

Example 4.4. We find the associated set of the set of positive integers $I = \mathbb{Z}^+$ as per the preceding construction. Choosing $B_1 = 1$ we get the following set,

$$X_{\mathbb{Z}^+} = \{1, 2, 6, 48, 240, 8640, 60480, 3870720, 104509440, 10450944000, \ldots\} \tag{11}$$

a set which coincides (by Lemma 4.2 and Theorem 4.1) with the set of cumulative products of all the divisors of the numbers $1, 2, \ldots, n$ (see Sloane [11], id.A092143). Note that by construction $n!|B_n$ for every n. Hence, we can define an abstract factorial function by setting $n!_a = B_n$ to find that for this factorial function the set of generalized factorials is given by the *set itself*, that is, this $X_{\mathbb{Z}^+}$ is self-factorial. In particular, equal consecutive factorials cannot occur by construction, and it follows from Lemma 3.1 that the number defined by the sum of the reciprocals of these B_n, i.e.,

$$e_a = 1 + \sum_{n=1}^{\infty} 1/\prod_{i=1}^{n} i^{\lfloor n/i \rfloor} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{48} + \frac{1}{240} + \ldots \approx 2.69179920$$

is irrational.

Remark 9. The previous example is a special case of a more general result which states that the associated set of the set $X = a\mathbb{Z}^+$, $a \in \mathbb{Z}^+$, is given by terms of the form

$$B_n = a^{\sum_{k=1}^{n} d(k)} \prod_{i=1}^{n} i^{\lfloor n/i \rfloor}.$$

This is readily ascertained using the representation theorem, Theorem 4.1, and Lemma 4.2.

Observe that infinitely many other integer sequences I have the property that $n!|B_n$ for all n and so such sequences can be used to define abstract factorial functions. For example, if we consider the set of all k-th powers of the integers, $I = \{n^k : n \in \mathbb{Z}^+\}$, $k \geq 2$, then another application of Lemma 4.2 shows that its associated set X_I with $B_1 = 1$ is given by terms of the form

$$B_n = \prod_{i=1}^{n} i^{k\lfloor n/i \rfloor}.$$

In these cases we can always define an abstract factorial function by writing $n!_a = B_n$.

Example 4.5. Let $q \in \mathbb{Z}^+$, $q \geq 2$ and consider the geometric progression $X = \{q^n : n \in \mathbb{N}\}$. Then the Bhargava factorials of this set are given simply by $n!_a = \prod_{k=1}^{n} (q^n - q^{k-1})$, [[2], Example 18].

Now the associated set X_q of this set X defined by setting $B_1 = q$ yields the set

$$X_q = \{1, q, q^4, q^8, q^{15}, q^{21}, q^{33}, q^{41}, q^{56}, q^{69}, q^{87}, q^{99}, \ldots\}. \tag{12}$$

whose $n-th$ term is $B_n = q^{a(n)}$ by Lemma 4.2, where $a(n) = \sigma(1) + \ldots + \sigma(n)$, is (n-times) the "average order of $\sigma(n)$", see also [[5], Section 18.3, p.239 and p. 266]. The average order of the arithmetic function $\sigma(n)$ is, in fact, the $a(n)$ defined here, its asymptotics appearing explicitly in [[5], Theorem 324]. Note that this sequence $a(n)$ appears in [[11], id.A024916] and that $n! \not| B_n$ generally, so this set is not self-factorial. However, one may still define infinitely many other abstract factorials on it as we have seen (e.g., $n!_a = n!\, B_n$; $n!_a = n!\, B_n B_{n+1}/B_1$; $n!_a = n!\, B_n B_{n+1} B_{n+2}/B_1 B_2$ etc.)

Example 4.6. Let $q \geq 2$ be an integer and consider the integer sequence $X = \{q^{n^2} : n \in \mathbb{N}\}$. The associated set X_q of this set X defined by setting $B_1 = q$ gives the set

$$X_q = \{1, q, q^6, q^{16}, q^{37}, q^{63}, q^{113}, q^{163}, q^{248}, q^{339}, q^{469}, q^{591}, \ldots\}. \tag{13}$$

where now the n$^{-\text{th}}$ term is $B_n = q^{a_2(n)}$ by Lemma 4.2, where $a_2(n) = \sum_{k=1}^{n} \sigma_2(k)$ and $\sigma_2(k)$ represents the sum of the *squares* of the divisors of k [[5], p.239].

The previous result generalizes nicely.

Example 4.7. Let $q \geq 2$, $k \geq 1$ be integers and consider the integer sequence $X = \{q^{n^k} : n \in \mathbb{N}\}$. In this case, the associated set X_q of this set X defined as usual by setting $B_1 = q$ gives the set whose n$^{-\text{th}}$ term is $B_n = q^{a_k(n)}$ by Lemma 4.2, where $a_k(n) = \sum_{i=1}^{n} \sigma_k(i)$ and $\sigma_k(i)$ is the sum of the k-th powers of the divisors of i [[5], p.239].

5. The associated set of the primes and highly composite numbers

As a final example we find an associated set for the set of primes and describe a few of its properties.

Example 5.1. Let $X = \{p_i : i \in \mathbb{Z}^+\}$ be the set of primes. Setting $B_1 = 2$ we obtain the characterization of this associated set

$$X_1 = \{2, 12, 120, 5040, 110880, 43243200, 1470268800, 1173274502400, \ldots\}$$

in the form, $X_1 = \{B_n\}$ where

$$B_n = 2^n\, 3^{\lfloor n/2 \rfloor}\, 5^{\lfloor n/3 \rfloor} \cdots {p_i}^{\lfloor n/i \rfloor} \cdots {p_n}^{\lfloor n/n \rfloor} = \prod_{i=1}^{n} {p_i}^{\lfloor n/i \rfloor}. \tag{14}$$

First we note that for each n the total number of prime factors of B_n is always equal to $d(1) + d(2) + \ldots d(n)$ where, as usual, $d(i)$ is the classical divisor function [[5],p.354]. Next, this particular associated set X_1 is actually contained within a class of numbers considered earlier by Ramanujan [8], namely the class of numbers of the form $\prod_{i=1}^{n} {p_i}^{a_i}$ where $a_1 \geq a_2 \geq \ldots \geq a_n$, a class which includes the *highly composite numbers* (abbr. hcn) he had already defined in 1915.

In addition, a delicate argument (again using the representation of the ordinary factorial function as a product over primes [[7], Theorem 27]) shows that for every positive integer n, $n! | B_n$. This now allows us to define an abstract factorial function by writing $n!_{_a} = B_n$. Since X_1 is a self-factorial set and there are no consecutive generalized factorials we conclude from Lemma 3.1 that

$$e_{_a} = 1 + \sum_{n=1}^{\infty} 1/2^n\, 3^{\lfloor n/2 \rfloor}\, 5^{\lfloor n/3 \rfloor} \cdots {p_i}^{\lfloor n/i \rfloor} \cdots {p_n}^{\lfloor n/n \rfloor} \approx 1.5918741,$$

is an irrational number.

We observe that the first 6 elements of our class X_1 are hcn but there is little hope of finding many more due to the following result.

Proposition 5.1. *The sequence defined by* (14) *contains only finitely many hcn.*

Remark 10. It is interesting to note that the first failure of the left side of (25) is when $n = 9$. Comparing all smaller hcn (i.e., those with $a_2 \leq 8$) with our sequence we see that there are no others (for a table of hcn see [[10],pp.151-152]); thus the

6 found at the beginning of the sequence are the only ones. The sequence B_n found here grows fairly rapidly: $B_n \geq 2^{n+1}p_1p_2\cdots p_n$ although this is by no means precise.

Actually more is true regarding Proposition 5.1. The next result shows that hcn are really elusive . . .

Proposition 5.2. *The integer sequences defined by taking any associated set of the set of primes, even associated sets of the associated sets of the set of primes etc. contain only finitely many hcn.*

Now we show that there are hcn that are divisible by arbitrarily large (ordinary) factorials.

Proposition 5.3. *Let $m \in \mathbb{Z}^+$. Then there exists a highly composite number N such that $m!|N$.*

Remark 11. It is difficult to expect Proposition 5.3 to be true for *all* hcn larger than N as can be seen by considering the hcn $N = 48$ where $4!|48$ but $4!$ does not divide the next hcn, namely, 60. However, the same argument shows that Proposition 5.3 is true for all those hcn larger than N for which the chosen prime p *does* appear in its prime factorization (which is not always the case: e.g., the largest prime in the prime decomposition of 27720 is 11 but the largest such prime for the next hcn, namely 45360, is 7).

Now we move on to the study of the associated set of a set of highly composite numbers. Using the methods described here it can be shown that the associated set of the set of all hcn defined by starting at $b_1 = 2$ contains only finitely many hcn in turn (we omit the proof). Still, the collection of all hcn contains those special ones derived from Proposition 5.3 numbers that we now put to use.

Terminology: We will denote by $H = \{h_n\}$ a collection of hcn with the property that $n!|h_n$ for each $n \in \mathbb{Z}^+$ (note that the existence of such a set is guaranteed by Proposition 5.3).

Proposition 5.4. *The associated set H_{h_1} of H is self-factorial. Furthermore, the series of reciprocals of various powers of these hcn, i.e.,*

$$\sum_{n=1}^{\infty} 1/h_1^{\lfloor n/1\rfloor} h_2^{\lfloor n/2\rfloor} h_3^{\lfloor n/3\rfloor} \cdots h_n^{\lfloor n/n\rfloor} \notin \mathbb{Q}.$$

To get irrationality results of the type presented here it merely suffices to have at our disposal an abstract factorial function, as then this factorial function will provide the definition of a self-factorial set. For example, the following theorem is obtained.

Theorem 5.2. *Let $q_n \in \mathbb{Z}^+$ be a given integer sequence satisfying $q_0 = 1$, $n!q_n$ is non-decreasing for all n, and for every $n \geq 1$, $q_i\, q_j | q_n$ for all i, j, $1 \leq i \leq j \leq n$ with $i + j = n$. Then the series*

$$\sum_{n=1}^{\infty} \frac{1}{n!q_n}$$

sums to an irrational number.

In this case, the abstract factorial function is given by $n!_a = n!q_n$ where $X = \{n!\, q_n\}$ is a self-factorial set.

6. Proofs

Proof. (Lemma 2.1) Assuming the contrary we let $!_a$ be such a factorial function and let $k \geq 2$ be an integer such that $b_k = b_{k+1} = 1$. Since the binomial coefficient

$$\binom{k+2}{k}_a = \frac{(k+2)!_a}{2!_a\, k!_a} = \frac{1}{2!_a} \in \mathbb{Z}^+,$$

by Definition 2.1(2), this implies that $2!_a|1$ for such k. On the other hand, $2!|2!_a$ by Definition 2.1(3), so we get a contradiction. □

Proof. (Lemma 2.2) Lemma 2.1 guarantees that $b_{k-1} \neq 1$. Hence $b_{k-1} \geq 2$. Assume, if possible, that $b_{k-1} = 2$. Since $(k+1)!_a = k!_a = 2(k-1)!_a$ and the generalized binomial coefficient

$$\binom{k+1}{k-1}_a = \frac{(k+1)!_a}{2!_a\,(k-1)!_a} = \frac{2}{2!_a}$$

is a positive integer, $2!_a$ must be equal to either 1 or 2. Hence, by hypothesis, it must be equal to 1. But then by Definition 2.1(3) 2! must divide $2!_a = 1$, so we get a contradiction. □

Proof. (Lemma 2.3) The sequence of generalized factorials $n!_a$ is non-decreasing by Remark 1, thus, in any case $\limsup_{k\to\infty} 1/b_k \leq 1$. Next, let $k_n \in \mathbb{Z}^+$, be a given infinite sequence. There are then two possibilities: Either there is a subsequence, denoted again by k_n, such that $k_n!_a = (k_n+1)!_a$ for infinitely many n, or every subsequence k_n has the property that $k_n!_a \neq (k_n+1)!_a$ except for finitely many n. In the first case we get (3). In the second case, since $k_n!_a$ divides $(k_n+1)!_a$ (by Definition 2.1) it follows that

$$(k_n+1)!_a \geq 2k_n!_a,$$

except for finitely many n and this now implies (4).

The final statement is supported by an example wherein X is the set of all (rational) primes, and the abstract factorial function is in the sense of Bhargava, [2]. In this case, the explicit formula derived in [[2],p.793] for these factorials can be used to show sharpness in the case where the indices in (4) are odd, since then $b_k = 2$ for all such k. □

Proof. (Proposition 2.2) Hypothesis 1 of Definition 2.1 is clear by definition of the Bhargava factorial. Hypothesis 2 of Definition 2.1 follows by [[2],Theorem 8] while hypothesis 3 of said definition follows by [[2],Lemma 13]. □

Proof. (Lemma 3.1) The quantity $0!_a = 1$ by definition, so we leave it out of the following discussion. Assume, on the contrary, that e_a is rational, that is, $E_a \equiv e_a - 1$ is rational. Then $E_a = a/b$, for some $a, b \in \mathbb{Z}^+$, $(a, b) = 1$. In addition,

$$E_a - \sum_{r=1}^{k} \frac{1}{r!_a} = \sum_{r=k+1}^{\infty} \frac{1}{r!_a}.$$

Let $k \geq b$, $k \in \mathbb{Z}^+$ and define the number α_k by setting

$$\alpha_k \quad \equiv \quad k!_a \left(E_a - \sum_{r=1}^{k} \frac{1}{r!_a} \right) = k!_a \left(\frac{a}{b} - \sum_{r=1}^{k} \frac{1}{r!_a} \right).$$

Since $k \geq b$ and $k!$ divides $k!_a$ (by Definition 2.1(3)) it follows that b divides $k!_a$ (since b divides $k!$ by our choice of k). Hence $k!_a a/b \in \mathbb{Z}^+$. Next, for $1 \leq r \leq k$ we have that $k!_a / r!_a \in \mathbb{Z}^+$ (by Definition 2.1(2)). Thus, $\alpha_k \in \mathbb{Z}^+$, for (any) $k \geq b$.

Note that,

$$\alpha_k \quad = \quad k!_a \sum_{r=k+1}^{\infty} \frac{1}{r!_a} = k!_a \left(\frac{1}{(k+1)!_a} + \frac{1}{(k+2)!_a} + \dots \right). \tag{15}$$

Let $L < 1/2$. For $\varepsilon > 0$ so small that $L + \varepsilon < 1/2$, we choose N sufficiently large so that for every $k \geq N$ we have $k!_a/(k+1)!_a < L + \varepsilon$. Then it is easily verified that

$$\frac{k!_a}{(k+i)!_a} < (L + \varepsilon)^i,$$

for every $i \geq 1$ and $k \geq N$. Since $L + \varepsilon < 1/2$ we see that

$$\alpha_k \leq (L + \varepsilon) \sum_{i=0}^{\infty} (L + \varepsilon)^i = \frac{L + \varepsilon}{1 - (L + \varepsilon)} < 1,$$

and this leads to a contradiction.

The case $L = 1/2$ proceeds as above except that now we note that equality in (4) implies that for every $\varepsilon > 0$, there exists an N such that for all $k \geq N$,

$$\frac{k!_a}{(k+1)!_a} \leq 1/2 + \varepsilon.$$

Hence, for all $k \geq N$,

$$\alpha_k \leq (1/2 + \varepsilon) \sum_{i=0}^{\infty} (1/2 + \varepsilon)^i = \frac{1/2 + \varepsilon}{1 - (1/2 + \varepsilon)}. \tag{16}$$

We now fix some $\varepsilon < 1/6$ and a corresponding N. Then the right-side of (16) is less than two. But for $k \geq N_o \equiv \max\{b, N\}$, α_k is a positive integer. It follows that $\alpha_k = 1$. Using this in (15) we get that for every $k \geq N_o$,

$$1 \quad = \quad k!_a \left(\frac{1}{(k+1)!_a} + \frac{1}{(k+2)!_a} + \dots \right). \tag{17}$$

Since the same argument gives that $\alpha_{X,k+1} = 1$, i.e.,

$$1 \quad = \quad (k+1)!_a \left(\frac{1}{(k+2)!_a} + \frac{1}{(k+3)!_a} + \dots \right), \tag{18}$$

comparing (17) and (18) we arrive at the relation $(k+1)!_a = 2k!_a$, for every $k \geq N_o$. Iterating this we find that, under the assumption of equality in (4) we have $(k+i)!_a = 2^i\, k!_a$, for each $i \geq 1$, and for all sufficiently large k. However, by Definition 2.1(3), $(k+i)!_a = n\, k!_a i!_a$ for some $n \in \mathbb{Z}^+$. Hence, $n_i\, i!_a = 2^i$, for every i, for some integer n_i depending on i. This, however, is impossible since, by Definition 2.1(4), $i!$ must divide $i!_a$. Thus, $i!$ must also divide 2^i for every i which is impossible. This contradiction proves the theorem.

□

Proof. (Lemma 3.2) Since $2!_a$ must be even by Definition 2.1(3) there are two cases: either $2!_a \neq 2$ or $2!_a = 2$.

Case 1: Let $2!_a \neq 2$. We proceed as in the preceding Lemma 3.1 up to (15). Thus the assumption that $e_a - 1$ is rational, $e_a - 1 = a/b$ implies that $\alpha_k \in \mathbb{Z}^+$ for any $k \geq b$ satisfying (15). We rewrite this more compactly below for ease of reference. Thus, using the notation in equation (2) above,

$$\begin{aligned} \alpha_k \quad &= \quad k!_a \sum_{r=k+1}^{\infty} \frac{1}{r!_a} = k!_a \left(\frac{1}{(k+1)!_a} + \frac{1}{(k+2)!_a} + \dots \right), \\ &= \quad 1/b_k + 1/b_k b_{k+1} + \sum_{n=3}^{\infty} 1/b_k b_{k+1} b_{k+2} \cdots b_{k+n-1}, \end{aligned} \tag{19}$$

Since the generalized factorials must have integral valued binomial coefficients by Definition 2.1(2), we see that the product $b_1 b_2 \cdots b_{n-1} = n!_a / 1!_a$ is a positive integer for every n. Hence, $\binom{n+k}{k}_a \in \mathbb{Z}^+$ is equivalent to saying that $n!_a | b_k b_{k+1} \cdots b_{k+n-1}$, for every $k \geq 0$ and $n \geq 1$. Since $n! | n!_a$ for all n by Definition 2.1(3), this means that

$$n! | b_k b_{k+1} \cdots b_{k+n-1}, \tag{20}$$

for every integer $k \geq 0$, $n \geq 1$.

By hypothesis there is an infinite sequence of equal consecutive factorials. Therefore, we can choose k sufficiently large so that $k \geq b$ and $b_{k+1} = 1$. Then (19) is satisfied for our k with the α_k there being a positive integer. With such a k at our disposal, we now use Lemma 2.2 which forces $b_k \geq 3$ (since $2!_a \neq 2$). Using this information along with (20) in (19) we get

$$\begin{aligned} \alpha_k \quad &\leq \quad 1/3 + 1/3 + \sum_{n=3}^{\infty} 1/b_k b_{k+1} b_{k+2} \cdots b_{k+n-1}, \\ &\leq \quad 2/3 + \sum_{n=3}^{\infty} 1/n! \\ &\leq \quad 2/3 + e - 2 - 1/2 \approx 0.8849... \end{aligned}$$

and this yields a contradiction.

Case 2: Let $2!_a = 2$. We proceed as in Case 1 up to (19) and then (20) without any changes. Once again, we choose $k \geq b$ and $b_{k+1} = 1$. Since $2 = 2!_a | b_k b_{k+1}$, we see that b_k must be a multiple of two. If $b_k = 2$, then (19) gives the estimate $\alpha_k \leq 1/2 + 1/2 + e - 2 - 1/2 \approx 1.218....$ However, since α_k is a positive integer, we must have $\alpha_k = 1$. Hence $b_k = 2$ is impossible on account of (19). Thus, $b_k \geq 4$. Now using this estimate once again in (19) we see that

$$\begin{aligned} 1 = \alpha_k &\leq 1/4 + 1/4 + \sum_{n=3}^{\infty} 1/b_k b_{k+1} b_{k+2} \cdots b_{k+n-1}, \qquad (21) \\ &\leq 1/2 + (e - 2 - 1/2) \approx 0.718... \qquad (22) \end{aligned}$$

and there arises another final contradiction. Hence e_a is irrational. □

Proof. (Theorem 4.1) Note that (7) holds for the first few n by inspection so we use an induction argument: Assume that

$$B_i = \prod_{k=1}^{i} b_k^{\lfloor i/k \rfloor}$$

holds for all $i \leq n-1$. Since we require $B_i B_j | B_n$ for every i, j, $0 \leq i \leq j \leq n$ and $i + j = n$, we note that $B_i B_{n-i} | B_n$ for $i = 0, 1, \ldots, \lfloor n/2 \rfloor$. On the other hand if this last relation holds for all such i then by the symmetry of the product involved we get $B_i B_j | B_n$ for every i, j, $0 \leq i \leq j \leq n$ and $i + j = n$. Now, writing $B_n = b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_n^{\alpha_n}$ where the $\alpha_i > 0$ by construction, we compare this with the expression for $B_i B_{n-i}$, that is

$$\begin{aligned} B_i B_{n-i} &= \prod_{j=1}^{i} b_j^{\lfloor i/j \rfloor} \prod_{j=1}^{n-i} b_j^{\lfloor (n-i)/j \rfloor}, \\ &= \prod_{j \geq 1} b_j^{\lfloor i/j \rfloor + \lfloor (n-i)/j \rfloor}, \end{aligned}$$

the product extending up to the maximum of the indices i, $n-i$. Comparison of the first and last terms of this product with the expression for B_n reveals that $\alpha_1 = n$ and $\alpha_n = 1$. For the other terms we observe that since for every i, $1 \leq i \leq \lfloor n/2 \rfloor$, $1 \leq j \leq n$,

$$\lfloor i/j \rfloor + \lfloor (n-i)/j \rfloor = \begin{cases} \lfloor n/j \rfloor, & \text{if } j | \; i \;, \\ \leq \lfloor n/j \rfloor, & \text{if } j \not| \; i, \end{cases}$$

it follows that $\lfloor n/j \rfloor$ is an attained upper bound for the left hand side, for all j, $1 \leq j \leq n$ and $1 \leq i \leq \lfloor n/2 \rfloor$. However, the divisibility criterion in the construction along with the minimal nature of the exponents concerned now forces $\alpha_j = \lfloor n/j \rfloor$ for all j under consideration, and this gives the form of B_n. □

Proof. (Lemma 4.1) The case $k = 0$ can be found in [[5], Theorem 320], while the case $k = 1$ is referred to in [[11], A024916]. Basically all we need to do is to keep track of the number of divisors of a given kind. For example, displaying a list of all the divisors from 1 to n before us, we see that the number 1 will appear $\lfloor n/1 \rfloor$ times, the number 2 will appear $\lfloor n/2 \rfloor$ times, and generally, the number j will appear $\lfloor n/j \rfloor$ times, for each j, $1 \leq j \leq n$. Thus the cumulative sum of all these divisors must be equal to $\sum_{i=1}^{n} i \lfloor n/i \rfloor$. However, this cumulative sum is also equal to $\sigma(1)+\sigma(2)+\ldots+\sigma(n)$ by definition, so the result follows for $k=1$. The general case is completely similar since the list now contains the k-th powers of each of the divisors but their number is otherwise the same. A similar argument thus leads to (8). □

Proof. (Lemma 4.2) Write down the list of all the divisors from 1 to n inclusively (as per Lemma 4.1). Of course, each integer i between 1 and n appears in this list a number of times. Actually, for such a given i there are $\lfloor n/i \rfloor$ multiples of the number i less than or equal to n. Hence $i^{\lfloor n/i \rfloor}$ divides our cumulative product by definition of the latter. Taking the product over all integers i shows that $\prod_{i=1}^{n} i^{\lfloor n/i \rfloor} | \alpha(n)$. But all the divisors of $\alpha(n)$ must also be in the list and so each must be a divisor of $\prod_{i=1}^{n} i^{\lfloor n/i \rfloor}$. The result follows. □

Proof. (Lemma 4.3) For let $X_{b_1} = \{B_n\}$ be one of its associated sets. By Theorem 4.1 its terms are necessarily of the form

$$B_n = {b_1}^{\lfloor n \rfloor} {b_2}^{\lfloor n/2 \rfloor} {b_3}^{\lfloor n/3 \rfloor} \cdots {b_n}^{\lfloor n/n \rfloor}.$$

Since $n!|b_n$ by hypothesis it follows that $n!|B_n$ as well, for all n, and so this set is self-factorial. If b_1 is replaced by any other element of I, then it is easy to see that $n!|B_n$ once again as all the exponents in the decomposition of B_n are at least one. □

Proof. (Proposition 5.1) This uses a deep result by Ramanujan [8] on the structure of hcn. Once it is shown that every hcn is of the form

$$q \equiv 2^{a_2} 3^{a_3} 5^{a_5} \cdots p^{a_p} \tag{23}$$

where $a_2 \geq a_3 \geq a_5 \geq \cdots \geq a_p \geq 1$ [[8], III.6-8], he goes on to show that

$$\lfloor \frac{\log p}{\log \lambda} \rfloor \leq a_\lambda \leq 2 \lfloor \frac{\log P}{\log \lambda} \rfloor, \tag{24}$$

for every prime index λ, [[8], III.6-10, eq.(54)], where P is the first prime after p. Now set $\lambda = 2$ in (24) and use the fact that for the n$^{\text{th}}$-term, B_n, the index of the prime 2 is n, i.e., $a_2 = n$. Since $p = p_n$ by the structure theorem for B_n, we have $P = p_{n+1}$. Since $p_n = \mathrm{O}(n \log n)$ for $n > 1$, [[7], Theorem 113], the right side of (24) now shows that

$$n \leq 2 \lfloor \frac{\log p_{n+1}}{\log 2} \rfloor = \mathrm{O}(\log(n)) + \mathrm{O}(\log\log(n)), \tag{25}$$

which is impossible for infinitely many n. The result follows. □

Proof. (Proposition 5.2) Let $X = \{p_n\}$ be the set of primes. Recall that an associated set is defined uniquely once we fix a value for b_1, some element of X. The choice $b_1 = 2, \ldots, b_n = p_n$ leads to the associated already discussed in Proposition 5.1. On the other hand, if $b_1 \neq 2$ then B_n can never be highly composite for any n by the structure theorem for hcn. We now consider the associated set X_2 of X_1 (itself the (main) associated set of X defined by setting $b_1 = p_1 = 2$ and whose elements are given by (14)). The elements of X_2 are necessarily of the form

$$\begin{aligned} B_{n,2} &= {B_1}^n {B_2}^{\lfloor n/2 \rfloor} {B_3}^{\lfloor n/3 \rfloor} \cdots {B_n}^{\lfloor n/n \rfloor}, \\ &= {p_1}^n ({p_1}^2 p_2)^{\lfloor n/2 \rfloor} ({p_1}^3 p_2 p_3)^{\lfloor n/3 \rfloor} \cdots ({p_1}^n {p_2}^{\lfloor n/2 \rfloor} {p_3}^{\lfloor n/3 \rfloor} \cdots p_n^{\lfloor n/n \rfloor})^{\lfloor n/n \rfloor}, \\ &= {p_1}^{\sum_{i=1}^n \lfloor i/1 \rfloor \, \lfloor n/i \rfloor} {p_2}^{\sum_{i=1}^n \lfloor i/2 \rfloor \, \lfloor n/i \rfloor} \cdots {p_n}^{\sum_{i=1}^n \lfloor i/n \rfloor \, \lfloor n/i \rfloor}, \\ &= {p_1}^{\sum_{i=1}^n \sigma(i)} \cdots p_n, \end{aligned}$$

where $\sigma(i)$ is the sum of the divisors of i (see Lemma 4.1). The assumption that for some n, $B_{n,2}$ is a hcn leads to the estimate (see (24))

$$\lfloor \log p_n / \log 2 \rfloor \le \sum_{i=1}^{n} \sigma(i) \le 2 \lfloor \log p_{n+1} / \log 2 \rfloor. \tag{26}$$

However, by Theorem 324 in [5], $\sum_{i=1}^n \sigma(i) = n^2\, \pi^2/12 + \mathrm{O}(n \log n)$. On the other hand, the right side of (26) is $\mathrm{O}(\log n) + \mathrm{O}(\log\log n)$. It follows that the right hand inequality in (26) cannot hold for infinitely many n, hence there can only be finitely many hcn in X_2.

Observe that the more iterations we make on the associated sets $X_1, X_2, \ldots, X_k$, the higher the order of the index of the prime 2 in the decomposition of the respective terms $B_{n,k}$, and this estimate cannot be compensated by the right side of an equation of the form (26). □

Proof. (Proposition 5.3) Since all the primes must appear in the sequence of hcn (when written as an increasing sequence) there exists a hcn of the form

$$N = 2^{a_2} 3^{a_3} 5^{a_5} \cdots p^{a_p}$$

with $p > e^m$ ($e = 2.718...$). Using the representation of the factorials as a product over primes we observe that

$$m! | N \quad \iff \quad a_\lambda \ge \sum_{j \ge 1} \lfloor m/\lambda^j \rfloor,$$

for every λ, where $\lambda = 2, 3, 5, \ldots, p$. In order to prove this, we note that (24) shows that it is sufficient to demonstrate that

$$\lfloor \frac{\log p}{\log \lambda} \rfloor \ge \sum_{j \ge 1} \lfloor m/\lambda^j \rfloor,$$

or since $p > e^m$ by hypothesis, that it is sufficient to show that

$$\lfloor \frac{m}{\log \lambda} \rfloor \ge \sum_{j \ge 1} \lfloor m/\lambda^j \rfloor,$$

for every prime $\lambda = 2, 3 \ldots, p$. The latter, however is true on account of the estimates

$$\lfloor \frac{m}{\log \lambda} \rfloor \ge \frac{m}{\lambda - 1} = \sum_{j \ge 1} m/\lambda^j \ge \sum_{j \ge 1} \lfloor m/\lambda^j \rfloor,$$

valid for all primes $\lambda = 2, 3, \ldots, p$. This completes the proof. □

Proof. (Proposition 5.4) Fix an associated set $H_1 = \{h_n\}$. Then H_1 contains terms of the form $B_n = \prod_{j=1}^{n} {h_j}^{\lfloor n/j \rfloor}$ by construction where the h_i are hcn in H. Since $n!|h_n$ Lemma 4.3 implies that the associated set H_1 is self-factorial. The conclusion about the irrationality now follows by Theorem 3.2 since $n!_a = B_n$ defines an abstract factorial function by construction of the respective associated sets. □

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