The Compositions of the Differential Operations and Gateaux Directional Derivative

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Abstract

In this paper we determine the number of the meaningful compositions of higher order of the differential operations and Gateaux directional derivative.

1 The compositions of the differential operations of the space $\mathbb{R}^3$

In the real three-dimensional space $\mathbb{R}^3$ we consider the following sets:

$$\mathrm{A}_0 = \{f : \mathbb{R}^3 \longrightarrow \mathbb{R} \mid f \in C^{\infty}(\mathbb{R}^3)\} \quad \text{and} \quad \mathrm{A}_1 = \{\vec{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \mid \vec{f} \in \vec{C}^{\infty}(\mathbb{R}^3)\}. \tag{1}$$

Then, over the sets A_0 and A_1 in the vector analysis, there are $m = 3$ differential operations of the first-order:

$$\begin{aligned}
\operatorname{grad} f &= \nabla_1 f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) : \mathrm{A}_0 \longrightarrow \mathrm{A}_1 \ , \\
\operatorname{curl} \vec{f} &= \nabla_2 \vec{f} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) : \mathrm{A}_1 \longrightarrow \mathrm{A}_1 \ , \\
\operatorname{div} \vec{f} &= \nabla_3 \vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} : \mathrm{A}_1 \longrightarrow \mathrm{A}_0 \ .
\end{aligned} \tag{2}$$

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Let us present the number of the meaningful compositions of higher order over the set $\mathcal{A}_3 = \{\nabla_1, \nabla_2, \nabla_3\}$. As a well-known fact, there are $m = 5$ compositions of the second-order:

$$\begin{aligned}
&\Delta f = \operatorname{div} \operatorname{grad} f = \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{grad} \operatorname{div} \vec{f} = \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \vec{f} = 0.
\end{aligned} \tag{3}$$

Malešević [2] proved that there are $m = 8$ compositions of the third-order:

$$\begin{aligned}
&\operatorname{grad} \operatorname{div} \operatorname{grad} f = \nabla_1 \circ \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl} \operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{div} \operatorname{grad} \operatorname{div} \vec{f} = \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \operatorname{grad} f = \nabla_3 \circ \nabla_2 \circ \nabla_1 f = 0,\\
&\operatorname{div} \operatorname{curl} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \circ \nabla_2 \vec{f} = 0,\\
&\operatorname{grad} \operatorname{div} \operatorname{curl} \vec{f} = \nabla_1 \circ \nabla_3 \circ \nabla_2 \vec{f} = \vec{0},\\
&\operatorname{curl} \operatorname{grad} \operatorname{div} \vec{f} = \nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f} = \vec{0}.
\end{aligned} \tag{4}$$

If we denote by $\mathtt{f}(k)$ the number of compositions of the k^{th}-order, then Malešević [3] proved:

$$\mathtt{f}(k) = F_{k+3}, \tag{5}$$

where F_k is k^{th} Fibonacci number.

2 The compositions of the differential operations and Gateaux directional derivative on the space $\mathbb{R}^3$

Let $f \in \mathrm{A}_0$ be a scalar function and $\vec{e} = (e_1, e_2, e_3) \in \mathbb{R}^3$ be a unit vector. Thus, the *Gateaux directional derivative* in direction $\vec{e}$ is defined by [1, p. 71]:

$$\operatorname{dir}_{\vec{e}} f = \nabla_0 f = \nabla_1 f \cdot \vec{e} = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3 : \mathrm{A}_0 \longrightarrow \mathrm{A}_0. \tag{6}$$

Let us determine the number of the meaningful compositions of higher order over the set $\mathcal{B}_3 = \{\nabla_0, \nabla_1, \nabla_2, \nabla_3\}$. There exist $m = 8$ compositions of the second-order:

$$
\begin{aligned}
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_0 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_1 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big),\\
&\Delta f = \operatorname{div} \operatorname{grad} f = \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_0 \circ \nabla_3 \vec{f} = \big(\nabla_1 \circ \nabla_3 \vec{f}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{div} \vec{f} = \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \vec{f} = 0;
\end{aligned}
\tag{7}
$$

that is, there exist $m = 16$ compositions of the third-order:

$$
\begin{aligned}
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_0 \circ \nabla_0 \circ \nabla_0 f,\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_1 \circ \nabla_0 \circ \nabla_0 f,\\
&\operatorname{div} \operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_3 \circ \nabla_1 \circ \nabla_0 f,\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \operatorname{grad} f = \nabla_0 \circ \nabla_3 \circ \nabla_1 f,\\
&\operatorname{grad} \operatorname{div} \operatorname{grad} f = \nabla_1 \circ \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl} \operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_0 \circ \nabla_0 \circ \nabla_3 \vec{f},\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_1 \circ \nabla_0 \circ \nabla_3 \vec{f},\\
&\operatorname{div} \operatorname{grad} \operatorname{div} \vec{f} = \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_2 \circ \nabla_1 \circ \nabla_0 \vec{f} = \vec{0},\\
&\operatorname{curl} \operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \operatorname{grad} f = \nabla_3 \circ \nabla_2 \circ \nabla_1 f = 0,\\
&\operatorname{div} \operatorname{curl} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \circ \nabla_2 \vec{f} = 0,\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \operatorname{curl} \vec{f} = \nabla_0 \circ \nabla_3 \circ \nabla_2 \vec{f} = 0,\\
&\operatorname{grad} \operatorname{div} \operatorname{curl} \vec{f} = \nabla_1 \circ \nabla_3 \circ \nabla_2 \vec{f} = \vec{0},\\
&\operatorname{curl} \operatorname{grad} \operatorname{div} \vec{f} = \nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f} = \vec{0}.
\end{aligned}
\tag{8}
$$

Using the method from the paper [3] let us define a binary relation σ "*to be in composition*": $\nabla_i \,\sigma\, \nabla_j = \top$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful. Thus, Cayley table of the relation σ is determined with

$$\begin{array}{c|cccc} \sigma & \nabla_0 & \nabla_1 & \nabla_2 & \nabla_3 \\ \hline \nabla_0 & \top & \top & \bot & \bot \\ \nabla_1 & \bot & \bot & \top & \top \\ \nabla_2 & \bot & \bot & \top & \top \\ \nabla_3 & \top & \top & \bot & \bot \end{array} \tag{9}$$

Let us form the graph according to the following rule: if $\nabla_i \,\sigma\, \nabla_j = \top$ let vertex ∇_j be under vertex ∇_i and let there exist an edge from the vertex ∇_i to the vertex ∇_j. Further on, let us denote by ∇_{-1} nowhere-defined function ϑ, where domain and range are the empty sets [2]. We shall define $\nabla_{-1} \,\sigma\, \nabla_i = \top$ $(i = 0, 1, 2, 3, 4)$. For the set $\mathcal{B}_3 \cup \{\nabla_{-1}\}$ the graph of the walks, determined previously, is a tree with the root in the vertex ∇_{-1}.

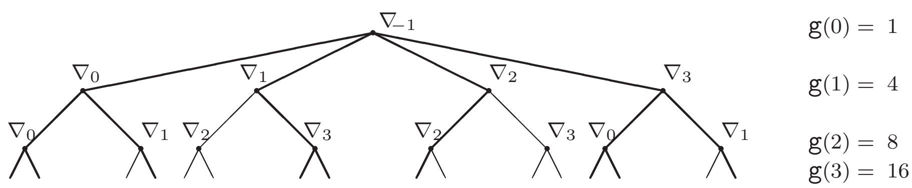

Fig. 1

Let $\mathsf{g}(k)$ be the number of the meaningful compositions of the k^{th}-order of the functions from $\mathcal{B}_3$. Let $\mathsf{g}_i(k)$ be the number of the meaningful compositions of the k^{th}-order beginning from the left by ∇_i. Then $\mathsf{g}(k) = \mathsf{g}_0(k) + \mathsf{g}_1(k) + \mathsf{g}_2(k) + \mathsf{g}_3(k)$. Based on the partial self similarity of the tree (Fig. 1) we get equalities

$$\begin{aligned} \mathsf{g}_0(k) &= \mathsf{g}_0(k-1) + \mathsf{g}_1(k-1), \\ \mathsf{g}_1(k) &= \mathsf{g}_2(k-1) + \mathsf{g}_3(k-1), \\ \mathsf{g}_2(k) &= \mathsf{g}_2(k-1) + \mathsf{g}_3(k-1), \\ \mathsf{g}_3(k) &= \mathsf{g}_0(k-1) + \mathsf{g}_1(k-1). \end{aligned} \tag{10}$$

Hence, a recurrence for $\mathsf{g}(k)$ can be derived as follows:

$$\mathsf{g}(k) = 2\,\mathsf{g}(k-1). \tag{11}$$

Based on the initial value $\mathsf{g}(1) = 4$, we can conclude:

$$\mathsf{g}(k) = 2^{k+1}. \tag{12}$$

3 The compositions of the differential operations of the space $\mathbb{R}^{\mathbf{n}}$

Let us present the number of the meaningful compositions of differential operations in the vector analysis of the space $\mathbb{R}^n$, where differential operations ∇_r $(r=1,\dots,n)$ are defined over non-empty corresponding sets A_s $(s=1,\dots,m$ and $m=\lfloor n/2 \rfloor$, $n \geq 3)$ according to the papers [3], [4]:

$$\mathcal{A}_n\ (n=2m)\!:\ \begin{array}{l} \nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1 \\ \nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2 \\ \vdots \\ \nabla_i : \mathrm{A}_{i-1} \to \mathrm{A}_i \\ \vdots \\ \nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m \\ \nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_{m-1} \\ \vdots \\ \nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j \\ \vdots \\ \nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1 \\ \nabla_n : \mathrm{A}_1 \to \mathrm{A}_0, \end{array} \qquad \mathcal{A}_n\ (n=2m+1)\!:\ \begin{array}{l} \nabla_1 : \mathrm{A}_0 \to \mathrm{A}_1 \\ \nabla_2 : \mathrm{A}_1 \to \mathrm{A}_2 \\ \vdots \\ \nabla_i : \mathrm{A}_{i-1} \to \mathrm{A}_i \\ \vdots \\ \nabla_m : \mathrm{A}_{m-1} \to \mathrm{A}_m \\ \nabla_{m+1} : \mathrm{A}_m \to \mathrm{A}_m \\ \nabla_{m+2} : \mathrm{A}_m \to \mathrm{A}_{m-1} \\ \vdots \\ \nabla_{n-j} : \mathrm{A}_{j+1} \to \mathrm{A}_j \\ \vdots \\ \nabla_{n-1} : \mathrm{A}_2 \to \mathrm{A}_1 \\ \nabla_n : \mathrm{A}_1 \to \mathrm{A}_0. \end{array} \tag{13}$$

Let us define *higher order differential operations* as the meaningful compositions of higher order of differential operations from the set $\mathcal{A}_n = \{\nabla_1, \dots, \nabla_n\}$. The number of the higher order differential operations is given according to the paper [3]. Let us define a binary relation ρ "*to be in composition*": $\nabla_i \,\rho\, \nabla_j = \top$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful. Thus, Cayley table of the relation ρ is determined with

$$\nabla_i \,\rho\, \nabla_j = \begin{cases} \top \ , & (j = i+1) \vee (i+j = n+1); \\ \bot \ , & \text{otherwise}. \end{cases} \tag{14}$$

Let us form the adjacency matrix $\mathtt{A} = [a_{ij}] \in \{0,1\}^{n \times n}$ associated with the graph, which is determined by the relation ρ. Thus, according to the paper [4], the following statement is true.

Theorem 3.1. *Let $P_n(\lambda) = |\mathtt{A} - \lambda \mathtt{I}| = \alpha_0 \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n$ be the characteristic polynomial of the matrix $\mathtt{A}$ and $v_n = [\,1\ \dots\ 1\,]_{1 \times n}$. If we denote by $\mathtt{f}(k)$ the number of the k^{th}-order differential operations, then the following formulas are true*:

$$\mathtt{f}(k) = v_n \cdot \mathtt{A}^{k-1} \cdot v_n^T \tag{15}$$

and

$$\alpha_0 \mathtt{f}(k) + \alpha_1 \mathtt{f}(k-1) + \cdots + \alpha_n \mathtt{f}(k-n) = 0 \quad (k > n). \tag{16}$$

Lemma 3.2. *Let $P_n(\lambda)$ be the characteristic polynomial of the matrix $\mathtt{A}$. Then the following recurrence is true*:

$$P_n(\lambda) = \lambda^2\big(P_{n-2}(\lambda) - P_{n-4}(\lambda)\big). \tag{17}$$

Lemma 3.3. *Let $P_n(\lambda)$ be the characteristic polynomial of the matrix $\mathtt{A}$. Then it has the following explicit representation:*

$$P_n(\lambda) = \begin{cases} \displaystyle\sum_{k=1}^{\lfloor\frac{n+2}{4}\rfloor+1} (-1)^{k-1}\binom{\frac{n}{2}-k+2}{k-1}\lambda^{n-2k+2} & ,\ n=2m; \\ \displaystyle\sum_{k=1}^{\lfloor\frac{n+2}{4}\rfloor+2} (-1)^{k-1}\left(\binom{\frac{n+3}{2}-k}{k-1}+\binom{\frac{n+3}{2}-k}{k-2}\lambda\right)\lambda^{n-2k+2} & ,\ n=2m+1. \end{cases} \tag{18}$$

The number of the higher order differential operations is determined by corresponding recurrence, which for dimension $n = 3, 4, 5, \ldots, 10$, we refer according to [3]:

Dimension:	Recurrence for the number of the k^{th}-order differential operations:
$n = 3$	$\mathtt{f}(k) = \mathtt{f}(k-1) + \mathtt{f}(k-2)$
$n = 4$	$\mathtt{f}(k) = 2\mathtt{f}(k-2)$
$n = 5$	$\mathtt{f}(k) = \mathtt{f}(k-1) + 2\mathtt{f}(k-2) - \mathtt{f}(k-3)$
$n = 6$	$\mathtt{f}(k) = 3\mathtt{f}(k-2) - \mathtt{f}(k-4)$
$n = 7$	$\mathtt{f}(k) = \mathtt{f}(k-1) + 3\mathtt{f}(k-2) - 2\mathtt{f}(k-3) - \mathtt{f}(k-4)$
$n = 8$	$\mathtt{f}(k) = 4\mathtt{f}(k-2) - 3\mathtt{f}(k-4)$
$n = 9$	$\mathtt{f}(k) = \mathtt{f}(k-1) + 4\mathtt{f}(k-2) - 3\mathtt{f}(k-3) - 3\mathtt{f}(k-4) + \mathtt{f}(k-5)$
$n = 10$	$\mathtt{f}(k) = 5\mathtt{f}(k-2) - 6\mathtt{f}(k-4) + \mathtt{f}(k-6)$

For considered dimensions $n = 3, 4, 5, \ldots, 10$, the values of the function $\mathtt{f}(k)$, for small values of the argument k, are given in the database of integer sequences [6] as sequences A020701 ($n = 3$), A090989 ($n = 4$), A090990 ($n = 5$), A090991 ($n = 6$), A090992 ($n = 7$), A090993 ($n = 8$), A090994 ($n = 9$), A090995 ($n = 10$), respectively.

4 The compositions of the differential operations and Gateaux directional derivative of the space $\mathbb{R}^{\mathbf{n}}$

Let $f \in A_0$ be a scalar function and $\vec{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$ be a unit vector. Thus, the *Gateaux directional derivative* in direction $\vec{e}$ is defined by [1, p. 71]:

$$\operatorname{dir}_{\vec{e}} f = \nabla_0 f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\, e_k : A_0 \longrightarrow A_0. \tag{19}$$

Let us extend the set of differential operations $\mathcal{A}_n = \{\nabla_1, \ldots, \nabla_n\}$ with Gateaux directional derivational to the set $\mathcal{B}_n = \mathcal{A}_n \cup \{\nabla_0\} = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$:

$$
\begin{array}{rl}
\mathcal{B}_n\ (n\!=\!2m)\!: & \nabla_0 : \mathrm{A}_0 \!\rightarrow\! \mathrm{A}_0 \\
& \nabla_1 : \mathrm{A}_0 \!\rightarrow\! \mathrm{A}_1 \\
& \nabla_2 : \mathrm{A}_1 \!\rightarrow\! \mathrm{A}_2 \\
& \vdots \\
& \nabla_i : \mathrm{A}_{i-1} \!\rightarrow\! \mathrm{A}_i \\
& \vdots \\
& \nabla_m : \mathrm{A}_{m-1} \!\rightarrow\! \mathrm{A}_m \\
& \nabla_{m+1} : \mathrm{A}_m \!\rightarrow\! \mathrm{A}_{m-1} \\
& \vdots \\
& \nabla_{n-j} : \mathrm{A}_{j+1} \!\rightarrow\! \mathrm{A}_j \\
& \vdots \\
& \nabla_{n-1} : \mathrm{A}_2 \!\rightarrow\! \mathrm{A}_1 \\
& \nabla_n : \mathrm{A}_1 \!\rightarrow\! \mathrm{A}_0,
\end{array}
\qquad
\begin{array}{rl}
\mathcal{B}_n\ (n\!=\!2m\!+\!1)\!: & \nabla_0 : \mathrm{A}_0 \!\rightarrow\! \mathrm{A}_0 \\
& \nabla_1 : \mathrm{A}_0 \!\rightarrow\! \mathrm{A}_1 \\
& \nabla_2 : \mathrm{A}_1 \!\rightarrow\! \mathrm{A}_2 \\
& \vdots \\
& \nabla_i : \mathrm{A}_{i-1} \!\rightarrow\! \mathrm{A}_i \\
& \vdots \\
& \nabla_m : \mathrm{A}_{m-1} \!\rightarrow\! \mathrm{A}_m \\
& \nabla_{m+1} : \mathrm{A}_m \!\rightarrow\! \mathrm{A}_m \\
& \nabla_{m+2} : \mathrm{A}_m \!\rightarrow\! \mathrm{A}_{m-1} \\
& \vdots \\
& \nabla_{n-j} : \mathrm{A}_{j+1} \!\rightarrow\! \mathrm{A}_j \\
& \vdots \\
& \nabla_{n-1} : \mathrm{A}_2 \!\rightarrow\! \mathrm{A}_1 \\
& \nabla_n : \mathrm{A}_1 \!\rightarrow\! \mathrm{A}_0.
\end{array}
\tag{20}
$$

Let us define *higher order differential operations with Gateaux derivative* as the meaningful compositions of higher order of the functions from the set $\mathcal{B}_n = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$. We determine the number of the higher order differential operations with Gateaux derivative by defining a binary relation σ "*to be in composition*":

$$
\nabla_i \,\sigma\, \nabla_j = \begin{cases} \top\,, & (i\!=\!0 \wedge j\!=\!0) \vee (i\!=\!n \wedge j\!=\!0) \vee (j\!=\!i\!+\!1) \vee (i\!+\!j\!=\!n\!+\!1); \\ \perp\,, & \text{otherwise.} \end{cases} \tag{21}
$$

Let us form the adjacency matrix $\mathtt{B} = [b_{ij}] \in \{\,0,1\}^{(n+1)\times n}$ associated with the graph, which is determined by relation σ. Thus, analogously to the paper [4], the following statement is true.

Theorem 4.1. *Let $Q_n(\lambda) = |\mathtt{B} - \lambda\mathtt{I}| = \beta_0\lambda^{n+1} + \beta_1\lambda^n + \cdots + \beta_{n+1}$ be the characteristic polynomial of the matrix $\mathtt{B}$ and $v_{n+1} = [\,1\ \ldots\ 1\,]_{1\times(n+1)}$. If we denote by $\mathsf{g}(k)$ the number of the k^{th}-order differential operations with Gateaux derivative, then the following formulas are true*:

$$
\mathsf{g}(k) = v_{n+1} \cdot \mathtt{B}^{k-1} \cdot v_{n+1}^T \tag{22}
$$

and

$$
\beta_0\mathsf{g}(k) + \beta_1\mathsf{g}(k-1) + \cdots + \beta_{n+1}\mathsf{g}(k-(n+1)) = 0 \quad (k > n\!+\!1). \tag{23}
$$

Lemma 4.2. *Let $Q_n(\lambda)$ and $P_n(\lambda)$ be the characteristic polynomials of the matrices $\mathtt{B}$ and $\mathtt{A}$ respectively. Then the following equality is true*:

$$
Q_n(\lambda) = \lambda^2 P_{n-2}(\lambda) - \lambda P_n(\lambda). \tag{24}
$$

Proof. Let us determine the characteristic polynomial $Q_n(\lambda) = |\mathtt{B} - \lambda\mathtt{I}|$ by

$$Q_n(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & 1 & \dots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & -\lambda & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & -\lambda \end{vmatrix}. \tag{25}$$

Expanding the determinant $Q_n(\lambda)$ by the first column we have

$$Q_n(\lambda) = (1-\lambda)P_n(\lambda) + (-1)^{n+2}D_n(\lambda), \tag{26}$$

where is

$$D_n(\lambda) = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \dots & -\lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -\lambda & 1 \end{vmatrix}. \tag{27}$$

Let us expand the determinant $D_n(\lambda)$ by the first row and then, in the next step, let us multiply the first row by -1 and add it to the last row. Then, we obtain the determinant of order $n-1$:

$$D_n(\lambda) = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & -\lambda & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -\lambda & 0 \end{vmatrix}. \tag{28}$$

Expanding the previous determinant by the last column we have

$$D_n(\lambda) = (-1)^n \begin{vmatrix} -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 & \dots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -\lambda \end{vmatrix}. \tag{29}$$

If we expand the previous determinant by the last row, and if we expand the obtained determinant by the first column, we have the determinant of order $n-4$:

$$D_n(\lambda) = (-1)^n \lambda^2 \begin{vmatrix} -\lambda & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 & \dots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -\lambda \end{vmatrix}. \tag{30}$$

In other words

$$D_n(\lambda) = (-1)^n \lambda^2 P_{n-4}(\lambda). \tag{31}$$

¿From equalities (31) and (26) there follows:

$$Q_n(\lambda) = (1-\lambda)P_n(\lambda) + \lambda^2 P_{n-4}(\lambda). \tag{32}$$

On the basis of Lemma 3.2. the following equality is true:

$$Q_n(\lambda) = \lambda^2 P_{n-2}(\lambda) - \lambda P_n(\lambda). \ \blacksquare \tag{33}$$

Lemma 4.3. *Let $Q_n(\lambda)$ be the characteristic polynomial of the matrix B. Then the following recurrence is true*:

$$Q_n(\lambda) = \lambda^2 \big(Q_{n-2}(\lambda) - Q_{n-4}(\lambda)\big). \tag{34}$$

Proof. On the basis of Lemma 3.2. and Lemma 4.2. there follows the statement. ■

Lemma 4.4. *Let $Q_n(\lambda)$ be the characteristic polynomial of the matrix B. Then it has the following explicit representation*:

$$Q_n(\lambda) = \begin{cases} (\lambda-2) \displaystyle\sum_{k=1}^{\lfloor \frac{n}{4} \rfloor + 1} (-1)^{k-1} \binom{\frac{n+1}{2} - k}{k-1} \lambda^{n-2k+2} & , \ n=2m+1; \\ \displaystyle\sum_{k=1}^{\lfloor \frac{n+3}{4} \rfloor + 2} (-1)^{k-1} \left(\binom{\frac{n}{2} - k+2}{k-1} + \binom{\frac{n}{2} - k+2}{k-2} \lambda \right) \lambda^{n-2k+3} & , \ n=2m. \end{cases} \tag{35}$$

Proof. On the basis of Lemma 3.3 and Lemma 4.2. there follows the statement. ■

The number of the higher order differential operations with Gateaux derivative is determined by corresponding recurrences, which for dimension $n=3,4,5,\ldots,10$, we can get by the means of [5]:

Dimension:	Recurrence for the num. of the k^{th}-order diff. operations with Gateaux derivative:
$n=\ 3$	$\mathrm{g}(k)=2\mathrm{g}(k-1)$
$n=\ 4$	$\mathrm{g}(k)=\mathrm{g}(k-1)+2\mathrm{g}(k-2)-\mathrm{g}(k-3)$
$n=\ 5$	$\mathrm{g}(k)=2\mathrm{g}(k-1)+\mathrm{g}(k-2)-2\mathrm{g}(k-3)$
$n=\ 6$	$\mathrm{g}(k)=\mathrm{g}(k-1)+3\mathrm{g}(k-2)-2\mathrm{g}(k-3)-\mathrm{g}(k-4)$
$n=\ 7$	$\mathrm{g}(k)=2\mathrm{g}(k-1)+2\mathrm{g}(k-2)-4\mathrm{g}(k-3)$
$n=\ 8$	$\mathrm{g}(k)=\mathrm{g}(k-1)+4\mathrm{g}(k-2)-3\mathrm{g}(k-3)-3\mathrm{g}(k-4)+\mathrm{g}(k-5)$
$n=\ 9$	$\mathrm{g}(k)=2\mathrm{g}(k-1)+3\mathrm{g}(k-2)-6\mathrm{g}(k-3)-\mathrm{g}(k-4)+2\mathrm{g}(k-5)$
$n=10$	$\mathrm{g}(k)=\mathrm{g}(k-1)+5\mathrm{g}(k-2)-4\mathrm{g}(k-3)-6\mathrm{g}(k-4)+3\mathrm{g}(k-5)+\mathrm{g}(k-6)$

For considered dimensions $n=3,4,5,\ldots,10$, the values of the function $\mathrm{g}(k)$, for small values of the argument k, are given in the database of integer sequences [6] as sequences A000079 ($n=3$), A090990 ($n=4$), A007283 ($n=5$), A090992 ($n=6$), A000079 ($n=7$), A090994 ($n=8$), A020714 ($n=9$), A129638 ($n=10$), respectively.

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