A CONJECTURE ON PRIMES AND A STEP TOWARDS JUSTIFICATION*

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Abstract. We put a new conjecture on primes from the point of view of its binary expansions and make a step towards justification.

1. Introduction and main results

Consider the partition of the set $\mathbb{N}$ into the following two disjoint subsets

$$\mathbb{N} = \mathbb{N}^e \cup \mathbb{N}^o, \tag{1}$$

where $\mathbb{N}^e(\mathbb{N}^o)$ is the set of positive integers which have even (odd) number of 1's in their binary expansions. These numbers are called the evil and the odious numbers respectively [9]. There are some results for these numbers and some applications of them in [1],[2],[3],[4], [5],[6].

Consider the same partition of the set $\mathbb{P}$ of prime numbers [10]:

$$\mathbb{P} = \mathbb{P}^e \cup \mathbb{P}^o. \tag{2}$$

For example, all the Fermat primes are evil while all the Mersenne primes > 3 are odious.

Using direct calculations up to 10^9 we noticed that among the primes not exceeding n the evil primes are never in majority except for the cases $n = 5$ *and* $n = 6$. Moreover, in the considered limits the excess of the odious primes is not monotone but increases on the whole with records on primes $2, 13, 41, 67, 79, 109, 131, 137, \ldots$

Let $\pi^e(x)(\pi^o(x))$ denote the number of the evil (odious) primes not exceeding x. Put

$$m_n = \min_{x \in (2^{n-1}, 2^n)} (\pi^o(x) - \pi^e(x)).$$

The following table shows that m_n increases monotonically.

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Table 1.

n	m_n	n	m_n
5	0	19	1353
6	2	20	1855
7	4	21	3659
8	7	22	5221
9	13	23	10484
10	19	24	14933
11	39	25	27491
12	54	26	35474
13	104	27	68816
14	139	28	97342
15	251	29	186405
16	334	30	265255
17	590		
18	716		

Therefore, the following conjecture seems plausible.

Conjecture 1. *For all* $n \in \mathbb{N},\ n \neq 5, 6$

$$\pi^e(n) \leq \pi^o(n); \tag{3}$$

moreover,

$$\lim_{n\to\infty} (\pi^o(n) - \pi^e(n)) = +\infty. \tag{4}$$

For a positive integer a, denote $\mu_a^e(n)(\mu_a^o(n))$ the number of *odd* evil (odious) nonnegative integers divisible by a and less than n.

Remark 1. *We include in this definition 0 (which is an evil integer) and use "less than" instead of "not exceeding" for the sake of more simplicity of the formulas which appear below.*

Put

$$\Delta_a^{odd}(n) = \mu_a^e(n) - \mu_a^o(n) \tag{5}$$

Theorem 1. *Let* $p, q, \ldots$ *denote odd primes. Then*

$$\pi^o(n) - \pi^e(n) = \varepsilon_n + \sum_{p \leq n} \Delta_p^{odd}(n) - \sum_{p<q\leq n} \Delta_{p,q}^{odd}(n) + \ldots, \tag{6}$$

where $|\varepsilon_n| \leq 4$.

In this article we make only the first step of investigation of $\pi^o(n) - \pi^e(n)$ with help of (6). Namely, by combinatorial methods we study in detail $\Delta_3^{odd}(n)$.

Let

$$\Delta_3^{odd}([a,b)) = \Delta_3^{odd}(b) - \Delta_3^{odd}(a) \tag{7}$$

Theorem 2. *1)* $\Delta_3^{(odd)}([0,2^n)) = 3^{\lfloor \frac{n}{2} \rfloor - 1},\ n \geq 2$

$$2)\Delta_3^{(odd)}([2^n, 2^n{+}2^m)) = \begin{cases} 0,\ n\ and\ m\ are\ even,\ \ 2 \leq m \leq n-2, \\ 3^{\frac{m-2}{2}},\ n\ is\ odd,\ m\ is\ even,\ \ 2 \leq m \leq n-1, \\ -3^{\frac{m-3}{2}},\ n\ is\ even,\ m\ is\ odd,\ \ 3 \leq m \leq n-1, \\ 2 \cdot 3^{\frac{m-3}{2}}, n\ and\ m\ are\ odd,\ 3 \leq m \leq n-2. \end{cases}$$

$$3)\Delta_3^{(odd)}([2^n{+}2^{n-2}, 2^n{+}2^{n-2}{+}2^m)) = \begin{cases} -3^{\frac{m-2}{2}},\ n\ and\ m\ are\ even,\ \ 2 \leq m \leq n-4, \\ 0,\ n\ is\ odd,\ m\ is\ even,\ \ 2 \leq m \leq n-3, \\ -2 \cdot 3^{\frac{m-3}{2}},\ n\ is\ even,\ m\ is\ odd,\ \ 3 \leq m \leq n-3, \\ 3^{\frac{m-3}{2}}, n\ and\ m\ are\ odd,\ 3 \leq m \leq n-4. \end{cases}$$

Consider together with $\Delta_3^{(odd)}([a,b))$ also $\Delta_3^{(even)}([a,b))$ which means the difference between the numbers of evil and odious *even* integers divisible by 3 on $[a,b)$. Put

$$\Delta_3([a,b)) = \Delta_3^{(odd)}([a,b)) + \Delta_3^{(even)}([a,b)) \tag{8}$$

Theorem 3. *1)*$\Delta_3([0,2^n)) = \begin{cases} 2 \cdot 3^{\frac{n}{2}-1},\ n\ is\ even \\ 3^{\frac{n-1}{2}},\ n\ is\ odd,\ n \geq 1 \end{cases}$

$$2)\Delta_3([2^n, 2^n + 2^m)) = \begin{cases} 3^{\lfloor \frac{m-1}{2} \rfloor},\ if\ n\ is\ odd, 1 \leq m \leq n-1, \\ 3^{\frac{m}{2}-1},\ if\ n\ and\ m\ are\ even,\ 2 \leq m \leq n-2, \\ 0,\ if\ n\ is\ even,\ m\ is\ odd, 1 \leq m \leq n-1, \end{cases}$$

$$3)\Delta_3([2^n{+}2^{n-2}, 2^n{+}2^{n-2}{+}2^m)) = \begin{cases} -3^{\lfloor \frac{m-1}{2} \rfloor},\ if\ n\ is\ even, 1 \leq m \leq n-3, \\ -3^{\frac{m}{2}-1},\ if\ n\ is\ odd,\ m\ is\ even,\ 2 \leq m \leq n-3, \\ 0,\ if\ n\ and\ m\ are\ odd, 1 \leq m \leq n-4 \end{cases}$$

At last, the following result is valid.

Theorem 4.

$$\lim_{n\to\infty} \frac{\ln \Delta_3^{(odd)}([0,n))}{\ln n} = \frac{\ln 3}{\ln 4}.$$

Using theorem 4 and simple heuristic arguments we put our conjecture in the following quantitative form.

Conjecture 2.

$$\lim_{n\to\infty} \frac{\ln(\pi^o(n) - \pi^e(n))}{\ln n} = \frac{\ln 3}{\ln 4}.$$

Conjecture 2 is illustrated by Table 2 in Section 3.

In the following Section we prove Theorems 1-4. Section 3 is devoted to some heuristic arguments which lead to Conjecture 2. Finally, in Section 4 we consider the increment of the excess of odiores primes on$(0, 2^n)(Table 3)$.

2. Proofs of results

A. Proof of Theorem 1.

Denote $\nu^e(n)(\nu^o(n))$ the number of evil (odious) nonnegative integers on interval $[0, n)$.

Lemma 1. *We have*

$$|\nu^o(n) - \nu^e(n)| \leq 1, \quad n \in \mathbb{N}. \tag{9}$$

Proof. The Lemma follows from the identity

$$\nu^e(2m) = \nu^o(2m), \quad m \in \mathbb{N}, \tag{10}$$

which is proved by induction.

Notice that (10) is satisfied for $m = 1$. Assuming that it is valid for $2m$ we prove (10) for $2(m+1)$. Indeed, let m has k 1's in the binary expansion. Then we have evidently

$$\nu^e(2m + 1) - \nu^o(2m + 1) = (-1)^k.$$

On the other hand, the last number in interval $[0, 2m+2)$, i.e. the number $2m + 1$ has $k + 1$ 1's and thus $\nu^e(2m + 2) - \nu^o(2m + 2) = 0$. ■

Let $\lambda^e(n)(\lambda^o(n))$ denote the number of *even* evil (odious) numbers less than n. At last, denote $\sigma^e(n)(\sigma^o(n))$ the number of evil (odious) *odd composite* numbers less than n.

For $n \geq 3$ we have

$$\pi^o(n) - \pi^e(n) + \sigma^o(n) - \sigma^e(n) + \lambda^o(n) - \lambda^e(n) - 1 = \nu^o(n) - \nu^e(n) + \delta_n, \tag{11}$$

where according to the definition of $\pi^o(n)(\pi^e(n))$, δ_n is 1, if n is an odious prime, -1, if n is an evil prime, 0-otherwise. Subtraction 1 in the left hand side of (11) is connected with the fact that only 2 is an odious prime and simultaneously is an odious even integer.

Using Lemma 1 and the evident identity

$$\lambda^o(n) - \lambda^e(n) = \nu^o(\frac{n}{2}) - \nu^e(\frac{n}{2}) \tag{12}$$

we find from (11)

$$\pi^o(n) - \pi^e(n) = \sigma^e(n) - \sigma^o(n) + \varepsilon_m, \tag{13}$$

where $|\varepsilon_m| \leq 4$.

At last, by inclusion-exclusion from (13) we obtain (6) ■

B.Proofs of Theorems 2-3. It is easy to see that for nonnegative integers $a < b$

$$\Delta_3^{(even)}([2a, 2b)) = \Delta_3([a, b)) \tag{14}$$

and consequently

$$\Delta_3^{(odd)}([2a, 2b)) = \Delta_3([2a, 2b)) - \Delta_3([a, b)). \tag{15}$$

Therefore, it is sufficient to prove Theorem 3 and by (14)-(15) we shall get also Theorem 2. For the proof of Theorem 3 we need a simple lemma.

Lemma 2. *Let for a nonnegative integer n, $i^{even}(n)(i^{odd}(n))$ denote the number of even (odd) powers of 2 in the binary representation of n. Then*

$$n \equiv 0(mod3) \Leftrightarrow i^{even}(n) \equiv i^{odd}(n)(mod3) \tag{16}$$

Proof. 1. Straightforward.

Proof of Theorem 3.

1a) let n be even, $n = 2m$. Consider all the nonnegative integers not exceeding $2^{2m} - 1$ which have $2m$ binary positions with numbering $0, 1, \dots, 2m-1$ beginning from the right. To find the difference between the numbers of evil and odious integers divisible by 3 not exceeding $2^{2m}-1$, let choose j even position for 1‘s (and $m - j$ even position for 0‘s) and according to Lemma 2 let choose $j + 3k$ $(k \geq 0)$ odd position for 1‘s (and the rest of the odd positions for 0‘s).

After that, vice versa, we choose j odd positions for 1‘s (and $n - j$ odd positions for 0‘s) and $j + 3k$ $(k \geq 1)$ even positions for 1‘s (and the rest of the even positions for 0‘s). Notice that, for each j the parity of the number of the chosen 1‘s is the same as the parity of k. Thus

$$\Delta_3([0, 2^{2m})) = \sum_{j\geq 0} \binom{m}{j}^2 + 2\sum_{k\geq 1}(-1)^k \sum_{j\geq 0} \binom{m}{j}\binom{m}{j+3k}. \tag{17}$$

Since (cf.[7],p.8)

$$\sum_{j\geq 0} \binom{m}{j}\binom{m}{j+3k} = \binom{2m}{m+3k}, \; k \geq 0, \tag{18}$$

then by (17)

$$\Delta_3([0, 2^{2m})) = \binom{2m}{m} + 2\sum_{k\geq 1}(-1)^k \binom{2m}{m+3k}. \tag{19}$$

To calculate $\sum_{k\geq 1}(-1)^k\binom{2m}{m+3k}$ in (19) we need some lemmas.

Lemma 3. *([7],p.86)*

$$\sum_{k=0}^{\lfloor \frac{m}{3} \rfloor} \binom{2m}{m+3k} = \frac{1}{3}(2^{2m-1}+1) + \frac{1}{2}\binom{2m}{m}. \tag{20}$$

Lemma 4.

$$\sum_{t=0}^{\lfloor \frac{m}{6} \rfloor} \binom{2m}{m+6t} = \frac{1}{2}\left(\frac{2^{2m-1}+1}{3} + 3^{m-1} + \binom{2m}{m}\right). \tag{21}$$

Proof. Denote the left hand side of (21) by $\sigma(m)$.

Let $m = 6l+s,\ \ 0 \le s \le 5$. Then

$$\sigma(m) = \sum_{t=0}^{l} \binom{12l+2s}{6l+s-6t} = \sum_{k=0}^{l} \binom{12l+2s}{6k+s} \tag{22}$$

Together with $\sigma(m)$ let consider the sum

$$\sigma_1(m) = \Sigma_{k=l+1}^{2l} \binom{12l+2s}{6k+s} = (2l-k=t) =$$

$$= \Sigma_{t=0}^{l-1} \binom{12l+2s}{12l-6t+s} = \Sigma_{t=0}^{l-1} \binom{12l+2s}{6t+s}, \tag{23}$$

From (22),(23) we conclude that

$$\sigma(m) = \sigma_1(m) + \binom{12l+2s}{6l+s}, \tag{24}$$

Consequently,

$$2\sigma(m) = \sigma(m) + \sigma_1(m) + \binom{12l+2s}{6l+s} = \sum_{k=0}^{2l} \binom{2m}{6k+s} + \binom{2m}{m}.$$

Thus,

$$\sum_{t=0}^{\lfloor \frac{m}{6} \rfloor} \binom{2m}{m+6t} = \frac{1}{2}\left(\sum_{k=0}^{\frac{m-s}{3}} \binom{2m}{6k+s} + \binom{2m}{m}\right), \tag{25}$$

where $0 \le s \le 5$.

Notice that, $\frac{m-s}{3}$ is the "natural" upper limit of the sum on the right hand side in (25).Indeed, in this sum $k \le \lfloor \frac{2m-s}{6} \rfloor = \lfloor \frac{12l+s}{6} \rfloor = 2l = \frac{m-s}{3}$. To calculate this sum we use the formula ([7], p.161)from which for $s = m-6t$ it follows that

$$\sum_{k=0}^{\frac{m-s}{3}} \binom{2m}{6k+s} = \frac{1}{6}\sum_{j=1}^{6} e^{\frac{\pi i}{3}(-jm)} \left(1+e^{\frac{\pi i}{3}j}\right)^{2m} =$$

$$= \frac{1}{6}\left(e^{-\frac{\pi m}{3}i}\left(1+e^{\frac{\pi i}{3}}\right)^{2m} + e^{-\frac{2\pi m}{3}i}\left(1+e^{\frac{2\pi i}{3}}\right)^{2m} + e^{-\frac{4\pi m}{3}i}\left(1+e^{\frac{4\pi i}{3}}\right)^{2m} + \right.$$

$$+e^{-\frac{5\pi m}{3}i}\left(1+e^{\frac{5\pi i}{3}}\right)^{2m}+2^{2m}\ )=\frac{1}{6}(\ e^{-\frac{\pi m}{3}i}\left(1+e^{\frac{\pi i}{3}}\right)^{2m}+e^{-\frac{2\pi m}{3}i}\left(1+e^{\frac{2\pi i}{3}}\right)^{2m}+$$

$$+e^{\frac{2\pi m}{3}i}\left(1+e^{-\frac{2\pi i}{3}}\right)^{2m}+e^{\frac{\pi m}{3}i}\left(1+e^{-\frac{\pi i}{3}}\right)^{2m}+2^{2m}\ )=\frac{1}{3}\left(2^{2m-1}+\right.$$

$$\left.+Re\left(e^{-\frac{\pi m}{3}i}\left(1+e^{\frac{\pi i}{3}}\right)^{2m}\right)+Re\left(e^{-\frac{2\pi m}{3}i}\left(1+e^{\frac{2\pi i}{3}}\right)^{2m}\right)\ \right).$$

Noticing that, $1+e^{\frac{\pi i}{3}}=\frac{3}{2}+\frac{\sqrt{3}}{2}i=\sqrt{3}e^{\frac{\pi i}{6}},\ 1+e^{\frac{2\pi i}{3}}=e^{\frac{\pi}{3}i}$ we have

$$\Sigma_{k=0}^{\frac{m-s}{3}}\binom{2m}{6k+s}=\frac{1}{3}\left(2^{2m-1}+3^m+1\right) \tag{26}$$

and by (25), (26) we obtain the lemma ■

Lemma 5.

$$\sum_{k\geq 0}(-1)^k\binom{2m}{m+3k}=3^{m-1}+\frac{1}{2}\binom{2m}{m}. \tag{27}$$

Proof. We have

$$\sum_{k\geq 0}(-1)^k\binom{2m}{m-3k}+\sum_{k\geq 0}\binom{2m}{m-3k}=2\sum_{j\geq 0}\binom{2m}{m-6j}$$

and by Lemmas 3, 4 we obtain the lemma ■

Now from (19) and Lemma 5 we find

$$\Delta_3\left([0,2^{2m})\right)=2\cdot 2^{m-1}.$$

1b) As opposed to the case 1a) here we have $2m-1$ positions from which m even and $m-1$ odd. Hence, by the same combinatorial arguments we find

$$\Delta_3\left([0,2^{2m-1})\right)=\sum_{j\geq 0}\binom{m}{j}\binom{m-1}{j}+$$

$$+\sum_{k\geq 1}(-1)^k\left(\sum_{j\geq 0}\binom{m}{j}\binom{m-1}{j+3k}+\sum_{j\geq 0}\binom{m-1}{j}\binom{m}{j+3k}\right) \tag{28}$$

Since (cf.[7],p.8)

$$\sum_{j\geq 0}\binom{m}{j}\binom{m-1}{j+3k}=\binom{2m-1}{m+3k}$$

$$\sum_{j\geq 0}\binom{m-1}{j}\binom{m}{j+3k}=\binom{2m-1}{m+3k-1}$$

then by (28) and Lemma 5 we have

$$\Delta_3\left([0,2^{2m-1})\right)=\binom{2m-1}{m}+\sum(-1)^k\binom{2m}{m+3k}=$$

$$= -\frac{1}{2}\binom{2m}{m} + \sum_{k\geq 0}(-1)^k\binom{2m}{m+3k} = 3^{m-1} \blacksquare$$

2a) Let m be even, $m = 2l$. Let, for definiteness, n be even. Choose $j-1$ of the last l even positions for 1's (and the rest $l-(j-1)$ positions for 0's) and according to Lemma 2, choose $j+3k$ ($k \geq 0$) of the last l odd positions for 1's (and $l-j-3k$ positions for 0's). After that, vice versa, we choose j of the last l odd positions for 1's and also $j-1+3k$ of the last l even positions for 1's and the rest of the positions for 0's. For each j the parity of the number of all 1's (including the 1 corresponding to 2^n) is the same as the parity of k. Thus,

$$\Delta_3\left([2^n, 2^n + 2^{2l})\right) = \sum_{j\geq 1}\binom{l}{j-1}\binom{l}{j} +$$

(29) $$+\sum_{k\geq 1}(-1)\left(\sum_{j\geq 1}\binom{l}{j-1}\binom{l}{j+3k} + \sum_{j\geq 0}\binom{l}{j}\binom{l}{j-1+3k}\right).$$

Since

$$\sum_{j\geq 1}\binom{l}{j-1}\binom{l}{j+3k} = \binom{2l}{l+3k+1},$$

$$\sum_{j\geq 0}\binom{l}{j}\binom{l}{j+3k-1} = \binom{2l}{l+3k-1},$$

then

$$\Delta_3\left([2^n, 2^n + 2^{2l})\right) = \binom{2l}{l+1} +$$

(30) $$+\sum_{k\geq 1}(-1)^k\left(\binom{2l}{l+3k-1} + \binom{2l}{l+3k+1}\right)$$

It is easy to verity that

(31) $$\binom{2l}{l+3k-1} + \binom{2l}{l+3k+1} = \binom{2l+2}{l+3k+1} - 2\binom{2l}{l+3k}.$$

Thus, using Lemma 5 for $m = l$ and $m = l+1$ we have

$$\Delta_3[2^n, 2^n + 2^{2l}) = \binom{2l}{l+1} + 3^l - \frac{1}{2}\binom{2(l+1)}{l+1} - 2\cdot 3^{l-1} + \binom{2l}{l} =$$

(32) $$= \binom{2l}{l+1} - \binom{2l+1}{l+1} + \binom{2l}{l} + 3^{l-1} = 3^{l-1}.$$

It is evident that in this case the validity of (29) does not depend on the parity of n.

2b) Let m be odd, $m = 2l+1, \quad l \geq 0$. As opposed to the case 2a here we have the last $2l+1$ positions from which $l+1$ are even and l are odd. Hence, by the same arguments we find

$$\Delta_3\left([2^n, 2^n+2^{2l-1})\right) = \sum_{j\geq 1}\binom{l}{j}\binom{l+1}{j-1} + \sum_{k\geq 1}(-1)^k\left(\sum_{j\geq 0}\binom{l}{j+3k+1}\binom{l+1}{j} + \right.$$

$$\left. + \sum_{j\geq 0}\binom{l+1}{j+3k-1}\binom{l}{j}\right), \quad \textit{if } n \textit{ is even}, \tag{33}$$

and

$$\Delta_3\left([2^n, 2^n+2^{2l-1})\right) = \sum_{j\geq 1}\binom{l}{j-1}\binom{l+1}{j} + \sum_{j\geq 1}(-1)^k\left(\sum_{j\geq 1}\binom{l+1}{j+3k}\binom{l}{j-1} + \right.$$

$$\left. + \sum_{j\geq 0}\binom{l}{j+3k-1}\binom{l+1}{j}\right), \quad \textit{if } n \textit{ is odd}. \tag{34}$$

Now by (33) for even n we have

$$\Delta_3\left([2^n, 2^n+2^{2l-1})\right) = \sum_{j\geq 0}\binom{l+1}{j}\binom{l}{j+1} +$$

$$+ \sum_{k\geq 1}(-1)^k\left(\binom{2l+1}{l+3k+2} + \binom{2l+1}{l+3k-1}\right) =$$

$$= \binom{2l+1}{l+2} + \sum_{k\geq 1}(-1)^k\binom{2l+1}{l+3k+2} + \sum_{k\geq 1}(-1)^k\binom{2l+1}{l+3k-1} =$$

$$= \binom{2l+1}{l+2} + \sum_{k\geq 1}(-1)\binom{2l+1}{l+3k+2} - \sum_{k\geq 0}(-1)^k\binom{2l+1}{l+3k+2} =$$

$$= \binom{2l+1}{l+2} - \binom{2l+1}{l+2} = 0.$$

and by (34) for odd n we have

$$\Delta_3\left([2^n, 2^n+2^{2l-1})\right) = \sum_{j\geq 0}\binom{l}{j}\binom{l+1}{j+1} + \sum_{k\geq 1}(-1)^k\left(\sum_{j\geq 0}\binom{l}{j}\binom{l+1}{j+3k+1} + \right.$$

$$\left. + \sum_{j\geq 0}\binom{l+1}{j}\binom{l}{j+3k-1}\right) = \binom{2l+1}{l+1} + \sum_{k\geq 1}(-1)^k\left(\binom{2l+1}{l+3k+1} + \right.$$

$$\left. + \binom{2l+1}{l+3k}\right) = \binom{2l+1}{l+1} + \sum_{k\geq 1}(-1)^k\binom{2l+2}{l+3k+1} =$$

$$= \binom{2l+1}{l+1} + \sum(-1)^k\binom{2(l+1)}{(l+1)-3k},$$

and by Lemma 5 for odd n we obtain

$$\Delta_3\left([2^n, 2^n+2^{2l-1})\right) = \binom{2l+1}{l+1} + 3^l - \frac{1}{2}\binom{2l+2}{l+1} = 3^l \quad \blacksquare$$

3) denote by $\Delta_{3,h}([a,b))$, $(h \in \mathbb{N})$, the difference between the numbers of evil and odious integers on $[a,b)$ having the form $3t+i$, $i=1,2$, where $i \equiv h(\mod 3)$

Lemma 6. *1)* $\Delta_{3,1}([0,2^n)) = \begin{cases} -3^{\frac{n}{2}-1}, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd \end{cases}$

2) $\Delta_{3,2}([0,2^n)) = -3^{\lfloor \frac{n-1}{2} \rfloor}$

Proof. Notice that,

$$(35) \qquad \Delta_3([2^n, 2^n+2^m)) = -\Delta_{3,2^n}([0,2^m))$$

Since by $\mod 3$

$$2^n \equiv \begin{cases} 1, & if\ n\ is\ even \\ 2, & if\ n\ is\ odd, \end{cases}$$

then by (35)

$$\Delta_{3,1}([0,2^m)) = \Delta_3([2^n, 2^n+2^m)) if\ n\ is\ even$$
$$\Delta_{3,2}([0,2^m)) = \Delta_3([2^n, 2^n+2^m)) if\ n\ is\ odd$$

and the lemma follows from the previous point ■.

Now we are able to complete the proof of Theorem 3.

a) Let n be even, $n=2t$.

We have

$$(36) \qquad \Delta_3([2^{2t}+2^{2t-2},\ 2^{2t}+2^{2t-2}+2m)) = \Delta_{3,2^{2t}+2^{2t-2}}([0,2^m)).$$

Since

$$2^{2t}+2^{2t-2} \equiv 5 \cdot 2^{2t-2} \equiv 2 \ (\mod 3),$$

then by (36) and by Lemma 6

$$\Delta_3([2^{2t}+2^{2t-2},\ 2^{2t}+2^{2t-2}+2^m)) = \Delta_{3,2}([0,2^m)) = -3^{\lfloor \frac{m-1}{2} \rfloor}.$$

b) Let now n be odd, $n=2t+1$. Since

$$2^{2t+1}+2^{2t-1} \equiv 5 \cdot 2^{2t-1} \equiv 1 \ (\mod 3)$$

then using Lemma 6 we have

$$\Delta_3([2^{2t+1}+2^{2t-1},\ 2^{2t+1}+2^{2t-1}+2^m)) = \Delta_{3,1}([0,2^m)) = \begin{cases} -3^{\frac{m}{2}-1}, & if\ m\ is\ even \\ 0, & if\ m\ is\ odd \end{cases}.$$

This completes the proof of both Theorem 3 and, in view of (15), Theorem 2 ■

Notice that,the results of Theorems 2,3 one can write in terms of the counting functions of the corresponding sequences. For example, let us consider the first points of these theorems. Let $\nu_3^e(n)(\nu_3^o(n))$ denote the number of the evil (odious) divisible by 3 nonnegative integers less than n. Then from the first point of Theorem 3 for $n \geq 1$ we have

$$\nu_3^e(2^n) = \begin{cases} \frac{1}{2}\left(\frac{2^n+1}{3} + 3^{\frac{n-1}{2}}\right), & if \;\; n \;\; is \;\; odd \\ \frac{2^{n-1}+1}{3} + 3^{\frac{n}{2}-1}, & if \;\; n \;\; is \;\; even; \end{cases}$$

$$\nu_3^o(2^n) = \begin{cases} \frac{1}{2}\left(\frac{2^n+1}{3} - 3^{\frac{n-1}{2}}\right), & if \;\; n \;\; is \;\; odd \\ \frac{2^{n-1}+1}{3} - 3^{\frac{n}{2}-1}, & if \;\; n \;\; is \;\; even. \end{cases}$$

Furthermore, let as above $\mu_3^e(n)(\mu_3^{(o)}(n))$ denote the number of the evil (odious) divisible by 3 nonnegative *odd* integers less than n. Then from the first point of Theorem 2 for $n \geq 2$ we have

$$\mu_3^e(2^n) = \frac{1}{2}\left(\left\lfloor \frac{2^{n-1}+1}{3} \right\rfloor + 3^{\lfloor \frac{n}{2} \rfloor - 1}\right),$$

$$\mu_3^o(2^n) = \frac{1}{2}\left(\left\lfloor \frac{2^{n-1}+1}{3} \right\rfloor - 3^{\lfloor \frac{n}{2} \rfloor - 1}\right).$$

Notice in addition that, Theorem 2 (Theorem 3) allows to calculate for any n the number $\Delta_3^{(odd)}([0,n))(\Delta_3([0,n)))$.

Indeed, let

$$(37) \quad \Delta_3^{(odd)} = \Delta_3^{(odd)}(2^{n_1} + 2^{n_2} + \ldots + 2^{n_k},\ 2^{n_1} + 2^{n_2} + \ldots + 2^{n_k} + 2^m).$$

Consider the sums

$$a = \sum_{i: n_i \equiv 0 (\mod 2)} 1, \;\; b = \sum_{i: n_i \equiv 1 (\mod 2)} 1.$$

Let

$$a \equiv \alpha (\mod 3), \; b \equiv \beta (\mod 3),$$

so that $0 \leq \alpha, \beta \leq 2$. Then for any integer $t > \frac{m}{2}$ we have

(38)

$$\Delta_3^{odd} = \begin{cases} (-1)^k \Delta_3^{odd}([0, 2^m)), \; if \;\; \alpha = \beta \\ (-1)^{k-1} \Delta_3^{odd}([2^{2t}, 2^{2t} + 2^m)), \; if \;\; \alpha - \beta = 1, \\ (-1)^{k-1} \Delta_3^{odd}([2^{2t+1}, 2^{2t+1} + 2^m)), \; if \;\; \alpha - \beta = -1, \\ (-1)^k \Delta_3^{odd}([2^{2t+2} + 2^{2t}, 2^{2t+2} + 2^{2t} + 2^m)), \; if \;\; \alpha - \beta = 2, \\ (-1)^k \Delta_3^{odd}([2^{2t+3} + 2^{2t+1}, 2^{2t+3} + 2^{2t+1} + 2^m)), \; if \;\; \alpha - \beta = -2. \end{cases}$$

(38) follows immediately from Lemma 2. The analogous equality is valid for Δ_3.

Example 1. $n=105$. *The interval [0, 105) contains 17 odd numbers divisible by 3. Among them there are 5 odious numbers (namely, 21, 69, 81, 87, 93) and 12 evil numbers. Thus,* $\Delta_3^{odd}([0,105))=7$.

Let find now this value by the algorithm. We have

$$[0,105)=[0,2^6)\cup[2^6,2^6+2^5)\cup[2^6+2^5,$$

$$2^6+2^5+2^3)\cup[2^6+2^5+2^3,2^6+2^5+2^3+1). \tag{39}$$

The last subset does not contain any odd number. By (38) we have

$$\Delta_3^{odd}([2^6+2^5,2^6+2^5+2^3))=$$

$$=\Delta_3^{odd}([0,2^3))\ \ (\textit{here } k=2,\ \alpha=\beta=1). \tag{40}$$

Therefore, by (39),(40) and Theorem 2 we find

$$\Delta_3^{odd}([0,105))=3^2-3+1=7\ \blacksquare$$

C.Proof of Theorem 4.

In view of (37)-(38) it is sufficient to prove Theorem 4 for the numbers of the form

$$a)2^n,\ b)2^n+2^m,\ \ m\le n-1,\ c)2^n+2^{n-2}+2^m,\ \ m\le n-3.$$

a) According to the point 1 of Theorem 2 we have

$$\lim_{n\to\infty}\frac{\ln\Delta_3^{odd}([0,2^n))}{\ln 2^n}=\lim_{n\to\infty}\frac{(\lfloor\frac{n}{2}\rfloor-1)\ln 3}{n\ln 2}=\frac{\ln 3}{\ln 4}.$$

b) According to the points 1 and 2 of Theorem 2 and taking into account that $m\in[2,n-1]$ we have

$$\Delta_3^{odd}([0,2^n+2^m))=\Delta_3^{odd}([0,2^n))+\Delta_3^{odd}([2^n,2^n+2^m))\le$$
$$\le 3^{\lfloor\frac{n}{2}\rfloor-1}+2\cdot 3^{\frac{m-3}{2}}\le 3^{\frac{n-2}{2}}+2\cdot 3^{\frac{n-4}{2}}\le c_1\cdot 3^{\frac{n}{2}}.$$

Therefore,

$$\limsup_{n\to\infty}\frac{\ln\Delta_3^{odd}([0,2^n+2^m))}{\ln(2^n+2^m)}\le\lim_{n\to\infty}\frac{\ln c_1+\frac{n}{2}\ln 3}{n\ln 2}=\frac{\ln 3}{\ln 4}.$$

On the other hand,

$$\Delta_3^{odd}([0,2^n+2^m))\ge 3^{\frac{n-1}{2}-1}-3^{\frac{m-3}{2}}\ge 3^{\frac{n-3}{2}}-3^{\frac{n}{2}-2}\ge 0.08\cdot 3^{\frac{n}{2}}.$$

Thus,

$$\liminf_{n\to\infty}\frac{\ln\Delta_3^{odd}([0,2^n+2^m))}{\ln(2^n+2^m)}\ge\lim_{n\to\infty}\frac{\ln 0.08+\frac{n}{2}\ln 3}{\ln 2+n\ln 2}=\frac{\ln 3}{\ln 4}.$$

c) Analogously, according to the points 1, 2 and 3 of Theorem 2 and taking into account that $m\in[2,n-3]$ we have

$$\Delta_3^{odd}([0, 2^n + 2^{n-2} + 2^m)) \le (c_1 + 1) \cdot 3^{\frac{n}{2}} + 3^{\frac{n-6}{2}} = c_2 3^{\frac{n}{2}},$$

$$\Delta_3^{odd}([0, 2^n + 2^{n-2} + 2^m)) \ge 0.08 \cdot 3^{\frac{n}{2}} - 2 \cdot 3^{\frac{n-6}{2}} \ge 0.005 \cdot 3^{\frac{n}{2}}$$

and we are done . ■

3. On Conjecture 2

Show that Conjecture 2 is a corollary of the following heuristic argument: the behavior of primes with the point of view the excess of the odious primes is proportionally similar to behavior of numbers not divisible by 2 and 3. Indeed, the number of the latter numbers less than n is $n - 1 - \lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{n-1}{3} \rfloor + \lfloor \frac{n-1}{6} \rfloor \sim \frac{n}{3}$. Thus, the excess $\delta(n)$ of the odious numbers not divisible by 2 and 3 and less than n equals

$$\delta(n) = (\nu^o(n) - \nu^e(n)) - (\lambda^o(n) - \lambda^e(n)) + \Delta_3(n) - \Delta_3^{even}(n)$$

and by (12) and Lemma 1

$$\delta(n) = \Delta_3^{odd}(n) + \varepsilon, \tag{41}$$

where $|\varepsilon| \le 2$.

Thus, by Theorem 4 we have

$$\lim_{n\to\infty} \frac{\ln \delta(n)}{\ln n} = \frac{\ln 3}{\ln 4}. \tag{42}$$

By the heuristic argument of the proportionality, we have

$$\pi^o(n) - \pi^e(n) \approx \frac{3\pi(n)}{n}\delta(n). \tag{43}$$

Now (42)-(43) is equivalent to Conjecture 2. ■

Table 2 compares on the powers of 4 the values of $x(n) = \frac{\ln(\pi^o(n)-\pi^e(n))}{\ln n}$ and $x^*(n) = \frac{\ln(\frac{3\pi(n)}{n}(\mu_3^e(n)-\mu_3^o(n))}{\ln n}$.

Table 2.

m	$x(4^m)$	$x^*(4^m)$	m	$x(4^m)$	$x^*(4^m)$
2	0.2500	0.3962	9	0.5983	0.5974
3	0.3333	0.4679	10	0.6153	0.6087
4	0.5574	0.5109	11	0.6237	0.6186
5	0.5322	0.5322	12	0.6318	0.6275
6	0.5736	0.5537	13	0.6364	0.6354
7	0.5792	0.5702	14	0.6439	0.6426

4. On the increment of the excess of the odious primes

In conclusion let us consider the absolute value of the increment of the excess of the numbers between the odious primes and the evil primes on intervals $(0, 2^n)$:

$$(44) \qquad \Delta(n) = \left|(\pi^o(2^n) - \pi^e(2^n)) - (\pi^o(2^{n-1}) - \pi^e(2^{n-1}))\right|.$$

By (41),(43), (44) and Theorem 1 we find

$$(45) \qquad \Delta(n) \approx \begin{cases} 3^{\frac{n-1}{2}}\left|\frac{\pi(2^{n-1})}{2^{n-1}} - \frac{\pi(2^n)}{2^n}\right|, \; if\ n\ is\ odd \\ 3^{\frac{n}{2}-1}\left(3\frac{\pi(2^n)}{2^n} - \frac{\pi(2^{n-1})}{2^{n-1}}\right), if\ n\ is\ even. \end{cases}$$

Notice that, by the Landau conjecture, $\pi(2n) \leq 2\pi(n),\ n \geq 3$ and therefore $\frac{\pi(2^{2n-1})}{2^{n-1}} \geq \frac{\pi(2^n)}{2^n},\ n \geq 2$. Unfortunately, this very plausible conjecture was proved until now only for sufficiently large n [8].

The following Table 4 illustrates the irregularity of the distribution of $\Delta(n)$ (44) in fact and by (45) for $n \geq 15$.

Table 3.

n	$\Delta(n)$	$by(45)$
15	58	19
16	492	421
17	111	42
18	1031	1114
19	110	98
20	3207	2990
21	158	238
22	8296	8118
23	1416	586
24	21790	22229
25	1246	1458
26	60294	61342
27	1570	3707
28	170024	170372

Notice that, although the phenomenon to a certain degree was explained it remains very impressive that in spite of the ratio of the numbers of primes in intervals $(2^{2t}, 2^{2t+1}),\ (2^{2t-1}, 2^{2t})$is less than 2 but the value of Δ (44) for $t \geq 8$ more that $8, 9, 29, \ldots, 48, 108, \ldots$ times as large!

Conclusive remarks. 1)On the one hand, it is interesting, using Theorem 1, to make the following steps towards justification of Conjecture 1. On the other hand, Conjecture 2 means that the influence of the rest of the other steps in totality is small. Nevertheless, the full proof most likely requires more strong methods.

2)It is interesting to investigate the behavior of primes from the considered point of view on the arithmetical progressions. For example, on the as one can show the excess of the odd evil integers of the form $3t + 2$ in interval $[5, 2^{2n-1})$ is equal to 3^{n-2}, while on interval $[5, 2^{2n})$ it is equal to 0. It is a topic for a separate article.

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