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THE TWISTED MELLIN TRANSFORM

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Abstract. The "twisted Mellin transform" is a slightly modified version of the usual classical Mellin transform on $L^2([0,\infty))$. In this short note we investigate some of its basic properties. From the point of view of combinatorics one of its most interesting properties is that it intertwines the differential operator, df/dx, with its finite difference analogue, $\nabla f = f(x) - f(x-1)$. From the point of view of analysis one of its most important properties is that it describes the asymptotics of one dimensional quantum states in Bargmann quantization.

1. Introduction

The standard Mellin Transform is defined by the formula

$$Mf(s) = \int_0^\infty f(x)x^{s-1}dx. \tag{1.1}$$

This paper has as its topic the following "twisted" version of (1.1):

$$\mathcal{M}f(s) = \frac{\int_0^\infty f(x)x^s e^{-x}\, dx}{\int_0^\infty x^s e^{-x}\, dx}. \tag{1.2}$$

This transform has a number of remarkable properties, the most remarkable perhaps being that it intertwines the standard differential operator, $\frac{d}{dx}$, and the finite difference analogue of $\frac{d}{dx}$:

$$\nabla f(x) = f(x) - f(x-1). \tag{1.3}$$

By a theorem of Mullin and Rota it is known that there exists an invertible operator intertwining the "umbral" calculi generated by $\frac{d}{dx}$ and ∇; but, as far as we know the explicit expression (1.2) for this intertwiner is new.

The main topic of this paper is an asymptotic formula for (1.2) when the function, f, in the integrand is a *symbol* of degree k, i.e. has the property that for any $r \in \mathbb{N}$ there exists a constant C_r such that

$$\left|\frac{d^r f}{dx^r}(x)\right| \le C_r x^{k-r}. \tag{1.4}$$

More explicitly, we will show that for such functions

$$\mathcal{M}f(x) \sim \sum_r f^{(r)}(x)g_r(x), \tag{1.5}$$

where $f^{(r)}(x) = \frac{d^r}{dx^r} f(x)$, and $g_r(x)$ is a polynomial of degree $[r/2]$ given by a simple recursion formula. In some joint work with V. Guillemin, now in progress, we will use this formula to obtain results about the spectral density functions of toric varieties.

A few words about the organization of this paper. In section 2 we will prove some elementary facts about the domain and range of $\mathcal{M}$, derive a twisted version of the standard inversion formula for the Mellin transform, prove that $\mathcal{M}$ has the intertwining property that we described above and compile a table of twisted Mellin transforms for most of the standard elementary functions. In section 3 we will use steepest descent techniques to derive (1.5) and give two rather different recipes for computing the g_r's, one analytic and one combinatorial. (By comparing these two recipes we obtain some curious combinatorial identities for the Stirling numbers of the first kind.)

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2. The Twisted Mellin transform

Let $\mathbb{C}$ be equipped with the Bargmann measure $\mu = e^{-|z|^2} dz d\bar{z}$. Given a function $f \in C^\infty(\mathbb{C})$, one would like to study the asymptotics of the spectral measure

$$T_k(f) = \mathrm{Tr}(\pi_k M_f \pi_k),$$

associated with the quantum eigenstate, z^k, as $k \to \infty$, where π_k is the orthogonal projection from $L^2(\mathbb{C}, \mu)$ onto the one dimensional subspace spanned by z^k, and M_f is the operator "multiplication by f". By averaging with respect to the $\mathbb{T}^1$-action, we can assume $f \in C^\infty(\mathbb{C})^{\mathbb{T}^1}$, i.e.

$$f(z) = f(r^2),$$

where $r = |z|$ is the modulus of complex number z.

For $k \in \mathbb{N}$, one has

$$\begin{aligned} T_k(f) = \frac{\langle f z^k, z^k \rangle_\mu}{\langle z^k, z^k \rangle_\mu} &= \frac{\int_0^\infty f(r^2) r^{2k+1} e^{-r^2}\, dr}{\int_0^\infty r^{2k+1} e^{-r^2}\, dr} \\ &= \frac{\int_0^\infty f(x) x^k e^{-x}\, dx}{\int_0^\infty x^k e^{-x}\, dx}. \end{aligned}$$

So the asymptotic properties of $T_k(f)$ can be deduced from asymptotic properties of the twisted Mellin transform

$$\mathcal{M}f(s) = f^{\#}(s) = \frac{\int_0^\infty f(x) x^s e^{-x}\, dx}{\int_0^\infty x^s e^{-x}\, dx} \tag{2.1}$$

where $f \in C^\infty(\mathbb{R}^+)$ and $s \geq 0$. Note that the denominator is the Gamma function $\Gamma(s+1)$, while the numerator is just the Mellin transform of the function $xe^{-x} f(x)$.

For this integral to converge, we will assume that f is of polynomial growth, i.e.

$$|f(x)| \leq C x^N \quad \text{for some } N. \tag{2.2}$$

Some basic properties of the transform are the following

Proposition 2.1. *Suppose $a, b \in \mathbb{R}$, $c > 0$, $n \in \mathbb{N}$, f is a function of polynomial growth, then*

(1) *For $g(x) = x^a f(x)$,*

$$\mathcal{M}g(s) = \frac{\Gamma(s+a+1)}{\Gamma(s+1)} \mathcal{M}f(s+a), \tag{2.3}$$

and for $g(x) = e^{-cx} f(x)$,

$$\mathcal{M}g(s) = (c+1)^{-s-1} \mathcal{M}f_c(s), \tag{2.4}$$

where $f_c(x)$ is the dilation, $f_c(x) = f(\frac{x}{c+1})$.

(2) *For $g(x) = \frac{df}{dx}(x)$,*

$$\mathcal{M}g(s) = \nabla \mathcal{M}f(s) := \mathcal{M}f(s) - \mathcal{M}f(s-1), \tag{2.5}$$

and more generally, for any $n \in \mathbb{N}$ and $g(x) = f^{(n)}(x)$,

$$\mathcal{M}g(s) = \nabla^n (\mathcal{M}f)(s) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathcal{M}f(s-i). \tag{2.6}$$

(3) *For $g(x) = f(x) \ln x$,*

$$\frac{d}{ds} \mathcal{M}f(s) = \mathcal{M}g(s) - \mathcal{M}f(s) \frac{\Gamma'(s+1)}{\Gamma(s+1)}. \tag{2.7}$$

(4) *For $g(x) = \int_0^x f(t)\, dt$,*

$$\mathcal{M}g(s) = \sum_{i=0}^{[s]-1} \mathcal{M}f(s-i) + \mathcal{M}g(s-[s]). \tag{2.8}$$

In particular,

$$\mathcal{M}g(n) = \sum_{i=0}^{n} \mathcal{M}f(i); \tag{2.9}$$

Proof. The assertion (1) is obvious.

To prove (2), we note that for $g(x) = f'(x)$,

$$\begin{aligned}
\mathcal{M}g(s) &= \frac{\int_0^\infty f'(x) x^s e^{-x}\, dx}{\int_0^\infty x^s e^{-x}\, dx} \\
&= \frac{\int_0^\infty f(x)(x^s - s x^{s-1}) e^{-x}\, dx}{\int_0^\infty x^s e^{-x}\, dx} \\
&= \frac{\int_0^\infty f(x) x^s e^{-x}\, dx}{\int_0^\infty x^s e^{-x}\, dx} - \frac{\int_0^\infty f(x) x^{s-1} e^{-x}\, dx}{\int_0^\infty x^{s-1} e^{-x}\, dx} \\
&= \mathcal{M}f(s) - \mathcal{M}f(s-1).
\end{aligned}$$

The property (2.6) is easily deduced from (2.5) by induction, and (3) is a direct computation.

To prove (4), we note that by integration by parts,

$$\mathcal{M}g(s) = \mathcal{M}f(s) + \mathcal{M}g(s-1), \tag{2.10}$$

which implies (2.8). As for (2.9), this follows from the obvious fact $\mathcal{M}g(0) = \mathcal{M}f(0)$. □

From the definition its easy to see that the twisted Mellin transform is smooth, i.e. it transform a smooth function to a smooth function. Moreover, it transforms a function which is of polynomial growth of degree N to a function which is of polynomial growth of degree N, and Schwartz functions to Schwartz functions:

Proposition 2.2. (1) *Suppose* $|f(x)| \le Cx^N$*, then* $|\mathcal{M}f(s)| \le C's^N$.

(2) $\mathcal{M}$ *maps Schwartz functions to Schwartz functions.*

Proof. (1) This comes from the definition:

$$|\mathcal{M}f(s)| \le \frac{\int_0^\infty Cx^N x^s e^{-x}\ dx}{\Gamma(s+1)} = C\frac{\Gamma(s+N+1)}{\Gamma(s+1)} \le C's^N.$$

(2) Suppose f is a Schwartz function, i.e. for any α, β, there is a constant $C_{\alpha,\beta}$ such that $\sup_x |x^\alpha \partial^\beta f(x)| \le C_{\alpha,\beta}$.

For $\beta = 0$, $|x^\alpha f(x)| \le C$ implies $|s^\alpha \mathcal{M}f(s)| \le C'$.

For $\beta = 1$, we apply (2.7) and the above result to get $|s^\alpha \frac{d}{ds}\mathcal{M}f(s)| \le C_\alpha$.

For $\beta \ge 1$, let $\psi(s) = \Gamma'(s)/\Gamma(s)$. Then by repeated applications of (2.7) one can see that $\frac{d^n}{ds^n}\mathcal{M}f(s)$ is a linear combination of the functions $\mathcal{M}g_i(s)\psi^{(j)}(s+1)$, where $g_i(x) = f(x)(\ln x)^i$ and

$$\psi^{(m)}(s+1) = \frac{d^m}{ds^m}\psi(s+1) \tag{2.11}$$

is the polygamma function, which is bounded for each m, as is clear from its integral representation:

$$|\psi^{(m)}(s+1)| = \left|(-1)^{m+1}\int_0^\infty \frac{t^m e^{-(s+1)t}}{1-e^{-t}}dt\right| \le \int_0^\infty \frac{t^m e^{-t}}{1-e^{-t}}dt = \zeta(m+1)\Gamma(m+1).$$

Thus by induction we easily deduce that $|s^\alpha \partial^\beta \mathcal{M}f(s)| \le C_{\alpha,\beta}$. □

Remark. Since the twisted Mellin transform transforms a Schwartz function to a Schwartz function, we can define the twisted Mellin transform on tempered distributions by duality.

We will next compute the twisted Mellin transform for some elementary functions such as polynomials, exponentials and trigonometric functions.

(a) For $f(x) = x^a$,

$$\mathcal{M}f(s) = \Gamma(s+a+1)/\Gamma(s+1). \tag{2.12}$$

In particular, if $f(x) = x^n$, n a positive integer, then

$$\mathcal{M}f(s) = s^{[n]} := (s+1)(s+2)\cdots(s+n). \tag{2.13}$$

Thus the twisted Mellin transform of a polynomial of degree n is again a polynomial of degree n.

(b) Suppose $a > 1$, then for $f(x) = a^{-x}$,

$$\mathcal{M}f(s) = (\ln a + 1)^{-1-s}. \tag{2.14}$$

More generally, if $f(x) = x^b a^{-x}$, then

$$\mathcal{M}f(s) = (\ln a + 1)^{-1-b-s}\Gamma(s+b+1)/\Gamma(s+1). \tag{2.15}$$

(c) For $f(x) = \frac{1}{1-e^{-x}}$,

$$\mathcal{M}f(s) = \zeta(s+1), \tag{2.16}$$

and as a corollary, for the Todd function $f(x) = \frac{x}{1-e^{-x}}$,

$$\mathcal{M}f(s) = (s+1)\zeta(s+2). \tag{2.17}$$

(d) For $f(x) = \ln x$, one gets from (2.7)

$$\mathcal{M}f(s) = \frac{\Gamma'(s+1)}{\Gamma(s+1)}, \tag{2.18}$$

and in general, for $f(x) = (\ln x)^n$,

$$\mathcal{M}f(s) = \frac{\Gamma^{(n)}(s+1)}{\Gamma(s+1)}. \tag{2.19}$$

(e) For the trigonometric functions $f(x) = \sin x$ and $g(x) = \cos x$,

$$\begin{aligned} \mathcal{M}f(s) &= \frac{1}{(\sqrt{2})^{s+1}} \sin \frac{(s+1)\pi}{4}, \\ \mathcal{M}g(s) &= \frac{1}{(\sqrt{2})^{s+1}} \cos \frac{(s+1)\pi}{4}. \end{aligned} \tag{2.20}$$

(*Proof.* Let $h(x) = e^{ix}$, then $\mathcal{M}h(s) = \frac{1}{(1-i)^{s+1}}$, which gives (2.20).)

Similarly for $f(x) = \sin(ax)$ and $g(x) = \cos(ax)$,

$$\begin{aligned} \mathcal{M}f(s) &= (1+a^2)^{-s} \sin(s \arctan a), \\ \mathcal{M}g(s) &= (1+a^2)^{-s} \cos(s \arctan a). \end{aligned} \tag{2.21}$$

Some concluding remarks:

(1) From the inversion formula for the Mellin transform, we obtain an inversion formula for the twisted Mellin transform:

$$f(x) = e^x \int_{c-\infty i}^{c+\infty i} \Gamma(s+1)\mathcal{M}f(s)x^{-s-1}ds. \tag{2.22}$$

and from the Parseval formula (c.f. [3]) a “Parsevel-like” formula for $\mathcal{M}$

$$\int_0^\infty f(x)g(x)x^2e^{-2x}\, dx = \frac{1}{2i}\int_{c-\infty i}^{c+\infty i} \mathcal{M}f(1-s)\mathcal{M}g(s)\frac{s(1-s)}{\sin \pi s}\, ds. \tag{2.23}$$

(2) Letting $(x)^{(n)} = x(x+1)\cdots(x+n-1)$, (2.13) becomes $\mathcal{M}f(s-1) = (s)^{(n)}$, where $f(x) = x^n$. Also expanding $s^{[n]}$ in terms of s^n, we get

$$s^{[n]} = (s+1)(s+2)\cdots(s+n) = \sum_{k=0}^{n} c(n+1,k+1)s^k, \tag{2.24}$$

where $c(n,k)$ is the signless Stirling number of first kind, c.f. [**?**]. Note that both $\{x^n\}$ and $\{x^{(n)}\}$ are a basis of the polynomial ring, thus $\mathcal{M}$ is a bijection from the polynomial ring to itself.

(3) Formula (2.5) tells us that $\mathcal{M}$ conjugates the differential operator, $\frac{d}{dx}$, to the backward difference operator (1.3). In combinatorics both $\frac{d}{dx}$ and the backward difference operator are "delta" operators, with the functions x^n and $x^{(n)}$ as their sequence of basic polynomials. Thus by a theorem of R.Mullin and G-C.Rota ([2]), the map $T : f(x) \mapsto \mathcal{M}f(s-1)$ is invertible and the map $S \mapsto TST^{-1}$ an automorphism of the algebra of shift-invariant operators onto the algebra of polynomials. Moreover, T maps every sequence of basic polynomials into a sequence of basic polynomials. Such an operator is called an *umbral* operator in the umbral calculus.

(4) If we replace the Bargmann measure, μ, by the generalized Bargmann measure $\mu_\alpha = e^{-\alpha|z|^2}\ dzd\bar{z}$, then we are naturally led, by the argument at the beginning of this section, to studying the "α-twisted Mellin transform"

$$\mathcal{M}_\alpha f(s) = \frac{\int_0^\infty f(x)x^s e^{-\alpha x}\ dx}{\int_0^\infty x^s e^{-\alpha x}\ dx}. \tag{2.25}$$

All the properties in Proposition 2.1 can be easily generalized to $\mathcal{M}_\alpha$. Moreover, it is easy to see that

$$\mathcal{M}_\alpha f_c(s) = \mathcal{M}_{\alpha/c} f(s), \tag{2.26}$$

where $f_c(x) = f(cx)$, so the α-twisted Mellin transform of a function can easily be computed from the twisted Mellin transform.

(5) The twisted Mellin transform can also be defined in higher dimension in a similar way. For the function $f(r_1, \cdots, r_d) = f_1(r_1)\cdots f_d(r_d)$, its twisted Mellin transform is just the product of the twisted Mellin transform of $f_1, \cdots, f_d$.

3. The Asymptotic Expansion.

We can rewrite the twisted Mellin transform as

$$\mathcal{M}f(s) = \frac{\int_0^\infty f(x)e^{s\log x - x}\ dx}{\int_0^\infty e^{s\log x - x}\ dx}. \tag{3.1}$$

For the phase function $\varphi(x,s) = s\log x - x$, we have

$$0 = \frac{\partial\varphi}{\partial x} \quad\Longrightarrow\quad x = s,$$

thus the function $\varphi_s(x) = \varphi(x,s)$ has a unique critical point at $x = s$. Moreover, this is a global maximum of $\varphi(x,s)$, since

$$\lim_{x\to+\infty} \varphi(x,s) = -\infty,$$

and

$$\frac{\partial^2\varphi}{\partial x^2} = -\frac{s}{x^2} < 0.$$

Hence if f is a symbol, we can apply the method of steepest descent to both denominator and numerator to get

$$\mathcal{M}f(s) \sim \sum_k g_k(s) f^{(k)}(s). \tag{3.2}$$

To compute the functions $g_k(s)$ consider the Taylor expansion of f,

$$f(x) = \sum_{r=0}^{\infty} \frac{1}{r!} f^{(r)}(s)(x-s)^r. \tag{3.3}$$

Applying $\mathcal{M}$ to (3.3) with s fixed we get

$$\mathcal{M}f(s) = \sum_{r=0}^{\infty} \frac{1}{r!} f^{(r)}(s) f_r(s), \tag{3.4}$$

where

$$f_r(s) = \frac{\int_0^\infty (x-s)^r x^s e^{-x}\, dx}{\int_0^\infty x^s e^{-x}\, dx} = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} s^{[i]} s^{r-i}. \tag{3.5}$$

For $r \le 5$ small,

$$\begin{aligned}
f_0(s) &= f_1(s) = 1,\\
f_2(s) &= 2 + s,\\
f_3(s) &= 6 + 5s,\\
f_4(s) &= 24 + 26s + 3s^2,\\
f_5(s) &= 120 + 154s + 35s^2.
\end{aligned}$$

which suggests that $f_r(s)$ is a polynomial of degree $[r/2]$ in s. We will give two separate proofs of this fact, one combinatorial and one analytic.

Proof 1. Putting (2.24) into (3.5), we get

$$\begin{aligned}
f_r(s) &= \sum_{i=0}^{r}\sum_{k=0}^{i} (-1)^{r-i}\binom{r}{i} c(i+1,k+1) s^{r-(i-k)}\\
&= \sum_{j=0}^{r}\left(\sum_{i=j}^{r} (-1)^{r-i}\binom{r}{i} c(i+1, i-j+1)\right) s^{r-j}.
\end{aligned}$$

On the other hand, by the definition of the Stirling number,

$$c(n+1, n+1-k) = c(n, n-k) + nc(n, n-(k-1)). \tag{3.6}$$

and from this recurrence relation we will show:

Lemma 3.1. *There are constants $C_{l,j}$, depending only on l, j, such that*

$$c(i+1, i+1-j) = \sum_{l=j}^{2j} C_{l,j}\ (i)_l, \tag{3.7}$$

where $(i)_l = i(i-1)\cdots(i-l+1)$ is the falling factorial.

Proof. This is true for $j = 0$, since $c(n+1, n+1) = 1$. Notice that

$$\sum_{i=l}^{n} (i)_l = l!\left(\binom{l}{l} + \binom{l+1}{l} + \cdots + \binom{n}{l}\right) = l!\binom{n+1}{l+1} = \frac{1}{l+1}(n+1)_{l+1}.$$

Now use induction and the recurrence relation (3.6). □

Now suppose $2j \le r$, then the coefficients of s^{r-j} in $f_r(s)$ is

$$\begin{aligned}\sum_{i=j}^{r}(-1)^{r-i}\binom{r}{i}c(i+1, i-j+1) =& \sum_{i=j}^{r}(-1)^{r-i}\binom{r}{i}\sum_{l=j}^{2j} C_{l,j}\ (i)_l \\ =& \sum_{l=j}^{2j} C_{l,j}(r)_l \sum_{i=l}^{r}(-1)^{r-i}\binom{r-l}{i-l} \\ =& 0,\end{aligned}$$

which proves that f_r is a polynomial of degree $[r/2]$.

Proof 2. First we derive a recurrence relation for $f_r(s)$. Using

$$\frac{d}{dx}(s\log x - x) = -\frac{x-s}{x}$$

and integration by parts we get

$$\begin{aligned}\Gamma(s+1)f_r(s) &= -\int_0^\infty e^{s\ln x - x}x(x-s)^{r-1}\frac{d}{dx}(s\ln x - x)\ dx \\ &= \int_0^\infty e^{s\ln x - x}\frac{d}{dx}(x(x-s)^{r-1})\ dx \\ &= \int_0^\infty e^{s\ln x - x}\frac{d}{dx}((x-s)^r + s(x-s)^{r-1})\ dx \\ &= r\int_0^\infty x^s e^{-x}(x-s)^{r-1}\ dx + (r-1)s\int_0^\infty x^s e^{-x}(x-s)^{r-2}\ dx,\end{aligned}$$

i.e.

$$f_r(s) = rf_{r-1}(s) + (r-1)sf_{r-2}(s). \tag{3.8}$$

Moreover, we can compute the initial conditions directly

$$f_1(s) = f_0(s) = 1. \tag{3.9}$$

Remark. The recurrence relation (3.8) also follows easily from (2.3) and (2.5). In fact, if we denote $h_r(x) = (x-s)^r$, then $f_r(s) = \mathcal{M}h_r(s)$, and thus

$$\begin{aligned} rf_{r-1}(s) &= \mathcal{M}h_r(s) - \mathcal{M}h_r(s-1) \\ &= f_r(s) - (\mathcal{M}(xh_{r-1})(s-1) - s\mathcal{M}h_{r-1}(s-1)) \\ &= f_r(s) - s(\mathcal{M}h_{r-1}(s) - \mathcal{M}h_{r-1}(s-1)) \\ &= f_r(s) - s(r-1)\mathcal{M}h_{r-2}(s). \end{aligned}$$

From (3.8), (3.9) and induction, it follows again that $f_r(s)$ is a polynomial of degree $[r/2]$. Thus coming back to (3.4) we have proved

Theorem 3.2. *For any symbolic function f, we have*

$$\mathcal{M}f(s) \sim \sum_r \frac{1}{r!} f^{(r)}(s) f_r(s), \tag{3.10}$$

where $f_r(s)$ is the polynomial of integer coefficients of degree $[r/2]$ given by (3.5).

The polynomials $f_r(s)$ have many interesting combinatorial properties:

(1) Since $f_r(s)$ is a polynomial of degree $[r/2]$, we can write

$$f_r(s) = \sum_{i=0}^{[r/2]} a_{r,i} s^i, \tag{3.11}$$

the coefficients satisfying the recurrence relation

$$a_{r,i} = ra_{r-1,i} + (r-1)a_{r-2,i-1} \tag{3.12}$$

and initial conditions

$$a_{r,0} = r!, \quad a_{2k,k} = (2k-1)!!,$$

which implies

$$\begin{aligned} a_{r,1} =& r!\left(\frac{1}{r} + \frac{1}{r-1} + \cdots + \frac{1}{2}\right), \\ a_{r,2} =& r!\left(\frac{(r-1)a_{r-2,1}}{r!} + \frac{(r-2)a_{r-3,1}}{(r-1)!} + \cdots + \frac{3a_{2,1}}{4!}\right), \end{aligned}$$

and in general

$$a_{r,k} = r!\left(\frac{(r-1)a_{r-2,k-1}}{r!} + \frac{(r-2)a_{r-3,k-1}}{(r-1)!} + \cdots + \frac{(2k-1)a_{2k-2,k-1}}{(2k)!}\right). \tag{3.13}$$

(2) The coefficients, $a_{r,i}$, of $f_r(s)$, are exactly those appeared as coefficients of polynomials used for exponential generating functions for diagonals of unsigned Stirling numbers of the first kind. More precisely, for fixed k, the exponential generating function for the sequence $\{c(n+1, n+1-k)\}_{n\geq 0}$ is given by (c.f. sequence A112486 in "The On-Line Encyclopedia of Integer Sequences")

$$\sum_{n=0}^{\infty} c_{n+1,n+1-k} \frac{x^n}{n!} = e^x \sum_{n=k}^{2k} \left(a_{n,n-k} \frac{x^n}{n!}\right).$$

(3) The sequence of functions f_r's have a pretty simple exponential generating function:

$$\begin{aligned}\sum_{r=0}^{\infty} f_r(s)\frac{x^r}{r!} &= \sum_{i=0}^{\infty}\sum_{r=i}^{\infty}(-1)^{r-i}\frac{1}{r!}\binom{r}{i}s^{[i]}s^{r-i}x^r \\ &= \left(\sum_{i=0}^{\infty}\frac{s^{[i]}x^i}{i!}\right)\left(\sum_{r=i}^{\infty}(-1)^{r-i}\frac{s^{r-i}x^{r-i}}{(r-i)!}\right) \\ &= \frac{e^{-sx}}{(1-x)^{1+s}}.\end{aligned}$$

(4) From the generating function above we get a combinatorial interpreting of $f_r(s)$ for integers s: $r!f_r(s)$ is the number of $r \times r$ $\mathbb{N}$-matrices with every row and column sum equal to $3+2s$ and with at most 2 nonzero entries in every row. (c.f. Exercise 5.62 of [4]).

(5) There are also other combinatorial interpreting for small value of s. For example, the sequence $f_r(1)$ count permutations w of $\{1, 2, \cdots, r+1\}$ such that $w(i+1) \neq w(i)+1$ (c.f. the sequence A000255 of "On-line Encyclopedia of Integer Sequences"). For $s = 2$, we have

$$f_r(2) = \frac{2^{-r^2}}{r!}\sum_{M \in D_r}(\det M)^4,$$

where D_r is the set of all $r \times r$ matrices of ± 1's. (c.f. Exercise 5.64(b) of [4]).

We will conclude by deriving a slight variant of the asymptotic expansion above, which will be needed for the application in [1]. Given a symbolic function f, consider the integral

$$A_N(f)(s) = \frac{\int_0^\infty f(x)x^{Ns}e^{-Nx}\,dx}{\int_0^\infty x^{Ns}e^{-Nx}\,dx}, \tag{3.14}$$

as $N \to \infty$. By definition, this is just the "N-twisted Mellin transform" $\mathcal{M}_N f(Ns)$, which, according to (2.26), equals $\mathcal{M}f_N(Ns)$, where $f_N(x) = f(x/N)$. Thus by Theorem 3.2,

$$A_N(f)(s) \sim \sum_k \left(\frac{1}{N}\right)^k f^{(k)}(s)g_k(Ns). \tag{3.15}$$

Note that since $g_k(x)$ is a polynomial of degree $[k/2]$, the above formula does give us an asymptotic expansion. In particular, we have

$$A_N(f)(s) = f(s) + \frac{1}{N}\left(f'(s) + f''(s)\frac{s}{2}\right) + \frac{1}{N^2}\left(f''(s) + f'''(s)\frac{5s}{6} + f^{(4)}(s)\frac{s^2}{8}\right) + O(N^{-3}).$$

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