CONVEX HULL REALIZATIONS OF THE MULTIPLIHEDRA

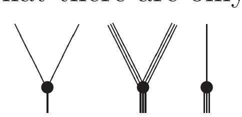
STEFAN FORCEY

Abstract. We present a simple algorithm for determining the extremal points in Euclidean space whose convex hull is the n^{th} polytope in the sequence known as the multiplihedra. This answers the open question of whether the multiplihedra could be realized as convex polytopes.

Contents

1. Introduction

Pictures in the form of painted binary trees can be drawn to represent the multiplication of several objects in a monoid, before or after their passage to the image of that monoid under a homomorphism. We use the term "painted" rather than "colored" to distinguish our trees with two edge colorings, "painted" and "unpainted," from the other meaning of colored, as in colored operad or multicategory. We will refer to the exterior vertices of the tree as the root and the leaves , and to the interior vertices as nodes. This will be handy since then we can reserve the term "vertices" for reference to polytopes. A partly painted binary tree is painted beginning at the root (the leaves are unpainted), and always painted in such a way that there are only three types of nodes. They are:



(1) (2) (3)

This limitation on nodes implies that painted regions must be connected, that painting must never end precisely at a trivalent node, and that painting must proceed up both branches of a trivalent node. To see the promised representation we let the left-hand, type (1) trivalent node above stand for multiplication in the domain; the middle, painted, type (2) trivalent node above stand for multiplication in the range; and the right-hand

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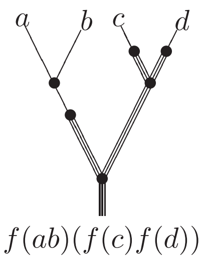
Thanks to XY-pic for the diagrams.

type (3) bivalent node stand for the action of the mapping. For instance, given a, b, c, d elements of a monoid, and f a monoid morphism, the following diagram represents the operation resulting in the product $f(ab)(f(c)f(d))$.



Of course in the category of associative monoids and monoid homomorphisms there is no need to distinguish the product $f(ab)(f(c)f(d))$ from $f(abcd)$. These diagrams were first introduced by Boardman and Vogt in [10] to help describe multiplication in (and morphisms of) monoids that are not strictly associative (and whose morphisms do not strictly respect that multiplication.) The n^{th} multiplihedron is a CW-complex whose vertices correspond to the unambiguous ways of multiplying and applying an A_∞-map to n ordered elements of an A_∞-space. Thus the vertices correspond to the binary painted trees with n leaves. The edges of the multiplihedra correspond to either an association $(ab)c \to a(bc)$ or to an application $f(ab) \to f(a)f(b)$.

The complexes now known as the multiplihedra $J(n)$ were first pictured by Stasheff, for $n \leq 4$ in [57]. They were introduced in order to approach a full description of the category of A_∞ spaces by providing the underlying structure for morphisms which preserved the structure of the domain space "up to homotopy" in the range. Recall that an A_∞ space itself is a monoid only "up to homotopy," and is recognized by a continuous action of the associahedra as described in [56]. Thus the multiplihedra are used to recognize the A_∞ maps. Stasheff described how to construct the 1-skeleton of these complexes, but stopped short of a full combinatorial description.

In [10] Boardman and Vogt took up the challenge of a complete description of the category of A_∞ spaces and maps (and their finite A_n versions.) Their approach was to use sequences of spaces of binary trees with interior edges given a length in $[0, 1]$. They show that the space of such unpainted trees with n leaves (under certain equivalence relations regarding length zero edges) is precisely the n^{th} associahedron. They then developed several equivalent versions of the space of painted binary trees with interior edges of length in $[0, 1]$. These are used to define maps between A_∞ spaces which preserve the multiplicative structure up to homotopy. A later definition of the same sort of map was published by Iwase and Mimura in [28]. They give the first detailed definition of the sequence of complexes $J(n)$ now known as the multiplihedra, and describe their combinatorial properties. The overall structure of the associahedra is that of a topological operad, with the composition given by inclusion. The multiplihedra together form a bimodule over this operad, with the action again given by inclusion. This is somewhat to be expected, since the spaces of painted trees form a bimodule over the operad of spaces of trees, where the compositions and actions are given by the grafting of trees, root to leaf. The study of the A_∞ spaces and their maps is still in progress. There is an open question about the correct way of defining composition of these maps in order to form a category. In [10] the obvious composition is shown not to be associative. There are also interesting questions about the extension of A_n-maps, as in [25], and about the transfer of A_∞ structure through these maps, as in [44]. In the latter there is an open question about canonical decompositions of the multiplihedra.

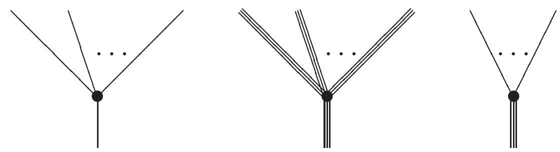
The multiplihedra have appeared in several areas related to deformation theory and A_∞ category theory. A diagonal map is constructed for these polytopes in [52]. This allows a functorial monoidal structure for certain categories of A_∞-algebras and A_∞-categories. A different, possibly equivalent, version of the diagonal is presented in [45]. The 3 dimensional version of the multiplihedron is called by the name Chinese lantern diagram in [63], and used to describe deformation of functors. There is a forthcoming paper by Woodward and Mau in which a new realization of the multiplihedra as moduli spaces of disks with additional structure is presented [48]. This realization promises to allow the authors and their collaborators to define A_n-functors as in [47].

The multiplihedra also appear in higher category theory. The definitions of bicategory and tricategory homomorphisms each include commuting pasting diagrams as seen in [35] and [21] respectively. The two halves of the axiom for a bicategory homomorphism together form the boundary of the multiplihedra $J(3)$, and the two halves of the axiom for a tricategory homomorphism together form the boundary of $J(4)$. Since weak n-categories can be understood as being the algebras of higher operads, these facts can be seen as the motivation for defining morphisms of operad (and n-operad) algebras in terms of their bimodules. This definition is mentioned in [6] and developed in detail in [26]. In the latter paper it is pointed out that the bimodules in question must be co-rings, which have a co-multiplication with respect to the bimodule product over the operad. There is also interest in the existence of a canonical parity structure on the multiplihedra which agrees with the well known such structure on the associahedra. This sort of structure allows a general interpretation of the polytopes as pasting diagrams in an n-category. In a future paper we plan to show how the convex hull realization will allow one to describe the parity structure. Roughly, one embeds a convex polyhedron "generically" in $\mathbf{R}^n$, so that the standard flag of $\mathbf{R}^n$ meets transversally all the finitely many flags one can build from incidence chains of faces. Then one can classify the $(k-1)$-dimensional cells on the boundary of a k-cell into positive and negative ones by a certain "scanning" procedure.

The purpose of this paper is to describe how to represent Boardman and Vogt's spaces of painted trees with n leaves as convex polytopes which are combinatorially equivalent to the CW-complexes described by Iwase and Mimura. Our algorithm for the vertices of the polytopes is flexible in that it allows an initial choice of a constant q between zero and one. In the limit as $q \to 1$ the convex hull approaches that of Loday's convex hull representation of the associahedra as described in [40]. The limit as $q \to 1$ corresponds to the case for which the mapping strictly respects the multiplication. In the limit as $q \to 0$ the convex hulls approach a newly discovered sequence of polytopes (see [51] for some of these in the axioms for pseudomonoids, and visit `http://faculty.tnstate.edu/sforcey/ct06.htm` to learn more about the entire sequence). The limit as $q \to 0$ represents the case for which multiplication in the domain of the morphism in question is strictly associative. The case for which multiplication in the range is strictly associative was found by Stasheff in [57] to yield the associahedra. It was long assumed that the case for which the domain was associative would likewise yield the associahedra, but we will demonstrate (in a sequel to this paper) that this is not so. Recall that when both the range and domain are strictly associative that the polytopes become the cubes.

2. Facets of the multiplihedra

Faces of the multiplihedra of dimension greater than zero correspond to painted trees that are no longer binary. Here are the three new types of node allowed in a general painted tree. They correspond to the the node types (1), (2) and (3) in that they are painted in similar fashion. They generalize types (1), (2), and (3) in that each has greater or equal valence than the corresponding earlier node type.



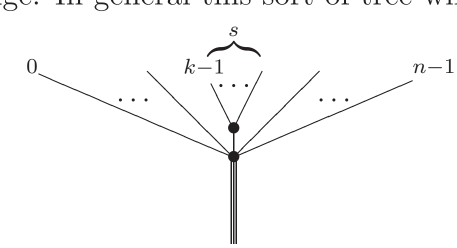
(4) (5) (6)

The recursive definition of the n^{th} multiplihedra is stated by describing the type and number of the facets, or $(n-2)$-dimensional cells, and then defining $\mathcal{J}(n)$ as the cone on the gluing together of these facets along $(n-3)$-dimensional cells with matching associated painted trees. Iwase and Mimura, however, rather than explicitly stating this recursive definition, give a geometric definition of the CW-complex and then prove all the combinatorial facts about the facets. The type and numbers of facets of the multiplihedra are described in [28]. Here we present an instructive proof in terms of the trees that correspond to the enumeration of facets in [28].

2.1. **Theorem.** *The number of facets of the n^{th} multiplihedron is:*

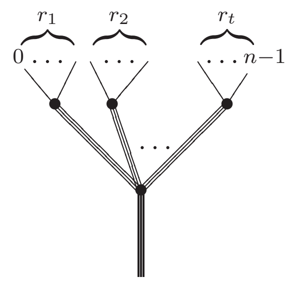
$$\frac{n(n-1)}{2} + 2^{(n-1)} - 1.$$

Proof. Recall that we refer to an unpainted tree with only one node as a corolla. A partly painted corolla is a painted tree with only one node, of type (6). A facet of the multiplihedron corresponds to a painted tree with only one, unpainted, interior edge, or to a tree with all its interior edges attached to a single painted node (type (2) or (5)). The former sort of tree is determined by a selection of s consecutive leaves of the partly painted corolla, $1 < s \leq n$, which will be the leaves of the subtree which has the sole interior edge as its root edge. In general this sort of tree will appear thus:



The number of these trees is $\frac{n(n-1)}{2}$. This follows easily from summing the ways of choosing $s-1$ consecutive "gaps between branches" of the corolla, corresponding to the choice of s consecutive leaves. Note that this count includes one more than the count of the facets of the associahedron, since it includes the possibility of selecting all n leaves. In [28] these *lower facets* are called $\mathcal{J}_k(r,s)$ where $r = n+1-s$ and k is the first "gap between branches" of the $s-1$ consecutive gaps (that is, $k-1$ is the first leaf of the s consecutive leaves.) We will therefore refer to these trees as lower trees. In the complex $\mathcal{J}(n)$ defined in [28] the lower facet $\mathcal{J}_k(r,s)$ is a combinatorial copy of the complex $\mathcal{J}(r) \times \mathcal{K}(s)$.

The trees with all interior (necessarily painted) edges attached to a single painted node will appear thus:



These are determined by choosing any size k proper subset of the "spaces between branches" of the partly painted corolla, $1 \leq k < n-1$. Each set of consecutive "spaces between branches" in that list of k chosen spaces determines a set of consecutive leaves which will be the leaves of a subtree (that is itself a partly painted corolla) with its root edge one of the painted interior edges. If neither of the adjacent spaces to a given branch are chosen, its leaf will be the sole leaf of a subtree that is a partly painted corolla with only one leaf. Thus we count these trees by $\sum_{k=0}^{n-2} \binom{n-1}{k} = 2^{(n-1)} - 1$. In [28] these *upper facets* are labeled $\mathcal{J}(t; r_1, \dots, r_t)$ where t is the number of painted interior edges and r_i is the number of leaves in the subtree supported by the i^{th} interior edge. We will therefore refer to these trees as upper trees. In the complex $\mathcal{J}(n)$ defined in [28] the upper facet $\mathcal{J}(t; r_1, \dots, r_t)$ is a combinatorial copy of the complex $\mathcal{K}(t) \times \mathcal{J}(r_1) \times \dots \times \mathcal{J}(r_t)$. □

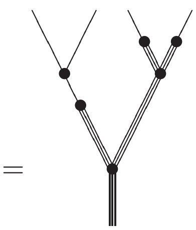
3. An algorithm for the extremal points

In [40] Loday gives an algorithm for taking the binary trees with n leaves and finding for each an extremal point in $\mathbf{R}^{n-1}$; together whose convex hull is $\mathcal{K}(n)$, the $(n-2)$-dimensional associahedron. Note that Loday writes formulas with the convention that the number of leaves is $n+1$, where we instead always use n to refer to the number of leaves. Given a (non-painted) binary n-leaved tree t, Loday arrives at a point $M(t)$ in $\mathbf{R}^{n-1}$ by calculating a coordinate from each trivalent node. These are ordered left to right based upon the ordering of the leaves from left to right. Following Loday we number the leaves $0, 1, \dots, n-1$ and the nodes $1, 2, \dots, n-1$. The i^{th} node is "between" leaf $i-1$ and leaf i where "between" might be described to mean that a rain drop falling between those leaves would be caught at that node. Each trivalent node has a left and right branch, which each support a subtree. To find the Loday coordinate for the i^{th} node we take the product of the number of leaves of the left subtree (l_i) and the number of leaves of the right subtree (r_i) for that node. Thus $M(t) = (x_1, \dots x_{n-1})$ where $x_i = l_i r_i$. Loday proves that the convex hull of the points thus calculated for all n-leaved binary trees is the n^{th} associahedron. He also shows that the points thus calculated all lie in the $n-2$ dimensional affine hyperplane H given by the equation $x_1 + \cdots + x_{n-1} = S(n-1) = \frac{1}{2}n(n-1)$.

We adjust Loday's algorithm to apply to painted binary trees as described above, with only nodes of type (1), (2), and (3), by choosing a number $q \in (0,1)$. Then given a painted binary tree t with n leaves we calculate a point $M_q(t)$ in $\mathbf{R}^{n-1}$ as follows: we begin by finding the coordinate for each trivalent node from left to right given by Loday's algorithm, but if the node is of type (1) (unpainted, or colored by the domain) then its new coordinate is found by further multiplying its Loday coordinate by q. Thus

$$M_q(t) = (x_1, \ldots x_{n-1}) \text{ where } x_i = \begin{cases} q l_i r_i, & \text{if node } i \text{ is type (1)} \\ l_i r_i, & \text{if node } i \text{ is type (2).} \end{cases}$$

Note that whenever we speak of the numbered nodes ($1, \ldots, n-1$ from left to right) of a binary tree, we are referring only to the trivalent nodes, of type (1) or (2). For an example, let us calculate the point in $\mathbf{R}^3$ which corresponds to the 4-leaved tree:

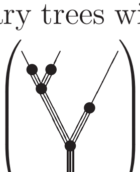
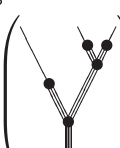
$t =$ 

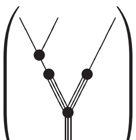
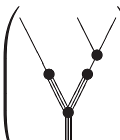
Now $M_q(t) = (q, 4, 1)$.

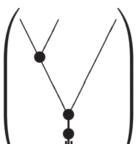
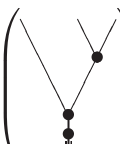
3.1. **Theorem.** *The convex hull of all the resulting points $M_q(t)$ for t in the set of n-leaved binary painted trees is the n^{th} multiplihedron. That is, our convex hull is combinatorially equivalent to the CW-complex $\mathcal{J}(n)$ defined by Iwase and Mimura, and is homeomorphic to the space of level (painted) trees defined by Boardman and Vogt.*

The proof will follow in section 5.

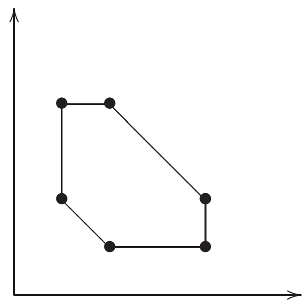
Here are all the painted binary trees with 3 leaves, together with their points $M_q(t) \in \mathbf{R}^2$.

M_q () $= (1,2)$, M_q () $= (2,1)$

M_q () $= (q,2)$ M_q () $= (2,q)$

M_q () $= (q,2q)$, M_q () $= (2q,q)$

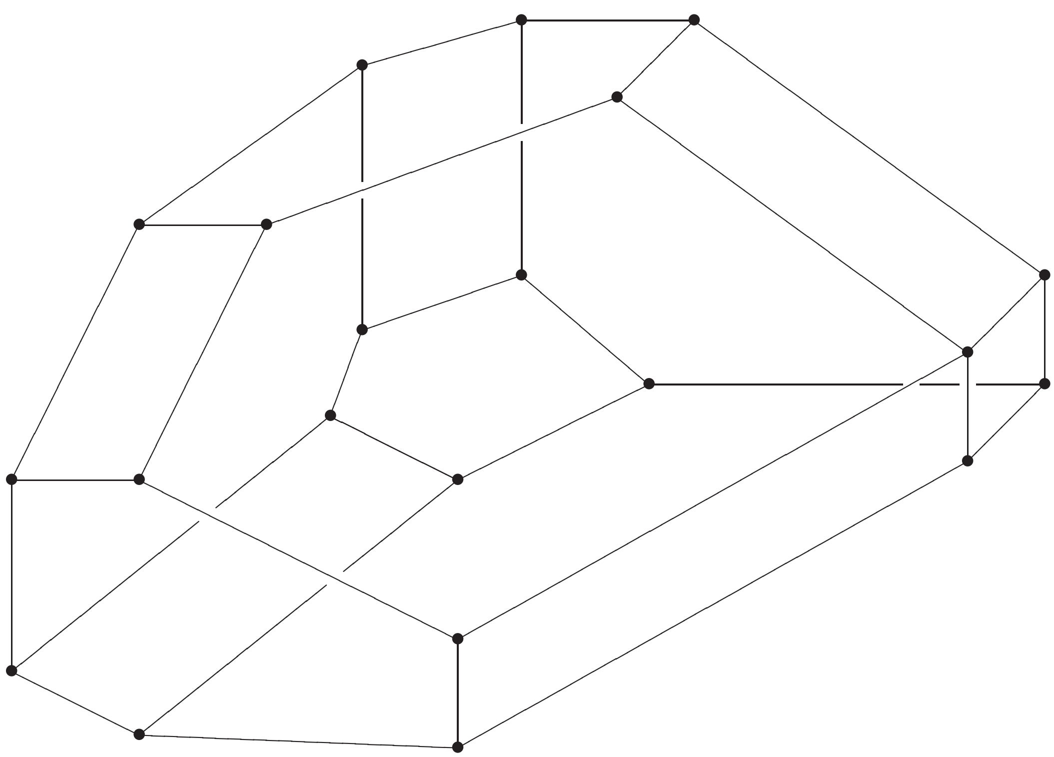
Thus for $q = \frac{1}{2}$ we have the six points $\{(1,2),(2,1),(\frac{1}{2},2),(2,\frac{1}{2}),(\frac{1}{2},1),(1,\frac{1}{2})\}$. Their convex hull appears as follows:



The list of vertices for $\mathcal{J}(4)$ based on painted binary trees with 4 edges, for $q = \frac{1}{2}$, is:

(1, 2 ,3)	(1/2 ,2 ,3)	(1/2 ,2/2 ,3)	(1/2, 1 ,3/2)	
(2, 1, 3)	(2 ,1/2 ,3)	(2/2 ,1/2 ,3)	(2/2, 1/2 ,3/2)	
(3 ,1 ,2)	(3, 1/2, 2)	(3 ,1/2 ,2/2)	(3/2, 1/2 ,2/2)	
(3, 2, 1)	(3 ,2, 1/2)	(3 ,2/2, 1/2)	(3/2 ,2/2 ,1/2)	
(1 ,4 ,1)	(1/2, 4, 1)	(1, 4, 1/2)	(1/2, 4, 1/2)	(1/2, 4/2 ,1/2)

...and here is the convex hull of these points.



The largest pentagonal facet of this picture corresponds to the bottom pentagonal facet in the drawing of $\mathcal{J}(4)$ on page 53 of [57], and to the pentagonal facet labeled $d_{(0,1)}$ in the diagram of $\mathcal{J}(4)$ in section 5 of [52].

To see a rotatable version of the convex hull which is the fourth multiplihedron, enter the following homogeneous coordinates into the Web Demo of polymake (with option visual), at `http://www.math.tu-berlin.de/polymake/index.html#apps/polytope`.

$$
\begin{array}{r}
POINTS \\
1\ 1\ 2\ 3 \\
1\ 1/2\ 2\ 3 \\
1\ 1/2\ 2/2\ 3 \\
1\ 1/2\ 1\ 3/2 \\
1\ 2\ 1\ 3 \\
1\ 2\ 1/2\ 3 \\
1\ 2/2\ 1/2\ 3 \\
1\ 2/2\ 1/2\ 3/2 \\
1\ 3\ 1\ 2 \\
1\ 3\ 1/2\ 2 \\
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1\ 3\ 2\ 1 \\
1\ 3\ 2\ 1/2 \\
1\ 3\ 2/2\ 1/2 \\
1\ 3/2\ 2/2\ 1/2 \\
1\ 1\ 4\ 1 \\
1\ 1/2\ 4\ 1 \\
1\ 1\ 4\ 1/2 \\
1\ 1/2\ 4\ 1/2 \\
1\ 1/2\ 4/2\ 1/2
\end{array}
$$

Before proving the main theorem in section 5 we mention a result about the counting of the binary painted trees with n leaves.

3.2. **Theorem.** *The number of vertices a_n of the n^{th} multiplihedron is given recursively by:*

$$a_n = C(n-1) + \sum_{i=1}^{n-1} a_i a_{n-i}$$

where $a_0 = 0$ and $C(n-1)$ are the Catalan numbers, which count binary (unpainted) trees as well as the vertices of the associahedron.

Proof. The Catalan numbers $C(n-1)$ count those vertices which correspond to the painted binary trees with n leaves which have only the root painted, that is only nodes of type (1) and (3). Now we count the trees for which the initial (lowest) trivalent node is painted (type (2)). Each of these consists of a choice of two painted binary subtrees whose root is the initial painted node, and whose leaves must sum to n. Thus we sum over the ways that n can be split into two natural numbers. □

3.3. *Remark.* This formula gives the sequence which begins: $0, 1, 2, 6, 21, 80, 322, 1348, 5814\ldots$. It is sequence A121988 of the On-line Encyclopedia of integer sequences. The recursive formula above yields the equation

$$A(x) = xc(x) + (A(x))^2$$

where $A(x)$ is the ordinary generating function of the sequence a_n above and $c(x)$ is the generating function for the Catalan numbers $C(n)$. (So $xc(x)$ is the generating function for the sequence $\{C(n-1)\}_{n=0}^{\infty}$.) Recall that $c(x) = \frac{1-\sqrt{1-4x}}{2x}$. Thus by use of the quadratic formula we have

$$A(x) = \frac{1 - \sqrt{2\sqrt{1-4x} - 1}}{2}.$$

It is not hard to check that therefore $A(x) = xc(x)c(xc(x))$. The Catalan transform of a sequence b_n with generating function $B(x)$ is defined in [4] as the sequence with generating function $B(xc(x))$. Since $xc(x)$ is the generating function of $C(n-1)$ then the number of vertices of the n^{th} multiplihedron is given by the Catalan transform of the Catalan numbers $C(n-1)$. Thus the theorems of [4] apply, for instance: a formula for the number of vertices is given by

$$a_n = \frac{1}{n}\sum_{k=1}^{n} \binom{2n-k-1}{n-1}\binom{2k-2}{k-1} \ ; \ a_0 = 0.$$

4. Spaces of painted trees

Boardman and Vogt develop several versions of the space of colored or painted trees with n leaves. Some are more useful than others for their purposes in proving theorems about A_∞ maps. We choose to focus on one version which has the advantage of reflecting the intuitive dimension of the multiplihedra. The points of this space are based on the binary painted trees with the three types of nodes pictured in the introduction. The leaves are always colored by the domain X (here we say unpainted), and the root is always colored by the range, Y (here we say painted).

To get a point of the space each interior edge of a given binary painted tree with n leaves is assigned a value in $[0,1]$. When none of the nodes are painted (that is, disallowing the second node type), and with the equivalence relations we will review shortly, this will become the space $SM\mathcal{U}(n,1)$ as defined in [10]. Allowing all three types of nodes gives the space

$$HW(\mathcal{U} \otimes \mathcal{L}_1)(n^0, 1^1).$$

(In [10] the superscripts denote the colors, so this denotes that there are n inputs colored "0" and one output colored "1." This is potentially confusing since these numbers are also used for edge lengths, and so in this paper we will denote coloring with the shaded edges and reserve the values to denote edge lengths.)

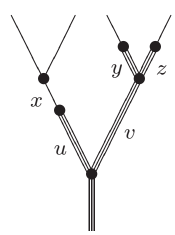
We want to consider the retract of this space to the level trees, denoted

$$LW(\mathcal{U} \otimes \mathcal{L}_1)(n^0, 1^1).$$

We will just call it $LW\mathcal{U}(n)$. One way to describe this space is by introducing relations on the lengths of edges. Edge lengths can be chosen freely from $[0,1]$ subject to the following conditions. At each trivalent node of a tree t there are two subtrees with their root that node. The left subtree is defined by the tree with its rooted edge the left-hand branch of that node and the right subtree is likewise supported by the righthand branch. The conditions are that for each node of type (2) we have an equation relating the painted interior edge lengths of the left subtree and the right subtree (interior with respect to the original t). Let $u_1 \dots u_k$ be the lengths of the painted interior edges of the left subtree and let $v_1 \dots v_j$ be the painted lengths of the right subtree. Let p_u be the number of leaves of the left subtree and let p_v be the number of leaves of the right subtree. The equation to be obeyed is

$$\frac{1}{p_u}\sum_{i=1}^{k} u_i = \frac{1}{p_v}\sum_{i=1}^{j} v_i.$$

For example consider the edge lengths assigned to the following tree:



The relations on the lengths then are the equations:

$$y = z \qquad \text{and} \qquad \frac{1}{2}u = \frac{1}{2}(v + y + z).$$

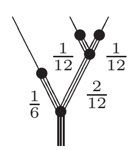
Note that this will sometimes imply that lengths of certain edges are forced to take values only from $[0,p], p < 1$.

We have now described a space corresponding to each painted binary tree. Before describing how to glue together all these subspaces for different trees to create the entire $LW\mathcal{U}(n)$ we show the following:

4.1. **Theorem.** *The dimension of the subspace of $LW\mathcal{U}(n)$ corresponding to a given binary painted tree is $n-1$.*

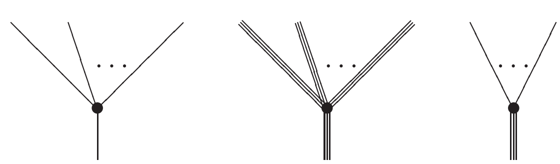
Proof. After assigning variables to the internal edges and applying the relations, the total number of free variables is at least the number of interior edges less the number of painted, type (2), nodes. This difference is always one less than the number of leaves. To see that the constraining equations really do reduce the number of free variables to $n-1$, notice what the equations imply about the painted interior edge lengths (the unpainted edge lengths are all free variables.) Beginning at the painted nodes which are closest to the leaves and setting equal to zero one of the two branches (a free variable) at each node it is seen that all the painted interior edge lengths are forced to be zero. Thus each painted node can only contribute one free variable–the other branch length must be dependent. In fact, given a painted binary tree with n leaves and k internal edges, the space of points corresponding to the allowed choices for the edge values of that tree is the intersection of an $(n-1)$-dimensional subspace of $\mathbf{R}^k$ with $[0,1]^k$. We see this simply by solving the system of homogeneous equations indicated by the type (2) nodes and restricting our solution to the lengths in $[0,1]$.

In fact, the intersection just described is an $(n-1)$-dimensional polytope in $\mathbf{R}^k$. We see that this is true since there is a point in the intersection for which each of the coordinates is in the range $(0, \frac{1}{2}]$. To see an example of such a point we consider edge lengths of our binary tree such that the unpainted edges each have length $\frac{1}{2}$ and such that the painted edges have lengths in $(0, \frac{1}{2}]$. To achieve the latter we begin at the first painted type (2) node above the root, and consider the left and right subtrees. If the left subtree has only one painted edge we assign that edge the length $\frac{p}{2n}$ where p is the number of leaves of the left subtree; but if not then we assign the root edge of the left subtree the length $\frac{p}{4n}$. We do the same for the right subtree, replacing p with the number of leaves of the right subtree. This proceeds inductively up the tree. At a given type (2) node if its left/right p'-leaved subtree has only one painted edge we assign that edge the length $\frac{p'}{d}$ where d is the denominator of the length assigned to the third edge (closest to the root) of the that node on the previous step; but if not then we assign the root edge of the left/right subtree the length $\frac{p'}{2d}$. This produces a set of lengths which obey the relations and are all $\leq \frac{1}{2}$. For example:



□

To describe the equivalence relations on our space we recall the trees with three additional allowed node types. They correspond to the the node types (1), (2) and (3) in that they are painted in similar fashion.

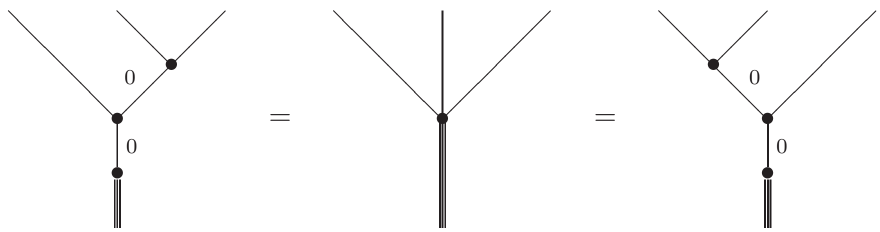


(4) (5) (6)

These nodes each have subtrees supported by each of their branches in order from left to right. The interior edges of each tree are again assigned lengths in [0, 1]. The requirements on edge lengths which we get from each node of type (5) of valence $j+1$ are the equalities:

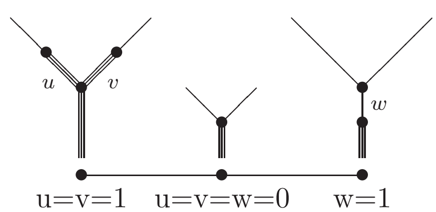
$$\frac{1}{p_1}\sum_{i=1}^{k_1} u_{1i} = \frac{1}{p_2}\sum_{i=1}^{k_2} u_{2i} = \cdots = \frac{1}{p_j}\sum_{i=1}^{k_j} u_{ji}$$

where $k_1 \dots k_j$ are the numbers of painted internal edges of each of the j subtrees, and $p_1 \dots p_j$ are the numbers of leaves of each of the subtrees. Now we review the equivalence relation on trees introduced in [10]. Two trees are equivalent if they reduce to the same tree after shrinking to points their respective edges of length zero. This is why we call the variable assigned to interior edges "length" in the first place. By "same tree" we mean possessing the same painted tree structure and having the same lengths assigned to corresponding edges. For example one pair of equivalence relations appears as follows:

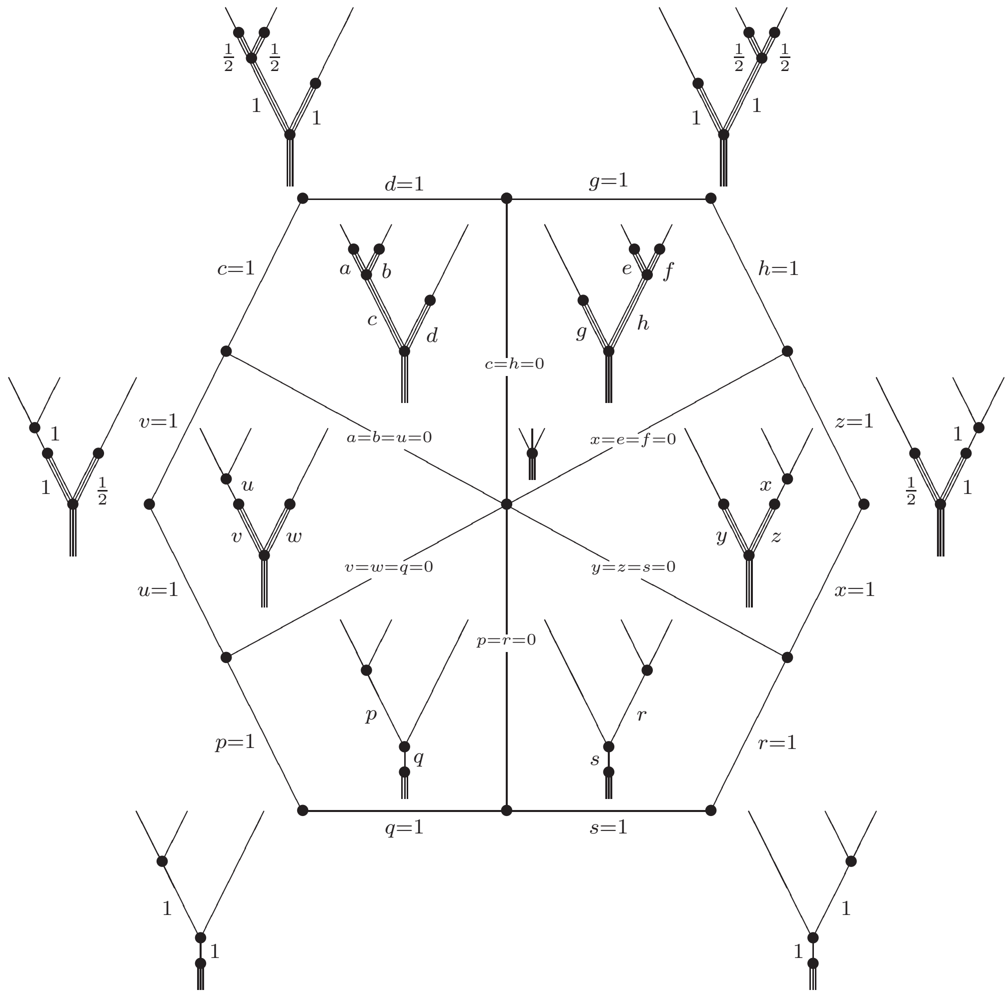


Note that an equivalence class of trees may always be represented by a binary tree, with only nodes of type (1), (2), and (3), since we can reduce the valence of nodes within an equivalence class by introducing extra interior edges of length zero. However we often represent the equivalence class with the tree that shows no zero edges.

Now the space of painted trees with n leaves is given the quotient topology of the standard topology of the disjoint union of $(n-1)$-dimensional polytopes in $\mathbf{R}^k$, one polytope for each binary painted tree, under the equivalence relation just described.
$LW\mathcal{U}(1)$ is just a single point. Here is the space $LW\mathcal{U}(2)$, where we require $u = v$:



And here is the space $LW\mathcal{U}(3)$:



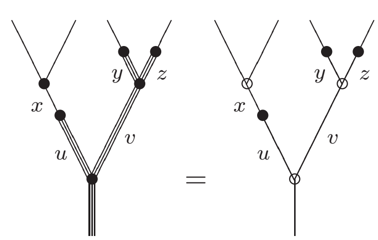
Note that the equations which the variables in $LW\mathcal{U}(3)$ must obey are:

$$a = b \qquad \text{and} \qquad d = \frac{1}{2}(a + b + c)$$

$$e = f \qquad \text{and} \qquad g = \frac{1}{2}(e + f + h)$$

$$w = \frac{1}{2}v \qquad \text{and} \qquad y = \frac{1}{2}z$$

In [48] the space of painted trees (bicolored metric ribbon trees) is described in a slightly different way. First, the trees are not drawn with painted edges, but instead the nodes of type (3) are indicated by color, and the edges between the root and those nodes can be assumed to be painted. The correspondence is clear: for example,



Secondly, the relations required of the painted lengths are different. In [48] it is required that the sum of the painted lengths along a path from the root to a leaf must always be the same. For example, for the above tree, the new relations obeyed in [48] are $u = v + y = v + z$. This provides the same dimension of $n - 1$ for the space associated to a single binary tree with n leaves as found in Theorem 4.1 in this paper.

Thirdly the topology on the space of painted trees with n leaves is described by first assigning lengths in $(0, \infty)$ and then defining the limit as some lengths in a given tree approach 0 as being the tree with those edges collapsed. This topology clearly is equivalent to the definition as a quotient space given here and in [10]. Thus we can use the results of [48] to show the following:

4.2. **Lemma.** *The space $LW\mathcal{U}(n)$ is homeomorphic to the closed ball in $\boldsymbol{R}^{n-1}$.*

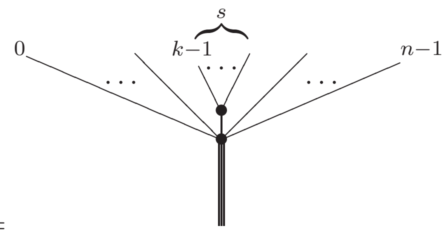
Proof. In [48] it is shown that the entire space of painted trees with n leaves with lengths in $[0, \infty)$ is homeomorphic to $\mathbf{R}_+^{n-1} \cup \mathbf{0}$. (This is done via a homeomorphism to the space of quilted disks.) Thus if the lengths are restricted to restricted to lie in $[0, 1]$ then the resulting space is homeomorphic to the closed ball in $\mathbf{R}^{n-1}$. □

5. Proof of Theorem 3.1

To demonstrate that our convex hulls are each combinatorially equivalent to the corresponding convex CW-complexes defined by Iwase and Mimura, we need only check that they both have the same vertex-facet incidence. We will show that for each n there is an isomorphism f between the vertex sets (0-cells) of our convex hull and $\mathcal{J}(n)$ which preserves the sets of vertices corresponding to facets; i.e. if S is the set of vertices of a facet of our convex hull then $f(S)$ is a vertex set of a facet of $\mathcal{J}(n)$. To demonstrate the existence of the isomorphism, noting that the vertices of $\mathcal{J}(n)$ correspond to the binary painted trees, we only need to check that the points we calculate from those binary painted trees are indeed the vertices of their convex hull. The isomorphism implied is the one that takes a vertex associated to a certain tree to the 0-cell associated to the same tree. Now a given facet of $\mathcal{J}(n)$ corresponds to a tree T which is one of the two sorts of trees pictured in the proof of Theorem 2.1. To show that our implied isomorphism of vertices preserves vertex sets of facets we need to show that that our facet is the convex hull of the points corresponding to the binary trees which are *refinements* of T. By refinement of painted trees we refer to the relationship: t refines t' if t' results from the collapse of some of the internal edges of t. Note that the two sorts of trees pictured in the proof of Theorem 2.1 are each a single collapse away from being the partly painted corolla.

The proofs of both key points will proceed in tandem, and will be inductive. The main strategy will be to define a dimension $n - 2$ affine hyperplane $H_q(t)$ in $\mathbf{R}^{n-1}$ for each of the upper an lower facet trees t (as drawn in the proof of Theorem 2.1), and then to show that these are the bounding hyperplanes of the convex hull. The definition of hyperplane will actually generalize our algorithm for finding a point $M_q(t)$ in $\mathbf{R}^{n-1}$ from a binary tree t with n leaves.

The lower facets $\mathcal{J}_k(r,s)$ correspond to lower trees such as:

$l(k,s) =$ 

These are assigned a hyperplane $H_q(l(k,s))$ determined by the equation

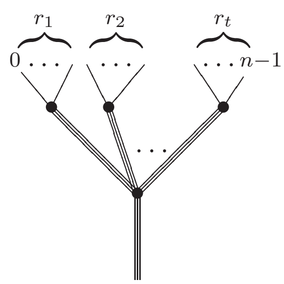
$$x_k + \cdots + x_{k+s-2} = \frac{q}{2}s(s-1).$$

Recall that r is the number of branches extending from the lowest node, and $r+s = n+1$. Thus $1 \le k \le r$. Notice that if $s = n$ (so $r = k = 1$) then this becomes the hyperplane given by

$$x_1 + \cdots + x_{n-1} = \frac{q}{2}n(n-1) = qS(n-1).$$

Therefore the points $M_q(t)$ for t a binary tree with only nodes type (1) and (3) will lie in the hyperplane $H_q(l(1,n))$ by Lemma 2.5 of [40]. (Simply multiply both sides of the relation proven there by q.) Also note that for $q = 1$ (thus disregarding the painting) that these hyperplanes are an alternate to the bounding hyperplanes of the associahedron defined by Loday using admissible shuffles. Our hyperplanes (for $q = 1$) each have the same intersection with the hyperplane H as does the corresponding hyperplane H_ω defined by Loday (for ω corresponding to the unpainted version of our tree $l(k,s)$.)

The upper facets $\mathcal{J}(t; r_1, \ldots, r_t)$ correspond to upper trees such as:

$u(t; r_1, \ldots, r_t) =$ 

These are assigned a hyperplane $H_q(u(t; r_1, \ldots, r_t))$ determined by the equation

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} = \frac{1}{2}\left(n(n-1) - \sum_{i=1}^{t} r_i(r_i-1)\right)$$

or equivalently:

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} = \sum_{1\le i<j\le t} r_i r_j.$$

Note that if $t = n$ (so $r_i = 1$ for all i) that this becomes the hyperplane given by

$$x_1 + \cdots + x_{n-1} = \frac{1}{2}n(n-1) = S(n-1).$$

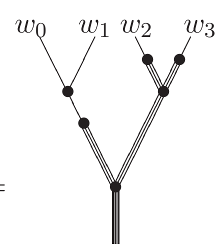
Therefore the points $M_q(t)$ for t a binary tree with only nodes type (2) and (3) will lie in the hyperplane H by Lemma 2.5 of [40].

Proof of Theorem 3.1:

In order to prove the theorem it turns out to be expedient to prove a more general result. This consists of an even more flexible version of the algorithm for assigning points to binary trees in order to achieve a convex hull of those points which is the multiplihedron. To assign points in $\mathbf{R}^{n-1}$ to the binary painted trees with n leaves, we not only choose a value $q \in (0,1)$ but also an ordered n-tuple of positive integers $w_0, \dots, w_{n-1}$. Now given a tree t we calculate a point $M_q^{w_0,\dots,w_{n-1}}(t)$ in $\mathbf{R}^{n-1}$ as follows: we begin by assigning the weight w_i to the i^{th} leaf. We refer to the result as a weighted tree. Then we modify Loday's algorithm for finding the coordinate for each trivalent node by replacing the number of leaves of the left and right subtrees with the sums of the weights of the leaves of those subtrees. Thus we let $L_i = \sum w_k$ where the sum is over the leaves of the subtree supported by the left branch of the i^{th} node. Similarly we let $R_i = \sum w_k$ where k ranges over the leaves of the the subtree supported by the right branch. Then

$$M_q^{w_0,\dots,w_{n-1}}(t) = (x_1, \dots x_{n-1}) \text{ where } x_i = \begin{cases} qL_iR_i, & \text{if node } i \text{ is type (1)} \\ L_iR_i, & \text{if node } i \text{ is type (2).} \end{cases}$$

Note that the original points $M_q(t)$ are recovered if $w_i = 1$ for $i = 0, \dots, n-1$. Thus proving that the convex hull of the points $M_q^{w_0,\dots,w_{n-1}}(t)$ where t ranges over the binary painted trees with n leaves is the n^{th} multiplihedron will imply the main theorem. For an example, let us calculate the point in $\mathbf{R}^3$ which corresponds to the 4-leaved tree:

$t =$ 

Now $M_q^{w_0,\dots,w_3}(t) = (qw_0w_1, (w_0+w_1)(w_2+w_3), w_2w_3)$. To motivate this new weighted version of our algorithm we mention that the weights $w_0, \dots, w_{n-1}$ are to be thought of as the sizes of various trees to be grafted to the respective leaves. This weighting is therefore necessary to make the induction go through, since the induction is itself based upon the grafting of trees.

5.1. **Lemma.** *For $q = 1$ the convex hull of the points $M_q^{w_0,\dots,w_{n-1}}(t)$ for t an n-leaved binary tree gives the n^{th} associahedron.*

Proof. Recall that for $q = 1$ we can ignore the painting, and thus for $w_i = 1$ for $i = 0, \dots, n-1$ the points we calculate are exactly those calculated by Loday's algorithm. Now for arbitrary weights $w_0, \dots, w_{n-1}$ we can form from each weighted tree t (those weights assigned to the respective leaves) a non-weighted tree t' formed by grafting a corolla with w_i leaves onto the i^{th} leaf of t. Note that for binary trees which are refinements of t' the coordinates which correspond to the nodes of t' below the grafting receive precisely the same value from Loday's algorithm which the corresponding nodes of the original weighted tree received from the weighted algorithm. Now since Loday's algorithm gives the vertices of the associahedra, then the binary trees which are refinements of t' give the vertices of $\mathcal{K}(n) \times \mathcal{K}(w_0) \times \cdots \times \mathcal{K}(w_{n-1})$. If we restrict our attention from the entire binary refinements of t' to the nodes of (the refinements of) the grafted corolla with w_i leaves we find the vertices of $\mathcal{K}(w_i)$. The definition of a cartesian product of polytopes guarantees that the vertices of the product are points which are cartesian products of the vertices of the operands. Polytopes are also combinatorially invariant under change of basis, and so we can rearrange the coordinates of our vertices to put all the coordinates corresponding to the nodes of (the refinements of) the grafted corollas at the end of the point, leaving the coordinates corresponding to the nodes below the graft in order at the beginning of the point. Thus the nodes below the grafting correspond to the vertices of $\mathcal{K}(n)$, and so the weighted algorithm (with $q = 1$) does give the vertices of $\mathcal{K}(n)$. □

5.2. **Lemma.** *For $q = 1$ the points $M_q^{w_0,\dots,w_{n-1}}(t)$ for t an n-leaved binary tree all lie in the $n-2$ dimensional affine hyperplane of $\mathbf{R}^{n-1}$ given by the equation:*

$$x_1 + \cdots + x_{n-1} = \sum_{1 \le i < j \le (n-1)} w_i w_j.$$

Proof. In Lemma 2.5 of [40] it is shown inductively that when $w_i = 1$ for $i = 1, \dots, n-1$ then the point $M_1^{1,\dots,1}(t) = M(t) = (x_1, \dots, x_{n-1})$ satisfies the equation $\sum_{i=1}^{n-1} x_i = \frac{1}{2}n(n-1)$. As in the proof of the previous lemma we replace the weighted tree t with the non-weighted t' formed by grafting an arbitrary binary tree with w_i leaves to the i^{th} leaf of t. Let $m = \sum_{i=1}^{n-1} w_i$. Thus the point $M_1^{1,\dots,1}(t') = M(t') = (x_1, \dots, x_m)$ satisfies the equation

$$\sum_{i=1}^{m-1} x_i = \frac{1}{2}m(m-1) = \frac{1}{2}\sum_{i=1}^{n-1} w_i(\sum_{i=1}^{n-1} w_i - 1).$$

Also the coordinates corresponding to the nodes of the grafted tree with w_i leaves sum up to the value $\frac{1}{2}w_i(w_i - 1)$. Thus the coordinates corresponding to the nodes below the graft, that is, the coordinates of the original weighted tree t, sum up to the difference:

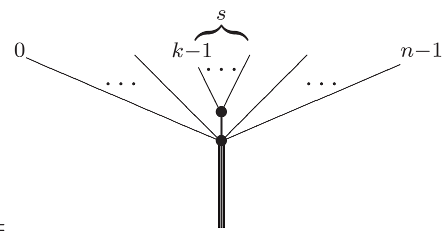
$$\frac{1}{2}\left(\sum_{i=1}^{n-1} w_i(\sum_{i=1}^{n-1} w_i - 1) - \sum_{i=1}^{n-1} w_i(w_i - 1)\right) = \sum_{1 \le i < j \le (n-1)} w_i w_j$$

□

Since we are proving that the points $M_q^{w_0,\dots,w_{n-1}}(t)$ are the vertices of the multiplihedron, we need to define hyperplanes $H_q^{w_0,\dots,w_{n-1}}(t)$ for this weighted version which we will show to be the the bounding hyperplanes when t has refinement degree 1.

5.3. **Definition.**

Recall that the lower facets $\mathcal{J}_k(r,s)$ correspond to lower trees such as:

$l(k,s) =$ 

These are assigned a hyperplane $H_q^{w_0,\dots,w_{n-1}}(l(k,s))$ determined by the equation

$$x_k + \cdots + x_{k+s-2} = q\left(\sum_{(k-1)\le i<j\le(k+s-2)} w_i w_j\right).$$

Recall that r is the number of branches from the lowest node, and $r+s=n+1$.

5.4. **Lemma.** *For any painted binary tree t the point $M_q^{w_0,\dots,w_{n-1}}(t)$ lies in the hyperplane $H_q^{w_0,\dots,w_{n-1}}(l(k,s))$ iff t is a refinement of $l(k,s)$. Also the hyperplane $H_q^{w_0,\dots,w_{n-1}}(l(k,s))$ bounds below the points $M_q^{w_0,\dots,w_{n-1}}(t)$ for t any binary painted tree.*

Proof. By Lemma 5.2 we have that any binary tree t which is a refinement of the lower tree $l(k,s)$ will yield a point $M_q^{w_0,\dots,w_{n-1}}(t)$ which lies in $H_q^{w_0,\dots,w_{n-1}}(l(k,s))$. To see this we simply note that the nodes in t associated to the coordinates $x_k,\dots,x_{k+s-2}$ in $M_q^{w_0,\dots,w_{n-1}}(t)$ will each be of type (1), and so we multiply by q on both sides of the equation proven in the Lemma.
We now demonstrate that if a binary tree t is not a refinement of a lower tree $l(k,s)$ then the point $M_q^{w_0,\dots,w_{n-1}}(t)$ will have the property that

$$x_k + \cdots + x_{k+s-2} > q\left(\sum_{(k-1)\le i<j\le(k+s-2)} w_i w_j\right).$$

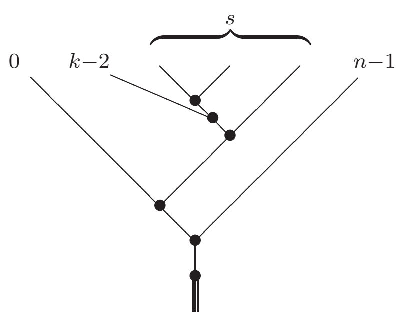
Recall that the trees which are refinements of $l(k,s)$ have all their nodes inclusively between k and $k+s-2$ of type (1). Now if t has these same $s-1$ nodes $k,\dots,k+s-2$ all type (1) and is not a refinement of $l(k,s)$ then there is no node in t whose deletion results in the separation of only the leaves $k-1,\dots,k+s-2$ from the rest of the leaves of t. Let t' be the subtree of t determined by taking as its root the node furthest from the root of t whose deletion results in the separation of all the leaves $k-1,\dots,k+s-2$ from the rest of the leaves of t. Thus t' will have more than just those s leaves, say those leaves of t labeled $k-p,\dots,k+q-2$ where $p\ge 1,\ q\ge s$ and at least one of the inequalities strict. Since the situation is symmetric we just consider the case where $q=s$ and $p>1$. Then we have an expression for the sum of all the coordinates whose nodes are in t' and can write:

$$(*)\ x_k + \cdots + x_{k+s-2} = q\left(\sum_{(k-p)\le i<j\le(k+s-2)} w_i w_j\right) - q(x_{k-p+1} + \cdots + x_{k-1}).$$

Notice that the first sum on the right hand side of $(*)$ contains

$$x_{k-p+1} + \cdots + x_{k-1} + \sum_{(k-1)\leq i<j\leq(k+s-2)} w_i w_j.$$

(There is no overlap between the coordinate values here and the sum since each of the terms in $x_{k-p+1}+\cdots+x_{k-1}$ contains a factor from $w_{k-p},\ldots,w_{k-2}$.) The first sum on the right hand side of $(*)$ also contains at least one term $w_m w_j$ where $(k-p) \leq m \leq (k-2)$ and where $w_m w_j$ dose not occur as a term in $x_{k-p+1} + \cdots + x_{k-1}$, else the leaf labeled by m would not lie in t'. Thus we have the desired inequality. Here is a picture of an example situation, where $p = 2$. Note that the key term $w_m w_j$ in the above discussion is actually $w_{k-2}w_{k+1}$ in this picture.

$t =$ 

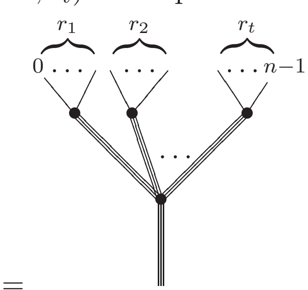
Now if in the situation for which there does not exist a node of t which if deleted would separate exactly the leaves $k-1,\ldots,k+s-2$ from the other leaves and root of t, there are also some of the nodes in $k,\ldots,k+s-1$ of type (2), the inequality still holds, and now to a greater degree since some of the factors of q are missing from the right hand side.

If there does exist a node of t which if deleted would separate exactly the leaves $k-1,\ldots,k+s-2$ from the other leaves and root of t, but t is not a refinement of $l(k,s)$ due to the painting (some of the nodes in $k,\ldots,k+s-1$ are of type (2)), then the inequality holds precisely because the only difference left to right is that the right hand side has fewer terms multiplied by the factor of q.

□

5.5. **Definition.**

Recall that the upper facets $\mathcal{J}(t; r_1,\ldots,r_t)$ correspond to upper trees such as:

$u(t; r_1,\ldots,r_t) =$ 

These are assigned a hyperplane $H_q^{w_0,\ldots,w_{n-1}}(u(t; r_1,\ldots,r_t))$ determined by the equation

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} = \sum_{1\leq i<j\leq t} R_i R_j.$$

where $R_i = \sum w_j$ where the sum is over the leaves of the i^{th} subtree (from left to right) with root the type (5) node; the index j goes from $(r_1+r_2+\cdots+r_{i-1})$ to $(r_1+r_2+\cdots+r_i-1)$ (where $r_0 = 0$.) Note that if $t = n$ (so $r_i = 1$ for all i) that this becomes the hyperplane given by

$$x_1 + \cdots + x_{n-1} = \sum_{1 \le i < j \le n-1} w_i w_j.$$

5.6. **Lemma.** *For any painted binary tree t the point $M_q^{w_0,\dots,w_{n-1}}(t)$ lies in the hyperplane $H_q^{w_0,\dots,w_{n-1}}(u(t;r_1,\dots,r_t))$ iff t is a refinement of $u(t;r_1,\dots,r_t)$. Also the hyperplane $H_q^{w_0,\dots,w_{n-1}}(u(t;r_1,\dots,r_t))$ bounds above the points $M_q^{w_0,\dots,w_{n-1}}(t)$ for t any binary painted tree.*

Proof. Now by by Lemma 5.2 we have that any binary tree t which is a refinement of the upper tree $u(t;r_1,\dots,r_t)$ will yield a point $M_q^{w_0,\dots,w_{n-1}}(t)$ which lies in $H_q^{w_0,\dots,w_{n-1}}(u(t;r_1,\dots,r_t))$. To see this we simply note that the coordinates $x_{r_1}, x_{(r_1+r_2)}, \dots, x_{(r_1+r_2+\cdots+r_{t-1})}$ in $M_q^{w_0,\dots,w_{n-1}}(t)$ will each be assigned the same value as if the original upper tree had had $r_i = 1$ for all i but where the weights given were $R_0, \dots R_{n-1}$.

We now demonstrate that if a binary tree T is not a refinement of an upper tree $u(t;r_1,\dots,r_t)$ then the point $M_q^{w_0,\dots,w_{n-1}}(T)$ will have the property that

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} < \sum_{1 \le i < j \le t} R_i R_j.$$

Recall that $R_i = \sum_j w_j$ where the sum is over the leaves of the i^{th} subtree (from left to right) with root the type (5) node; the index j goes from $(r_1 + r_2 + \cdots + r_{i-1})$ to $(r_1 + r_2 + \cdots + r_i - 1)$ (where $r_0 = 0$.) If T is not a refinement of $u(t;r_1,\dots,r_t)$ then for some of the partitioned sets of r_i leaves in the partition $r_1,\dots,r_t$ it is true that there does not exist a node of T which if deleted would separate exactly the leaves in that set from the other leaves and root of T. Thus the proof here will use the previous result for the lower trees. First we consider the case for which T is entirely painted–it has only type (2) nodes. Now by Lemma 5.2 the total sum of the coordinates of $M_q^{w_0,\dots,w_{n-1}}(T)$ will be equal to $\sum_{1\le i<j\le n-1} w_i w_j$. Consider a (partitioned) set of r_m leaves (starting with leaf $k-1$) in the partition $r_1,\dots,r_t$ for which there does not exist a node of T which if deleted would separate exactly the leaves in that set from the other leaves and root of T. (Here $k - 1 = r_1 + r_2 + \cdots + r_{m-1}$) Let P_m be the sum of the $r_m - 1$ coordinates $x_k + \cdots + x_{k+r_m-2}$. We have by the same argument used for lower trees that

$$P_m > \sum_{(k-1) \le i < j \le (k+r_m-2)} w_i w_j.$$

Now for this T, for which some of the partitioned sets of r_i leaves in the partition $r_1,\dots,r_t$ there does not exist a node of T which if deleted would separate exactly the leaves in that set from the other leaves and root of T, we have:

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} = \sum_{1\le i<j\le n-1} w_i w_j - \sum_{m=1}^{t} P_m < \sum_{1\le i<j\le t} R_i R_j.$$

If a tree T' has the same branching structure as T but with some nodes of type (1) then the argument still holds since the argument from the lower trees still applies. Now for a tree T whose branching structure is a refinement of the branching structure of the upper tree $u(t; r_1, \ldots, r_t)$, but which has some of its nodes $r_1, (r_1+r_2), \ldots, (r_1+r_2+\cdots+r_{t-1})$ of type (1), the inequality holds simply due to the application of some factors q on the left hand side. □

Proof. of Theorem 3.1: Now we may proceed with our inductive argument. The base case of $n = 2$ leaves is trivial to check. The points in $\mathbf{R}^1$ are w_0w_1 and qw_0w_1. Their convex hull is a line segment, combinatorially equivalent to $\mathcal{J}(2)$. Now we assume that for all $i < n$ and for arbitrary $q \in (0,1)$ and for positive integer weights $w_0, \ldots, w_{i-1}$, that the convex hull of the points $\{M_q^{w_0,\ldots,w_{i-1}}(t) \mid t \text{ is a painted binary tree with } i \text{ leaves}\}$ in $\mathbf{R}^{i-1}$ is combinatorially equivalent to the complex $\mathcal{J}(i)$, and that the points $M_q^{w_0,\ldots,w_{i-1}}(t)$ are the vertices of the convex hull. Now for $i = n$ we need to show that the equivalence still holds. Recall that the two items we plan to demonstrate are that the points $M_q^{w_0,\ldots,w_{n-1}}(t)$ are the vertices of their convex hull and that the facet of the convex hull corresponding to a given lower or upper tree T is the convex hull of just the points corresponding to the binary trees that are refinements of T. The first item will be seen in the process of checking the second.

Given an n-leaved lower tree $l(k,s)$ we have from Lemma 5.4 that the points corresponding to binary refinements of $l(k,s)$ lie in an $n-2$ dimensional hyperplane $H_q^{w_0,\ldots,w_{n-1}}(l(k,s))$ which bounds the entire convex hull. To see that this hyperplane does indeed contain a facet of the entire convex hull we use the induction hypothesis to show that the dimension of the convex hull of just the points in $H_q^{w_0,\ldots,w_{n-1}}(l(k,s))$ is $n-2$. Recall that the tree $l(k,s)$ is the result of grafting an unpainted s-leaved corolla onto leaf $k-1$ of an r-leaved partly painted corolla. Thus the points $M_q^{w_0,\ldots,w_{n-1}}(t)$ for t a refinement of $l(k,s)$ have coordinates $x_k, \ldots, x_{k+s-1}$ which are precisely those of the associahedron $\mathcal{K}(s)$, by Lemma 5.1 (after multiplying by q). Now considering the remaining coordinates, we see by induction that they are the coordinates of the multiplihedron $\mathcal{J}(r)$. This is by process of considering their calculation as if performed on an r-leaved weighted tree t' formed by replacing the subtree of t (with leaves $x_{k-1}, \ldots, x_{k+s-1}$) with a single leaf of weight $\sum_{j=k-1}^{k+s-1} w_j$. Now after a change of basis to reorder the coordinates, we see that the points corresponding to the binary refinements of $l(k,s)$ are the vertices of a polytope combinatorially equivalent to $\mathcal{J}(r) \times \mathcal{K}(s)$ as expected. Since $r + s = n+1$ this polytope has dimension $r - 1 + s - 2 = n - 2$, and so is a facet of the entire convex hull.

Given an n-leaved upper tree $u(t, r_1, \ldots, r_t)$ we have from Lemma 5.6 that the points corresponding to binary refinements of $u(t, r_1, \ldots, r_t)$ lie in an $n-2$ dimensional hyperplane $H_q^{w_0,\ldots,w_{n-1}}(u(t, r_1, \ldots, r_t))$ which bounds the entire convex hull. To see that this hyperplane does indeed contain a facet of the entire convex hull we use the induction hypothesis to show that the dimension of the convex hull of just the points in $H_q^{w_0,\ldots,w_{n-1}}(u(t, r_1, \ldots, r_t))$ is $n-2$. Recall that the tree $u(t, r_1, \ldots, r_t)$ is the result of grafting partly painted r_i-leaved corollas onto leaf i of a t-leaved completely painted corolla. Thus the points $M_q^{w_0,\dots,w_{n-1}}(t)$ for T a refinement of $u(t, r_1, \dots, r_t)$ have coordinates corresponding to the nodes in the i^{th} subtree which are precisely those of the multiplihedron $\mathcal{J}(r_i)$, by the inductive hypothesis. Now considering the remaining coordinates, we see by Lemma 5.1 that they are the coordinates of the associahedron $\mathcal{J}(t)$. This is by process of considering their calculation as if performed on an t-leaved weighted tree T' formed by replacing each (grafted) subtree of T (with r_i leaves) with a single leaf of weight $\sum_j w_j$, where the sum is over the r_i leaves of the i^{th} grafted subtree. Now after a change of basis to reorder the coordinates, we see that the points corresponding to the binary refinements of $u(t, r_1, \dots, r_t)$ are the vertices of a polytope combinatorially equivalent to $\mathcal{K}(t) \times \mathcal{J}(r_1) \times \cdots \times \mathcal{J}(r_t)$ as expected. Since $r_1 + \cdots + r_t = n$ this polytope has dimension $t - 2 + (r_1 - 1) + (r_2 - 1) + \cdots + (r_t - 1) = n - 2$, and so is a facet of the entire convex hull.

Since each n-leaved binary painted tree is a refinement of some upper and or or lower trees, then the point associated to that tree is found as a vertex of some of the facets of the entire convex hull, and thus is a vertex of the convex hull. This completes the proof. Since the dimension of $\mathcal{J}(n)$ is $n - 1$, we have also shown that our convex hull is homeomorphic to the space of painted trees $LW\mathcal{U}(n)$.

□

A picture of the convex hull giving $\mathcal{J}(4)$ is also available at
http://faculty.tnstate.edu/sforcey/ct06.htm.

The convex hull for $\mathcal{J}(5)$ with 80 vertices is also pictured there as a Schlegel diagram generated by polymake.

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E-mail address: sforcey@tnstate.edu

Department of Physics and Mathematics, Tennessee State University, Nashville, TN 37209, USA,