arXiv:0707.4633v1 [math.CO] 31 Jul 2007

GENERATING TREES FOR PERMUTATIONS AVOIDING GENERALIZED PATTERNS

SERGI ELIZALDE

Abstract. We construct generating trees with one, two, and three labels for some classes of permutations avoiding generalized patterns of length 3 and 4. These trees are built by adding at each level an entry to the right end of the permutation, which allows us to incorporate the adjacency condition about some entries in an occurrence of a generalized pattern. We use these trees to find functional equations for the generating functions enumerating these classes of permutations with respect to different parameters. In several cases we solve them using the kernel method and some ideas of Bousquet-Mélou [4]. We obtain refinements of known enumerative results and find new ones.

1. Introduction

1.1. **Generalized pattern avoidance.** We denote by $\mathcal{S}_n$ the symmetric group on $\{1, 2, \ldots, n\}$. Let n and k be two positive integers with $k \leq n$, and let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{S}_n$ be a permutation. A generalized pattern σ is obtained from a permutation $\sigma_1\sigma_2 \cdots \sigma_k \in \mathcal{S}_k$ by choosing, for each $j = 1, \ldots, k-1$, either to insert a dash - between σ_j and σ_{j+1} or not. More formally, $\sigma = \sigma_1\varepsilon_1\sigma_2\varepsilon_2 \cdots \varepsilon_{k-1}\sigma_k$, where each ε_j is either the symbol - or the empty string. With this notation, we say that π *contains* (the generalized pattern) σ if there exist indices $i_1 < i_2 < \cdots < i_k$ such that (i) for each $j = 1, \ldots, k-1$, if ε_j is empty then $i_{j+1} = i_j + 1$, and (ii) for every $a, b \in \{1, 2, \ldots, k\}$, $\pi_{i_a} < \pi_{i_b}$ if and only if $\sigma_a < \sigma_b$. In this case, $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}$ is called an *occurrence* of σ in π.

If π does not contain σ, we say that π *avoids* σ, or that it is σ*-avoiding*. For example, the permutation $\pi = 3542716$ contains the pattern 12-4-3 because it has the subsequence 3576. On the other hand, π avoids the pattern 12-43. We denote by $\mathcal{S}_n(\sigma)$ the set of permutations in $\mathcal{S}_n$ that avoid σ. More generally, if $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ is a collection of generalized patterns, we say that a permutation π is Σ-avoiding if π is σ-avoiding for all $\sigma \in \Sigma$. We denote by $\mathcal{S}_n(\Sigma)$ the set of Σ-avoiding permutations in $\mathcal{S}_n$.

We use the word *length* to refer to the number of letters in a permutation, so that $\mathcal{S}_n$ is the set of permutations of length n. A *class* will consist of a set (e.g., all permutations avoiding a given pattern) together with a function (e.g., the length). Given a permutation $\pi \in \mathcal{S}_n$, we will write $r(\pi) = \pi_n$ to denote the rightmost entry of π. In all our generating functions, the variable t will mark the length of the permutation.

1.2. **Generating trees.** Generating trees are a useful tool for enumerating classes of pattern-avoiding permutations (see, for example, [17, 18]). The nodes at each level of the generating tree are indexed by permutations of a given length. It is common in the literature to define the children of a permutation π of length n to be those permutations that are obtained by inserting the entry $n+1$ in $\pi = \pi_1\pi_2 \cdots \pi_n$ in such a way that the new permutation is still in the class. In this paper we consider a variation of this definition. Here, the children of a permutation π of length n are obtained by appending an entry to the right of π, and adding one to all the entries in π that were greater than or equal to the new entry. For example, if the entry 3 is appended to the right of $\pi = 24135$, the child that we obtain is 251463. Adding the new entry to the right of the permutation makes these trees well-suited to enumerate permutations avoiding generalized patterns, as we will see throughout the paper. We will refer to these trees as *rightward generating trees*. This kind of generating trees has been used in [3] to enumerate permutations avoiding sets of three generalized patterns of length three with one dash, such as $\{1\text{-}23, 2\text{-}13, 1\text{-}32\}$.

For some classes of permutations, a label (ℓ) can be associated to each node of the tree in such a way that the number of children of a permutation and their labels depend only on the label of the parent. For example, in the tree for 1-2-3-avoiding permutations, we can label each node π with $m =$

$\min\{\pi_i : \exists j < i \text{ with } \pi_j < \pi_i\}$ (or $m = n+1$ if $\pi = n \cdots 21$). Then, the children of a permutation with label (m) have labels $(m+1), (2), (3), \ldots, (m)$, corresponding to the appended entry being $1, 2, 3, \ldots, m$, respectively. This succession rule, together with the fact that the root ($\pi = 1 \in \mathcal{S}_1$) has label (2), completely determines the tree. From this rule one can derive a functional equation for the generating function that enumerates the permutations by their length and the label of the corresponding node in the tree for this class of 1-2-3-avoiding permutations. For generating trees with one label, these equations are well understood and their solutions are algebraic series. This is the case of the generating trees obtained in [3], for example.

In other cases, however, one label is not enough to describe the generating tree in terms of a succession rule. Generating trees with two labels were used in [4] to enumerate restricted permutations. In fact, the inspiration for the present paper and many of the ideas used come from Bousquet-Mélou's work. One difference is that here trees are constructed by adding at each level an entry to the right end of the permutation, which allows us to keep track of elements occurring in adjacent positions. In Section 4 we consider some classes of permutations whose rightward generating tree has three labels for each node.

1.3. **Organization of the paper.** In this paper we enumerate several families of permutations that avoid generalized patterns. What ties together the results in the different sections is the technique that we use to obtain them. The strategy consists of building a rightward generating tree for the family of permutations, translating the succession rule into a set of functional equations, and applying the kernel method to them. We have tried this strategy for a number of classes of permutations, and we have found it to work in several cases, which we include here. This is why the sets of generalized patterns that we discuss may seem somewhat arbitrary. For other patterns one can construct similar generating trees with two or three labels, but we have not been able to solve the corresponding functional equations for the generating function, so we have not included these examples here. In any event, this paper is not meant to be an exhaustive study of the sets of patterns for which this technique would work.

In general, we have looked for sets of patterns for which the rightward generating tree of the class of permutations avoiding them has a simple succession rule, once appropriate labels are chosen. In some cases, we have chosen patterns based on the elegance of their enumerating sequence, like in Section 2.2. In others, we have chosen patterns whose corresponding generating function has zero radius of convergence, as is the case in Sections 3.6, 4.1, and 4.2. These seem to be the first instances of generating functions with zero radius of convergence that arise from generating trees and the kernel method.

We have classified the sets of studied patterns depending on how many labels are needed to describe the generating tree. In Section 2 we consider some families of permutations where the tree can be described with one label, which is the value of the rightmost entry in the permutation. The results in this section are new, and all involve permutations that avoid the pattern 2-1-3. This makes the succession rules easier because this restriction prevents a permutation with rightmost value r to have a child with rightmost value greater than $r+1$.

In Section 3 we study classes of permutations where each node of the generating tree bears a pair of labels. For most of them we get rational or algebraic generating functions, and their enumeration has been done in the literature using different techniques. Section 4 contains some of the main results of the paper. We find ordinary generating functions for $\{1\text{-}23, 3\text{-}12, 34\text{-}21\}$-avoiding and $\{1\text{-}23, 34\text{-}21\}$-avoiding permutations. Both families are described by generating trees with three labels.

Additional motivation for the study of these families of permutations comes from trying to understand the possible asymptotic behaviors of the number of permutations avoiding generalized patterns (see [10]). An asymptotic analysis of the coefficients of the generating functions for $\{1\text{-}23, 3\text{-}12\}$-avoiding, $\{1\text{-}23, 34\text{-}21\}$-avoiding, and $\{1\text{-}23, 3\text{-}12, 34\text{-}21\}$-avoiding permutations that we have found may reveal that their asymptotic growth is strictly smaller than that of Bell numbers but strictly greater than exponential. This would be the first known instance of a family of pattern-avoiding permutations that exhibits such a behavior.

2. Generating trees with one label

In this section we enumerate classes of pattern-avoiding permutations whose rightward generating trees can be described by a succession rule involving only one label for each node. For some of these classes, rightward generating trees are not the only way to obtain the results, but they are a tool that works in all these cases.

The classes in this section avoid the pattern 2-1-3. Note that avoiding this pattern is equivalent to avoiding the generalized pattern 2-13. Indeed, if π contains an occurrence of 2-1-3, say $\pi_i\pi_j\pi_k$ with $\pi_j < \pi_i < \pi_k$, then there must be some index ℓ with $j \leq \ell < k$ such that $\pi_\ell < \pi_i$ and $\pi_{\ell+1} > \pi_i$, so $\pi_i\pi_\ell\pi_{\ell+1}$ is an occurrence of 2-13. For any class of permutations that avoid this pattern, the corresponding rightward generating tree has the property that the appended entry at each level can never be more than one unit larger than the entry appended at the previous level.

2.1. **$\{2\text{-}1\text{-}3, \overline{2}\text{-}31\}$-avoiding permutations.** A permutation π is said to avoid the *barred* pattern $\overline{2}$-31 if every descent in π (an occurrence of the generalized pattern 21) is part of an occurrence of 2-31; equivalently, for any index i such that $\pi_i > \pi_{i+1}$ there is an index $j < i$ such that $\pi_i > \pi_j > \pi_{i+1}$. The bar indicates that the 2 is forced whenever a 31 occurs. For example, the permutation 4627513 avoids $\overline{2}$-31, but 2475613 does not.

We use M_n to denote the n-th Motzkin number. Recall that $\sum_{n\geq 0} M_n t^n = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$. The next result seems to be a new interpretation of the Motzkin numbers.

Proposition 2.1. *The number of $\{2\text{-}1\text{-}3, \overline{2}\text{-}31\}$-avoiding permutations of size n is M_{n-1}.*

Proof. Consider the rightward generating tree for $\{2\text{-}1\text{-}3, \overline{2}\text{-}31\}$-avoiding permutations. Labeling each permutation with its rightmost entry $r = r(\pi)$, this tree is described by the succession rule

$$\begin{array}{l} (1) \\ (r) \longrightarrow (1)\ (2)\ \cdots\ (r-1)\ (r+1). \end{array}$$

Indeed, the new entry appended to the right of π cannot be greater than $\pi_n + 1$ in order for the new permutation to be 2-1-3-avoiding, and it cannot be π_n because then it would create an occurrence of 21 that is not part of an occurrence of 2-31.

Defining $D(t,u) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,\overline{2}\text{-}31)} u^{r(\pi)} t^n$, the succession rule above gives the following equation for the generating function:

$$\left(1 - \frac{t}{u-1} - tu\right) D(t,u) = tu - \frac{tu}{u-1} D(t,1). \tag{1}$$

The next step is to apply the kernel method. This technique, which has been part of mathematical folklore for decades, has recently been systematized in [1, 2, 5]. Of the two values of u as a function of t that cancel the term multiplying $D(t,u)$ on the left hand side, $u_0 = u_0(t) = \frac{1+t-\sqrt{1-2t-3t^2}}{2t}$ is a well-defined formal power series in t. Substituting $u = u_0$ in (1) gives

$$D(t,1) = u_0 - 1 = \frac{1-t-\sqrt{1-2t-3t^2}}{2t},$$

which is the generating function for the Motzkin numbers with the indices shifted by one. □

There is also a bijective proof of Proposition 2.1. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n(2\text{-}1\text{-}3, \overline{2}\text{-}31)$, we can construct a Dyck path of size n (i.e., a sequence of n Us and n Ds so that no prefix contains more Ds than Us) as follows. A right-to-left maximum of π is an entry π_i such that $\pi_i > \pi_j$ for all $j > i$. Let $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_m}$ be the right-to-left maxima of π, with $i_1 < i_2 < \cdots < i_m = n$. Consider the Dyck path

$$\varphi(\pi) = U^{i_1} D^{\pi_{i_1}-\pi_{i_2}} U^{i_2-i_1} D^{\pi_{i_2}-\pi_{i_3}} U^{i_3-i_2} \ldots D^{\pi_{i_{m-1}}-\pi_{i_m}} U^{i_m-i_{m-1}} D^{\pi_{i_m}},$$

where exponentiation indicates repetition of a step. This map is a bijection between 2-1-3-avoiding permutations and Dyck paths (see [14]), and it is not hard to see that the condition if π being $\overline{2}$-31-avoiding is equivalent to the requirement that the path contains no three consecutive steps UDU. So, we have a bijection between $\mathcal{S}_n(2\text{-}1\text{-}3, \overline{2}\text{-}31)$ and UDU-free Dyck paths of size n.

To finish the proof, we next describe a bijection due to Callan [7] between UDU-free Dyck paths of size n and Motzkin paths of length $n-1$ (i.e., sequences of $n-1$ steps U, D, and H with the same number of Us and Ds and so that no prefix contains more Ds than Us). We say that a U and a D in a Dyck path are matched if the D is to the right of the U and the letters between them form a Dyck path. Note that each step is matched with exactly another one. Given a UDU-free Dyck path, first append a D to it. Now, for each D that is immediately preceded and followed by D steps, delete it and replace its matching U with an H. Next, replace each occurrence of UDD with a D. Finally, delete the D that was appended to the path. This produces a Motzkin path of length $n-1$. The composition of these two bijections completes the bijective proof of Proposition 2.1.

2.2. **$\{$2-1-3$, \overline{2}^o$-31$\}$-avoiding permutations.** Extending the notion of barred patterns, we say that a permutation π avoids the pattern $\overline{2}^o$-31 if every descent in π is the '31' part of an odd number of occurrences of 2-31; equivalently, for any index i such that $\pi_i > \pi_{i+1}$, the number of indices $j < i$ such that $\pi_i > \pi_j > \pi_{i+1}$ is odd.

Proposition 2.2. *The number of* $\{$2-1-3$,\overline{2}^o$-31$\}$*-avoiding permutations of size* n *is*

$$|\mathcal{S}_n(\text{2-1-3}, \overline{2}^o\text{-31})| = \begin{cases} \frac{1}{2k+1}\binom{3k}{k} & \text{if } n = 2k, \\ \frac{1}{2k+1}\binom{3k+1}{k+1} & \text{if } n = 2k+1. \end{cases} \tag{2}$$

Proof. The rightward generating tree for $\{$2-1-3$,\overline{2}^o$-31$\}$-avoiding permutations is given by the succession rule

$$\begin{array}{l}(1)\\ (r) \longrightarrow \cdots (r-3)\ (r-1)\ (r+1),\end{array}$$

that is, the labels of the children of a node labeled r are the numbers $1 \le j \le r+1$ such that $r-j$ is odd. Let $J(t,u) = \sum_{n\ge1} \sum_{\pi\in\mathcal{S}_n(\text{2-1-3},\overline{2}^o\text{-31})} u^{r(\pi)}t^n = \sum_{r\ge1} J_r(t)u^r$, and let $J^e(t,u) = \sum_{r \text{ even}} J_r(t)u^r$. The succession rule translates into the following functional equation:

$$\left(1 - \frac{tu^3}{u^2-1}\right) J(t,u) = tu - \frac{tu^2}{u^2-1} J(t,1) + \frac{tu(u-1)}{u^2-1} J^e(t,1). \tag{3}$$

The kernel $1 - \frac{tu^3}{u^2-1}$ as a function in the variable u has three zeroes, two of which are complex conjugates. Denote them by $u_1 = a(t)+b(t)i$ and $u_2 = \bar{u_1} = a(t)-b(t)i$. Adding the equations $0 = u_i^2 - 1 - u_iJ(t,1) + (u_i-1)J^e(t,1)$ for $i = 1,2$, we get

$$a(t)J(t,1) = a(t)^2 - b(t)^2 - 1 + (a(t)-1)J^e(t,1),$$

and subtracting them gives

$$J(t,1) = 2a(t) + J^e(t,1).$$

Solving this system of equations for J, we get that $J(t,1) = 2a(t) - a(t)^2 - b(t)^2 - 1$. Plugging in the values of $a(t)$ and $b(t)$ yields the expression

$$J(t,1) = \frac{(2-3t)f(t)^2 + (9t-2-g(t))f(t) + (2-6t)g(t) + 54t^2 - 18t - 4}{3tf(t)^2},$$

where $g(t) = \sqrt{3(27t^2-4)}$ and $f(t) = [12tg(t) - 108t^2 + 8]^{1/3}$. It is easy to check that $J = J(t,1)$ is a root of the polynomial $tJ^3 + (3t-2)J^2 + (3t-1)J + t = 0$. Using the Lagrange inversion formula, one sees that its coefficients are given by (2), which is sequence A047749 from the On-Line Encyclopedia of Integer Sequences [16]. Observe that we can also obtain an expression for $J(t,u)$ using (3) and the fact that $J^e(t,1) = -a(t)^2 - b(t)^2 - 1$. □

It is also possible to give a direct bijective proof of Proposition 2.2 that does not use rightward generating trees. A well-known combinatorial interpretation of the numbers (2) is that they enumerate lattice paths from $(0,0)$ to $(n, \lfloor n/2 \rfloor)$ with steps $E = (1,0)$ and $N = (0,1)$ that never go above the line $y = x/2$. We next describe a bijection from $\mathcal{S}_n(\text{2-1-3}, \overline{2}^o\text{-31})$ to these paths. Let $\pi = \pi_1\pi_2\cdots\pi_n \in$

$\mathcal{S}_n(\text{2-1-3}, \overline{2}^o\text{-31})$. Let $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_m}$ be the right-to-left maxima of π, with $i_1 < i_2 < \cdots < i_m = n$. We claim that the condition that π is $\{\text{2-1-3}, \overline{2}^o\text{-31}\}$-avoiding guarantees that all the differences $\pi_{i_j} - \pi_{i_{j+1}}$ are even. To see this, fix j and let $\mathcal{O}$ be the set of entries a such that $a\pi_{i_j}\pi_{i_j+1}$ is an occurrence of 2-31. Since π avoids $\overline{2}^o$-31, the cardinality of $\mathcal{O}$ is odd. Now, every $a \in \mathcal{O}$ must satisfy $a > \pi_{i_{j+1}}$. This is obvious if $i_j + 1 = i_{j+1}$, and otherwise it follows from the fact that if $a < \pi_{i_{j+1}}$, then $a\pi_{i_j+1}\pi_{i_{j+1}}$ would be an occurrence of 2-1-3. On the other hand, any entry a with $\pi_{i_{j+1}} < a < \pi_{i_j}$ must appear to the left of π_{i_j}, since π_{i_j} and $\pi_{i_{j+1}}$ are consecutive right-to-left maxima, and so $a \in \mathcal{O}$. Thus, $\mathcal{O}$ is precisely the set of integers strictly between π_{i_j} and $\pi_{i_{j+1}}$, which implies that $\pi_{i_j} - \pi_{i_{j+1}}$ is even. Now, for $j = 1, \dots, m-1$, let $a_j = (\pi_{i_j} - \pi_{i_{j+1}})/2$. Let $a_m = \lfloor \pi_{i_m}/2 \rfloor$. We map π to the following path from $(0,0)$ to $(n, \lfloor n/2 \rfloor)$:

$$E^{i_1} N^{a_1} E^{i_2-i_1} N^{a_2} E^{i_3-i_2} N^{a_3} \cdots E^{i_m-i_{m-1}} N^{a_m}.$$

It can be checked that this map is a bijection. For example, if $\pi = 4675123$, we have $\pi_{i_1}\pi_{i_2}\pi_{i_3} = \pi_3\pi_4\pi_7 = 753$, so the corresponding path from $(0,0)$ to $(7,3)$ is $EEENENEEEN$.

Aside from lattice paths, the sequence $d_n := |\mathcal{S}_n(\text{2-1-3}, \overline{2}^o\text{-31})|$ from (2) is also known to enumerate symmetric ternary trees with $3n$ edges and symmetric diagonally convex directed polyominoes of area n. These numbers have also appeared before in connection to pattern-avoiding permutations. It is shown in [6] that the number of 2143-avoiding Dumont permutations of the second kind of length $2n$ is $d_n d_{n+1}$ (see [6] for definitions). The sequence d_n enumerates what the authors call *lower boards*, which are 2-1-3-avoiding permutations of length n whose diagram fits in a certain shape. A bijection between such permutations and $\mathcal{S}_n(\text{2-1-3}, \overline{2}^o\text{-31})$ can be established by composing our bijection into lattice paths with the one from [6].

Analogously to the definition for the pattern $\overline{2}^o$-31, we say that a permutation π avoids the pattern $\overline{2}^e$-31 if every occurrence of 21 in π is part of an even number of occurrences of 2-31. We can also enumerate $\{\text{2-1-3}, \overline{2}^e\text{-31}\}$-avoiding permutations.

Proposition 2.3. *The number of* $\{\text{2-1-3}, \overline{2}^e\text{-31}\}$*-avoiding permutations of size* n *is*

$$\frac{1}{n}\sum_{k=0}^{\lfloor n/2 \rfloor}\left[2\binom{n}{2k}\binom{n-k}{k-1} + \frac{n}{n-k}\binom{n}{2k+1}\binom{n-k}{k}\right].$$

Proof. Let $Q(t) = \sum_{n\geq 1} |\mathcal{S}_n(\text{2-1-3}, \overline{2}^e\text{-31})|\, t^n$. An argument similar to the proof of Proposition 2.2 shows that

$$Q(t) = \frac{(2-4t)\tilde{f}(t)^2 + (-2+12t-7t^2-\tilde{g}(t))\tilde{f}(t) + (2-8t)\tilde{g}(t) + 8t^3 + 46t^2 - 8t - 4}{3t\tilde{f}(t)^2},$$

where $\tilde{g}(t) = \sqrt{3(-5t^4 + 24t^3 - 4t^2 + 12t - 4)}$ and $\tilde{f}(t) = [4(3t\tilde{g}(t) - 11t^3 - 12t^2 - 6t + 2)]^{1/3}$. It follows that $Q = Q(t)$ is a root of the polynomial $tQ^3 + (4t-2)Q^2 + (4t-1)Q + t = 0$. Applying Lagrange inversion we get the stated formula. □

2.3. **$\{$2-1-3, 2-3-41, 3-2-41$\}$-avoiding permutations.** The rightward generating tree for this class of permutations has a simple succession rule. This allows us to enumerate them easily. Let $K(t,u) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(\text{2-1-3},\text{2-3-41},\text{3-2-41})} u^{r(\pi)} t^n = \sum_{r\geq 1} K_r(t) u^r$.

Proposition 2.4. *The generating function for* $\{\text{2-1-3}, \text{2-3-41}, \text{3-2-41}\}$*-avoiding permutations where* u *marks the value of the rightmost entry is*

$$K(t,u) = \frac{1 - t - 2tu - \sqrt{1-2t-3t^2}}{2t(\frac{1}{u} + 1 + u) - 2}.$$

Proof. The succession rule for this class of permutations is

$$(r) \longrightarrow \begin{cases} (1)\ (2) & \text{if } r = 1, \\ (r-1)\ (r)\ (r+1) & \text{if } r > 1, \end{cases}$$

with the root labeled (1). This translates into the functional equation

$$\left[1 - t\left(\frac{1}{u} + 1 + u\right)\right] K(t,u) = tu - tK_1(t). \tag{4}$$

Applying the kernel method we find that $K_1(t) = \frac{1-t-\sqrt{1-2t-3t^2}}{2t}$, and substituting back into (4) we get the expression for $K(t,u)$. □

The generating function $K(t,1)$ also enumerates $\{1\text{-}3\text{-}2, 123\text{-}4\}$-avoiding permutations, as shown in [15, Example 2.6]. However, no direct bijection between $\mathcal{S}_n(2\text{-}1\text{-}3, 2\text{-}3\text{-}41, 3\text{-}2\text{-}41)$ and $\mathcal{S}_n(1\text{-}3\text{-}2, 123\text{-}4)$ seems to be known.

3. Generating trees with two labels

The generating trees in all the examples in the previous section were described using one label for each node. This will not be the case in the families of permutations in this section. However, we will use the same technique of translating the succession rule into a set of functional equations and applying the kernel method to them. This method is what unifies the different classes of permutations studied in this paper.

Here we enumerate some classes of permutations whose rightward generating tree has a succession rule that can be described using a pair of labels for each node. These trees give rise to functional equations with three variables. Even though no method is known to solve them in general, in this section we present special cases where we have been able to solve the corresponding equations.

In a few cases, one of the two labels is the length of the permutation. When that happens, the functional equations have only two variables, but the variable t appears multiplied by another variable, which makes these equations more difficult than the ones in Section 2.

Note that for the classes that we consider in this section, the enumeration of the permutations by their length has already been done by different authors [8, 9, 11, 12, 15, 18]. Our contribution is a refined enumeration of these permutations by several parameters, and also the fact that our results are obtained using the unifying framework of rightward generating trees.

3.1. $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations.

It was shown in [8] that $|\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23)| = M_n$. A bijection between $\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23)$ and the set of Motzkin paths of length n was given in [11]. Clearly the sets $\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23)$ and $\mathcal{S}_n(2\text{-}1\text{-}3, 12\text{-}3)$ are equinumerous, since a permutation $\pi_1\pi_2\cdots\pi_n$ is $\{1\text{-}3\text{-}2, 1\text{-}23\}$-avoiding exactly when $(n+1-\pi_n)\cdots(n+1-\pi_2)(n+1-\pi_1)$ is $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding. In this section we recover the formula for $|\mathcal{S}_n(2\text{-}1\text{-}3, 12\text{-}3)|$ using a generating tree with two labels. This method provides a refined enumeration of $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations by two new parameters: the value of the last entry and the smallest value of the top of an ascent.

Let $\mathcal{T}_1$ be the rightward generating tree for the set of $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations. Given any $\pi \in \mathcal{S}_n$, define the parameter

$$\ell(\pi) = \begin{cases} n+1 & \text{if } \pi = n(n-1)\cdots 21, \\ \min\{\pi_i : i > 1,\ \pi_{i-1} < \pi_i\} & \text{otherwise.} \end{cases} \tag{5}$$

Let each permutation π be labeled by the pair $(\ell, r) = (\ell(\pi), r(\pi))$. Note that since π avoids 12-3, then necessarily $\ell \geq r$.

Lemma 3.1. *The rightward generating tree $\mathcal{T}_1$ for $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$(2,1)$$

$$(\ell, r) \longrightarrow \begin{cases} (\ell+1,1)\ (\ell+1,2)\ \cdots\ (\ell+1,\ell) & \text{if } \ell = r, \\ (\ell+1,1)\ (\ell+1,2)\ \cdots\ (\ell+1,r)\ (r+1,r+1) & \text{if } \ell > r. \end{cases}$$

Proof. The permutation obtained by appending an entry to the right of $\pi \in \mathcal{S}_n(2\text{-}1\text{-}3, 12\text{-}3)$ is 2-1-3-avoiding if and only if the appended entry is at most $r(\pi)+1$, and it is 12-3-avoiding if and only if the appended entry is at most $\ell(\pi)$. The labels of the children are obtained by looking at how the values of (ℓ, r) change when the new entry is added. □

We will use this generating rule to obtain a formula for the generating function

$$M(t,u,v) := \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,12\text{-}3)} u^{\ell(\pi)} v^{r(\pi)}\ t^n.$$

For fixed ℓ and r, let $M_{\ell,r}(t) = \sum_{n\geq 1} |\{\pi \in \mathcal{S}_n(2\text{-}1\text{-}3, 12\text{-}3) : \ell(\pi) = \ell, r(\pi) = r\}|\ t^n$. Note that $M(t,u,v) = \sum_{\ell,r} M_{\ell,r}(t) u^\ell v^r$.

Proposition 3.2. *The generating function for* $\{2\text{-}1\text{-}3, 12\text{-}3\}$*-avoiding permutations where* u *and* v *mark the parameters* ℓ *and* r *defined above is*

$$M(t,u,v) = \frac{[(1-u)v + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 - ((1-u)v + tu + t^2u^2v)\sqrt{1-2t-3t^2})]u^2 v}{2(1-u-tu(1-u)+t^2u^2)(1-uv+tuv+t^2u^2v^2)},$$

where $c_1 = 2-u-v-uv+2u^2v$, $c_2 = u(-1+(2-u)v+2(u-1)v^2)$, $c_3 = u^2v(-3+2v-2uv)$, *and* $c_4 = -2u^3v^2$.

Substituting $u = v = 1$ in the above expression we recover the generating function for the Motzkin numbers.

Proof. The coefficient of t^n in $M(t,u,v)$ is the sum of $u^\ell v^r$ over all the pairs (ℓ, r) of labels that appear at level n of the tree. By Lemma 3.1, the children of a node with labels (ℓ,ℓ) contribute $u^{\ell+1}v + u^{\ell+1}v^2 + \cdots + u^{\ell+1}v^\ell$ to the next level, and the children of a node with labels (ℓ, r) with $\ell > r$ contribute $u^{\ell+1}v + u^{\ell+1}v^2 + \cdots + u^{\ell+1}v^r + u^{r+1}v^{r+1}$. It follows that

$$M(t,u,v) = tu^2v + t\sum_\ell M_{\ell,\ell}(t)u^{\ell+1}(v+v^2+\cdots+v^\ell) + t\sum_{\ell>r} M_{\ell,r}(t)[u^{\ell+1}(v+v^2+\cdots+v^r) + u^{r+1}v^{r+1}]. \tag{6}$$

It will be convenient to define

$$M_>(t,u,v) := \sum_{n\geq 1} \sum_{\substack{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,12\text{-}3)\\ \text{with } \ell(\pi)>r(\pi)}} u^{\ell(\pi)} v^{r(\pi)}\ t^n \quad \text{and} \quad M_=(t,u,v) := \sum_{n\geq 1} \sum_{\substack{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,12\text{-}3)\\ \text{with } \ell(\pi)=r(\pi)}} (uv)^{\ell(\pi)}\ t^n,$$

so that $M(t,u,v) = M_>(t,u,v) + M_=(t,u,v)$. Taking from (6) only the pairs (ℓ, r) with $\ell > r$, we get

$$\begin{aligned} M_>(t,u,v) &= tu^2v + t\sum_\ell M_{\ell,\ell}(t)u^{\ell+1}(v+v^2+\cdots+v^\ell) + t\sum_{\ell>r} M_{\ell,r}(t)[u^{\ell+1}(v+v^2+\cdots+v^r)] \\ &= tu^2v + t\sum_\ell M_{\ell,\ell}(t)u^{\ell+1}\frac{v^{\ell+1}-v}{v-1} + t\sum_{\ell>r} M_{\ell,r}(t)u^{\ell+1}\frac{v^{r+1}-v}{v-1} \\ &= tu^2v + \frac{tuv}{v-1}\left[M_=(t,u,v) - M_=(t,u,1) + M_>(t,u,v) - M_>(t,u,1)\right]. \end{aligned} \tag{7}$$

Similarly, taking from (6) only the pairs (ℓ, r) with $\ell = r$,

$$M_=(t,u,v) = t\sum_{\ell>r} M_{\ell,r}(t)u^{r+1}v^{r+1} = tuv\sum_{\ell>r} M_{\ell,r}(t)(uv)^r = tuv\ M_>(t,1,uv). \tag{8}$$

Using in (7) the expression of $M_=$ in terms of $M_>$ given in (8), we get

$$M_>(t,u,v) = tu^2v + \frac{tuv}{v-1}\left[tuv\ M_>(t,1,uv) - tu\ M_>(t,1,u) + M_>(t,u,v) - M_>(t,u,1)\right]. \tag{9}$$

Substituting $u = 1$ in this equation and collecting the terms in $M_>(t,1,v)$, we have

$$\left(1 - \frac{t^2v^2}{v-1} - \frac{tv}{v-1}\right) M_>(t,1,v) = tv - \frac{t(t+1)v}{v-1} M_>(t,1,1). \tag{10}$$

Now we apply the kernel method, substituting $v = v_0 = v_0(t) = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$ in (10) to obtain

$$M_>(t,1,1) = \frac{v_0 - 1}{t+1} = \frac{1 - t - 2t^2 - \sqrt{1-2t-3t^2}}{2t^2(t+1)}.$$

Plugging this expression for $M_>(t,1,1)$ back into (10) we get that

$$M_>(t,1,v) = \frac{(1 - t - 2t^2 v - \sqrt{1-2t-3t^2})v}{2t(1 - v + tv + t^2v^2)}. \tag{11}$$

If we write equation (9) as

$$\left(1 - \frac{tuv}{v-1}\right) M_>(t,u,v) = tu^2v + \frac{tuv}{v-1}\left[tuv\ M_>(t,1,uv) - tu\ M_>(t,1,u) - M_>(t,u,1)\right], \tag{12}$$

we can apply again the kernel method, taking $v = v_1 = v_1(t,u) = \frac{1}{1-tu}$. This cancels the left hand side and gives

$$M_>(t,u,1) = \frac{[2(1-u) + u^2 - t(1+2t)u^2 + (1-2u)\sqrt{1-2t-3t^2})]tu^2}{2(1 - u + tu + t^2u^2)(1 - u - tu(1-u) + t^2u^2)}$$

using (11). Substituting back into (12) and using (11) again we get that

$$M_>(t,u,v) = \frac{[2 - u - uv + u^2v + tu(v-1) - t(1+2t)u^2v + (1-2u)\sqrt{1-2t-3t^2})]tu^2v}{2(1 - u - tu(1-u) + t^2u^2)(1 - uv + tuv + t^2u^2v^2)}.$$

Finally, combining it with the fact that

$$M(t,u,v) = M_>(t,u,v) + M_=(t,u,v) = M_>(t,u,v) + tuv\ M_>(t,1,uv),$$

we obtain the desired expression for $M(t,u,v)$. □

We have encountered two classes of pattern-avoiding permutations enumerated by the Motzkin numbers, namely

$$|\mathcal{S}_{n+1}(2\text{-}1\text{-}3, \overline{2}\text{-}31)| = |\mathcal{S}_n(2\text{-}1\text{-}3, 12\text{-}3)| = M_n \tag{13}$$

(see Proposition 2.1). In Section 2.1 we described a bijection φ between 2-3-1-avoiding permutations and Dyck paths. A permutation π is $\overline{2}$-31-avoiding if and only if $\varphi(\pi)$ is a UDU-free Dyck path. It is not hard to check (see [11]) that π is 12-3-avoiding if and only if $\varphi(\pi)$ is UUU-free. Next we give a bijection between UDU-free Dyck paths of size $n+1$ and UUU-free Dyck paths of size n, reproving equation (13).

Given a UDU-free Dyck path, mark each D that is immediately preceded and followed by a D (and also the rightmost D if it is preceded by a D). Move left each one of the marked Ds so that it immediately follows its matching U. Finally, delete the rightmost peak (i.e., occurrence of UD). This gives a DDD-free Dyck path, which can be easily turned into a UUU-free one by reversing it, that is, reading the steps from right to left and exchanging Us and Ds.

To show that this map is the desired bijection, we now describe its inverse. Given a DDD-free Dyck path, we first reverse it and then append a peak UD to it. Define the height of a step to be the number of Us minus the number of Ds preceding it. Mark each D step in an occurrence of UDU. For each marked step, if h is its height, move it to the right so that it immediately precedes the next D step with height $h-1$ (if $h = 1$, then move it to the end). This produces the original UDU-free Dyck path.

As an example of this bijection, consider the UDU-free path $U\bar{U}UUDD\bar{U}UD\bar{D}\bar{D}D\bar{U}UD\bar{D}$. The marked Ds and their matching Us are distinguished with a bar. The DDD-free Dyck path that we obtain is $UU\bar{D}UUDDU\bar{D}UDDU\bar{D}$ (the barred Ds are the steps that have been moved), and its reversal is $UDUUDUDUUDDUDD$. Applying the inverse map moves the barred Ds back to their original position.

3.2. $\{2\text{-}1\text{-}3, 32\text{-}1\}$**-avoiding permutations.** It is known [8] that $|\mathcal{S}_n(2\text{-}1\text{-}3, 32\text{-}1)| = 2^{n-1}$. Here we use rightward generating trees with two labels to recover this fact, and to refine it with two parameters: the value of the last entry and the largest value of the bottom of a descent. Given any $\pi \in \mathcal{S}_n$, define

$$h(\pi) = \begin{cases} 0 & \text{if } \pi = 12\cdots n, \\ \max\{\pi_i : i > 1,\ \pi_{i-1} > \pi_i\} & \text{otherwise.} \end{cases}$$

To each $\{2\text{-}1\text{-}3, 32\text{-}1\}$-avoiding permutation π we assign the pair of labels $(h, r) = (h(\pi), r(\pi))$. Note that since π avoids 32-1, then necessarily $h \le r$.

Lemma 3.3. *The rightward generating tree for $\{2\text{-}1\text{-}3, 32\text{-}1\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l} (0,1) \\ (h,r) \longrightarrow (h+1,h+1)\ (h+2,h+2)\ \cdots\ (r,r)\ (h,r+1). \end{array}$$

Proof. When we append an entry i to a $\{2\text{-}1\text{-}3, 32\text{-}1\}$-avoiding permutation, the new permutation is 2-1-3-avoiding if and only if $i \le r(\pi)+1$, and it is 32-1-avoiding if and only if $i > h(\pi)$. The list of labels of the children obtained by appending an i satisfying these two conditions is the right hand side of the rule. □

Let

$$N(t,u,v) = \sum_{n\ge1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,32\text{-}1)} u^{h(\pi)} v^{r(\pi)}\ t^n = \sum_{h,r} N_{h,r}(t) u^h v^r.$$

Proposition 3.4. *The generating function for $\{2\text{-}1\text{-}3, 32\text{-}1\}$-avoiding permutations where u and v mark the parameters h and r defined above is*

$$N(t,u,v) = \frac{tv(1-t+tu-tuv)}{(1-tv)(1-t-tuv)}.$$

Proof. By Lemma 3.3, the children of a node with labels (h, r) contribute $u^{h+1}v^{h+1} + u^{h+2}v^{h+2} + \cdots + u^r v^r + u^h v^{r+1}$ to the next level. It follows that

$$\begin{aligned} N(t,u,v) &= tv + t\sum_{h,r} N_{h,r}(t)\left[\frac{(uv)^{r+1}-(uv)^{h+1}}{uv-1} + u^h v^{r+1}\right] \\ &= tv + tvN(t,u,v) + \frac{tuv[N(t,1,uv) - N(t,uv,1)]}{uv-1}. \qquad (14) \end{aligned}$$

Substituting $u = 1$ and $v = 1$ separately gives a system of two equations in $N(t, 1, *)$ and $N(t, *, 1)$ that can be easily solved. □

The above result can indeed be obtained as well without using rightward generating trees. The recursive structure of 2-1-3-avoiding permutations (i.e., they are of the form $\sigma 1 \tau$, where σ and τ are 2-1-3-avoiding and every entry in σ is larger than every entry in τ) can be used to obtain an equation satisfied by $N(t, u, v)$ and to deduce the above formula without much difficulty.

3.3. $\{2\text{-}1\text{-}3, 34\text{-}21\}$**-avoiding permutations.** The labels that will be convenient to use to describe the rightward generating tree for this class are $(s, r) = (s(\pi), r(\pi))$, where

$$s(\pi) = \begin{cases} 0 & \text{if } \pi = n(n-1)\cdots 21, \\ \max\{\pi_i :\ \pi_i < \pi_{i+1}\} & \text{otherwise,} \end{cases} \qquad (15)$$

and $r(\pi) = \pi_n$ as usual.

Lemma 3.5. *The rightward generating tree for $\{2\text{-}1\text{-}3, 34\text{-}21\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l} (0,1) \\ (s,r) \longrightarrow \begin{cases} (s+1,1)\ (s+1,2)\ \cdots\ (s+1,s)\ (s,s+1)\ (r,r+1) & \textit{if } s<r, \\ (s+1,r+1) & \textit{if } s>r. \end{cases} \end{array}$$

Proof. First note that the 2-1-3-avoiding condition implies that the new entry appended to π has to be at most $r+1$. If $s<r$ and π_i is an entry to the right of s, then $\pi_i > s$, otherwise $s\pi_i r$ would be an occurrence of 2-1-3. In fact, we also know that $\pi_i \neq s+1$, unless $\pi_i = r = s+1$, because otherwise the entry following $s+1$ would be greater than it, contradicting the definition of s. So, unless $r = s+1$, the entry $s+1$ precedes s, so the appended entry cannot be greater than $s+1$, otherwise it would create a 2-1-3. This explains the labels in the case $s<r$. If $s>r$, the appended entry has to be greater than r for the new permutation to be 34-21-avoiding. □

Let

$$K(t,u,v) := \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(\text{2-1-3},\text{34-21})} u^{s(\pi)} v^{r(\pi)}\ t^n = \sum_{s,r} K_{s,r}(t) u^s v^r,$$

and let $K_<(t,u,v)$ and $K_>(t,u,v)$ be defined similarly, with the sum running only over permutations with $s(\pi)<r(\pi)$ and $s(\pi)>r(\pi)$, respectively, so that $K(t,u,v) = K_<(t,u,v) + K_>(t,u,v)$.

Proposition 3.6. *The generating function for* $\{$2-1-3, 34-21$\}$*-avoiding permutations where* u *and* v *mark the parameters* s *and* r *defined above is*

$$K(t,u,v) = \frac{tv[1-(1+u+uv)t + (u^2+uv+u^2v)t^2]}{(1-t-tu)(1-t-tuv)(1-tuv)}. \tag{16}$$

Proof. By Lemma 3.5, the generating functions $K_<$ and $K_>$ satisfy

$$\begin{aligned}
K_<(t,u,v) &= tv + t\sum_{s<r} K_{s,r}(t)(u^s v^{s+1} + u^r v^{r+1}) = tv + tv[K_<(t,uv,1) + K_<(t,1,uv)],\\
K_>(t,u,v) &= t\sum_{s<r} K_{s,r}(t) u^{s+1}(v+\cdots+v^s) + t\sum_{s>r} K_{s,r}(t) u^{s+1} v^{r+1}\\
&= \frac{tuv}{v-1}[K_<(t,uv,1) - K_<(t,u,1)] + tuvK_>(t,u,v).
\end{aligned}$$

Substituting first $u=1$ and then $v=1$ in the first equation, we get two equations involving $K_<(t,1,w)$ and $K_<(t,w,1)$ that can be easily solved to give

$$K_<(t,u,v) = \frac{tv}{1-t-tuv}.$$

The second equation then implies that

$$K_>(t,u,v) = \frac{u^2vt^3}{(1-t-tu)(1-t-tuv)(1-tuv)},$$

and the proposition follows. □

Corollary 3.7. *The number of* $\{$2-1-3, 34-21$\}$*-avoiding permutations of size* n *is* $(n-1)2^{n-2}+1$.

Proof. Taking $u=v=1$ in (16), we get that

$$K(t,1,1) = \frac{t(1-3t+3t^2)}{(1-t)(1-2t)^2}.$$

The coefficient of t^n in the series expansion of this rational function is $(n-1)2^{n-2}+1$. □

It is not hard to show that $\mathcal{S}_n(\text{2-1-3},\text{34-21}) = \mathcal{S}_n(\text{2-1-3},\text{3-4-2-1}) = \mathcal{S}_n(\text{1-3-2},\text{3-4-2-1})$. This last set of permutations was enumerated by West [18], and Corollary 3.7 agrees with his result.

3.4. $\{1\text{-}2\text{-}34, 2\text{-}1\text{-}3\}$**-avoiding permutations.** The generating function for these permutations appears in [15]. In fact, it is easy to see that $\mathcal{S}_n(1\text{-}2\text{-}34, 2\text{-}1\text{-}3) = \mathcal{S}_n(1\text{-}2\text{-}3\text{-}4, 2\text{-}1\text{-}3)$, and the latter set of permutations was studied in [18], where it is shown that they are counted by the Fibonacci numbers F_{2n-1}. Here we derive the generating function and obtain a refinement of it using a rightward generating tree with two labels.

Let the labels of a permutation π be the pair $(m, r) = (m(\pi), r(\pi))$, where $r(\pi) = \pi_n$ and

$$m(\pi) = \begin{cases} n+1 & \text{if } \pi = n(n-1)\cdots 21, \\ \min\{\pi_i : \exists j < i \text{ with } \pi_j < \pi_i\} & \text{otherwise.} \end{cases}$$

Note that we always have $m(\pi) \le r(\pi)$ unless $r = 1$, and that if $m(\pi) = r(\pi)$, then $\pi = n(n-1)\cdots 312$, so $m = r = 2$.

Lemma 3.8. *The rightward generating tree for* $\{1\text{-}2\text{-}34, 2\text{-}1\text{-}3\}$*-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l} (2,1) \\ (m,r) \longrightarrow \begin{cases} (m+1,1)\ (2,2) & \textit{if } r = 1, \\ (3,1)\ (2,2)\ (2,3) & \textit{if } m = r = 2, \\ (m+1,1)\ (2,2)\ (m,m+1)\ \cdots\ (m,r) & \textit{if } m < r. \end{cases} \end{array}$$

Proof. As usual, the appended entry has to be at most $r+1$ for the permutation to avoid 2-1-3. In the case that $m < r$, this entry cannot be greater than r in order to avoid 1-2-34. The labels are now obtained by looking at how the parameter m changes after appending the new entry. $\square$

Let $H(t,u,v) := \sum_{n\ge 1} \sum_{\pi\in\mathcal{S}_n(1\text{-}2\text{-}34,2\text{-}1\text{-}3)} u^{m(\pi)} v^{r(\pi)}\, t^n$, and let $H_1(t,u,v)$, $H_=(t,u,v)$, and $H_<(t,u,v)$ be defined similarly, with the summation restricted to permutations with $r(\pi) = 1$, $m(\pi) = r(\pi)$, and $m(\pi) < r(\pi)$, respectively, so that $H(t,u,v) = H_1(t,u,v) + H_=(t,u,v) + H_<(t,u,v)$.

Proposition 3.9. *The generating function for* $\{1\text{-}2\text{-}34, 2\text{-}1\text{-}3\}$*-avoiding permutations where u and v mark the parameters m and r defined above is*

$$H(t,u,v) = \frac{tu^2v[1 + (v-3)t + (1+u-v-uv+v^2)t^2 + uv(1-v)t^3]}{(1-3t+t^2)(1-tu)}.$$

Proof. From Lemma 3.8 we get the following functional equations defining H_1, $H_=$, and $H_<$.

$$H_1(t,u,v) = tu^2v + tuvH(t,u,1) \tag{17}$$

$$H_=(t,u,v) = tu^2v^2H(t,1,1) \tag{18}$$

$$H_<(t,u,v) = tvH_=(t,u,v) + \frac{tv}{v-1}[H_<(t,u,v) - H_<(t,uv,1)] \tag{19}$$

Combining (18) and (19), introducing a variable $w = uv$, and defining $\tilde{H}_<(t,w,v) = H_<(t,\frac{w}{v},v)$, we get

$$\left(1 - \frac{tv}{v-1}\right)\tilde{H}_<(t,w,v) = t^2vw^2H(t,1,1) - \frac{tv}{v-1}\tilde{H}_<(t,w,1). \tag{20}$$

The kernel is canceled with $v = \frac{1}{1-t}$, giving an expression for $\tilde{H}_<(t,w,1)$ in terms of $H(t,1,1)$, which plugged back into (20) yields

$$H_<(t,u,v) = \tilde{H}_<(t,uv,v) = \frac{t^2u^2v^3}{1-t}H(t,1,1). \tag{21}$$

On the other hand, adding equations (17) and (18) and using that $H_1(t,u,v) + H_=(t,u,v) = H(t,u,v) - H_<(t,u,v)$, we get

$$H_<(t,u,v) = H(t,u,v) - tu^2v - tuvH(t,u,1) - tu^2v^2H(t,1,1),$$

which combined with (21) gives a simple expression relating $H(t,u,v)$, $H(t,u,1)$ and $H(t,1,1)$. In this expression, the substitution $u=v=1$ gives

$$H(t,1,1)=\frac{t(1-t)}{1-3t+t^2},$$

and the substitution $v=1$ puts $H(t,u,1)$ in terms of $H(t,1,1)$. All together produces the desired formula for $H(t,u,v)$. □

3.5. **$\{12\text{-}34, 2\text{-}1\text{-}3\}$-avoiding permutations.** It was proved in [15] that the generating function for permutations avoiding $\{12\text{-}34, 2\text{-}1\text{-}3\}$ is $\frac{1-2t-t^2-\sqrt{1-4t+2t^2+t^4}}{2t^2}$. Using the labels (ℓ,r) defined as in (5), we can construct a generating tree with two labels for this class of permutations. The proof of the following lemma is straightforward and analogous to that of Lemma 3.1.

Lemma 3.10. *The rightward generating tree for $\{12\text{-}34, 2\text{-}1\text{-}3\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l}(2,1)\\ (\ell,r)\longrightarrow\begin{cases}(\ell+1,1)\ (\ell+1,2)\ \cdots\ (\ell+1,r)\ (r+1,r+1) & \text{if } \ell>r,\\ (\ell+1,1)\ (\ell+1,2)\ \cdots\ (\ell+1,\ell)\ (\ell,\ell+1) & \text{if } \ell=r,\\ (\ell+1,1)\ (\ell+1,2)\ \cdots\ (\ell+1,\ell)\ (\ell,\ell+1)\ (\ell,\ell+2)\ \cdots\ (\ell,r) & \text{if } \ell<r.\end{cases}\end{array}$$

Let $F(t,u,v):=\sum_{n\ge1}\sum_{\pi\in\mathcal{S}_n(12\text{-}34,2\text{-}1\text{-}3)} u^{\ell(\pi)}v^{r(\pi)}\,t^n$, and let $F_>(t,u,v)$, $F_=(t,u,v)$, and $F_<(t,u,v)$ be defined similarly, with the summation restricted to permutations with $\ell(\pi)>r(\pi)$, $\ell(\pi)=r(\pi)$, and $\ell(\pi)<r(\pi)$, respectively. By definition, $F(t,u,v)=F_>(t,u,v)+F_=(t,u,v)+F_<(t,u,v)$.

Proposition 3.11. *The generating function for $\{12\text{-}34, 2\text{-}1\text{-}3\}$-avoiding permutations where u and v mark the parameters ℓ and r defined above is*

$$F(t,u,v)=\frac{u^2v[p_1(t,u,v)+p_2(t,u,v)\sqrt{1-4t+2t^2+t^4}]}{2[(1+tuv)^2-uv-t-uvt^2][1+(u+t)(tu-1)]},$$

where

$$\begin{aligned}p_1(t,u,v) &= (1-u)v+(2-u-4v+2uv+v^2+2u^2v-uv^2)t+(-4+u+6v+uv-3v^2-6u^2v+3u^2v^2)t^2\\ &\quad+(2+u-4v-5uv+3v^2+4u^2v+4uv^2-4u^2v^2-2uv^3-2u^3v^2+2u^2v^3)t^3\\ &\quad+(-u+v+4uv-v^2-4uv^2-u^2v^2+2uv^3+2u^3v^2-2u^3v^3)t^4-uv(v-1)(2uv-1)t^5,\\ p_2(t,u,v) &= (u-1)v+[(u-1)v(v-2)-u]t+(u-v+v^2-u^2v^2)t^2+uv(1-v)t^3.\end{aligned}$$

Note that this expression becomes much simpler if we ignore the parameter r, that is,

$$F(t,u,1)=\frac{u^2(1-2tu-t^2-\sqrt{1-4t+2t^2+t^4})}{2[1+(u+t)(tu-1)]},$$

and coincides with the result from [15] if we ignore both parameters:

$$F(t,1,1)=\frac{1-2t-t^2-\sqrt{1-4t+2t^2+t^4}}{2t^2}.$$

Proof. From Lemma 3.10 we get the following functional equations defining $F_>$, $F_=$, and $F_<$.

$$\begin{aligned}F_>(t,u,v) &= tu^2v+\frac{tuv}{v-1}[F_>(t,u,v)-F_>(t,u,1)\\ &\qquad +F_=(t,u,v)-F_=(t,u,1)+F_<(t,uv,1)-F_<(t,u,1)], &(22)\\ F_=(t,u,v) &= tuvF_>(t,1,uv), &(23)\\ F_<(t,u,v) &= tvF_=(t,u,v)+\frac{tv}{v-1}[F_<(t,u,v)-F_<(t,uv,1)]. &(24)\end{aligned}$$

We can introduce a variable $w = uv$ in (24), and apply the kernel method with $v = \frac{1}{1-t}$ to get that

$$F_<(t, w, 1) = \frac{t^2 w F_>(t, 1, w)}{1-t}.$$

Using this expression together with (23) in (22), we get an equation that involves only $F_>$:

$$\left(1 - \frac{tuv}{v-1}\right) F_>(t,u,v) = tu^2v - \frac{tuv}{v-1}\left[F_>(t,u,1) + \frac{tuv}{1-t}F_>(t,1,uv) - \frac{tu}{1-t}F_>(t,1,u)\right]. \tag{25}$$

Substituting $u = 1$, it becomes

$$\frac{(1 - v - t + 2tv - vt^2 + v^2t^2)}{(1-v)(1-t)} F_>(t,1,v) = tv + \frac{tv}{(1-v)(1-t)} F_>(t,1,1). \tag{26}$$

We apply the kernel method again, this time with $v = \frac{1-2t+t^2-\sqrt{1-4t+2t^2+t^4}}{2t^2}$ to cancel the left hand side of (26), which yields

$$F_>(t,1,1) = \frac{1 - 3t + t^2 + t^3 + (t-1)\sqrt{1 - 4t + 2t^2 + t^4}}{2t^2}.$$

Now we can use (26) to obtain a formula for $F_>(t, 1, v)$. Applying again the kernel method in (25), with $v = \frac{1}{1-tu}$, we get an expression for $F_>(t, u, 1)$ in terms of $F_>(t, 1, u)$ and $F_>(t, 1, \frac{u}{1-tu})$, and therefore a formula for $F_>(t, u, 1)$. Substituting back into (25), we get a formula for $F_>(t, u, v)$. From this it is straightforward to obtain formulas for $F_<(t, u, v)$ and $F_=(t, u, v)$ as well, and the result follows. □

3.6. **{1-23, 3-12}-avoiding permutations.** Generating trees with two labels can be used to obtain the generating function for the number of {1-23, 3-12}-avoiding permutations. These permutations were studied in [9], where it was shown that if we let $b_n = |\mathcal{S}_n(\text{1-23}, \text{3-12})|$, then these numbers satisfy the recurrence $b_{n+2} = b_{n+1} + \sum_{k=0}^{n} \binom{n}{k} b_k$. Here we obtain an ordinary generating function without going through the recurrence. The labels are particularly easy in this case because we can take one of them to be just the length n of the permutation. The labels of $\pi \in \mathcal{S}_n$ are then (r, n), where $r = \pi_n$ as usual. The advantage of having one of the labels be n is that we do not need an extra variable for this label in the generating function, since it is already encoded in the exponent of the variable t.

Lemma 3.12. *The rightward generating tree for {1-23, 3-12}-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l} (1,1) \\ (r,n) \longrightarrow \begin{cases} (1,n+1)\ (n+1,n+1) & \text{if } r = 1, \\ (1,n+1)\ (2,n+1)\ \cdots\ (r,n+1) & \text{if } r > 1. \end{cases} \end{array}$$

Proof. The appended element cannot be larger than the rightmost entry of π, except where this entry is 1, in which case the appended element can be the new largest one. □

Let $P(t,u) := \sum_{n\geq 1} \sum_{\pi \in \mathcal{S}_n(\text{1-23},\text{3-12})} u^{r(\pi)}\, t^n = \sum_r P_r(t) u^r$.

Proposition 3.13. *The generating function for {1-23, 3-12}-avoiding permutations is*

$$P(t,1) = \sum_{k\geq 1} \frac{t^{2k-1}(1-(k-1)t)}{(1-t)^2(1-2t)^2\cdots(1-kt)^2}.$$

Proof. From Lemma 3.12 we get

$$P(t,u) = tu + \frac{tu}{u-1}(P(t,u) - uP_1(t) - P(t,1) + P_1(t)) + tu(P_1(t) + P_1(tu)). \tag{27}$$

Using that $P_1(t) = t + t\,P(t,1)$ and collecting the terms with $P(t,u)$, we get

$$\left(1 - \frac{tu}{u-1}\right) P(t,u) = tu + t^2u^2 + t^2u^2 P(tu,1) + \left(t^2u + \frac{tu(t-1-tu)}{u-1}\right) P(t,1). \tag{28}$$

Substituting $u = \frac{1}{1-t}$ gives

$$P(t,1) = \frac{t}{(1-t)^2}\left(1 + tP(\frac{t}{1-t},1)\right),$$

and by iterated application of this formula,

$$\begin{aligned} P(t,1) &= \frac{t}{(1-t)^2}\left(1+\frac{t^2(1-t)}{(1-2t)^2}\left(1+\frac{t^2(1-2t)}{(1-t)(1-3t)^2}\left(1+\frac{t^2(1-3t)}{(1-2t)(1-4t)^2}\left(1+\cdots\right)\right)\right)\right) \\ &= \frac{t}{(1-t)^2}+\frac{t^3}{(1-t)(1-2t)^2}+\frac{t^5}{(1-t)^2(1-2t)(1-3t)^2}+\frac{t^7}{(1-t)^2(1-2t)^2(1-3t)(1-4t)^2}+\cdots, \end{aligned}$$

which is the formula above. If we substitute this expression back into (28) we get the refined formula for $P(t,u)$. □

A very similar argument can be applied to 1-23-avoiding permutations, which are known to be enumerated by the Bell numbers [8]. Our approach in this case gives essentially the same functional equation that is derived in [13] using what the authors call the *scanning-elements algorithm*.

Rightward generating trees and the kernel method can also be used to produce a functional equation for the ordinary generating function of 123-avoiding permutations. We omit this result here because a more direct way to enumerate these permutations was already given in [12].

4. Generating trees with three labels

In this section we include two instances of permutations avoiding generalized patterns where the rightward generating tree can be described by a succession rule with three labels. One of these labels is the length of the permutation, so that the functional equations that we obtain have three variables instead of four. However, the fact that the variable t appears multiplied by another variable adds some difficulty to the equations.

To the best of our knowledge, the two classes of restricted permutations considered in this section have never been enumerated before.

4.1. $\{1\text{-}23, 3\text{-}12, 34\text{-}21\}$-avoiding permutations.

Given a permutation $\pi \in \mathcal{S}_n$, let $s(\pi)$ be defined as in (15). We associate to π the triple of labels $(s,r,n) = (s(\pi), r(\pi), n)$.

Lemma 4.1. *The rightward generating tree for* $\{1\text{-}23, 3\text{-}12, 34\text{-}21\}$*-avoiding permutations is specified by the following succession rule on the labels:*

$$(0,1,1)$$

$$(s,r,n) \longrightarrow \begin{cases} (s+1,1,n+1)\ (s+1,2,n+1)\ \cdots\ (s+1,s,n+1) & \\ \qquad (s,s+1,n+1)\ (s,s+2,n+1)\ \cdots\ (s,r,n+1) & \text{if } s<r\neq 1, \\ (0,1,n+1)\ (1,n+1,n+1) & \text{if } (s,r)=(0,1), \\ (s,n+1,n+1) & \text{if } s>r=1, \\ \emptyset & \text{if } s>r>1. \end{cases}$$

Proof. If $r>1$, the appended entry has to be at most r for the new permutation to avoid 1-23. If $s>r$, it has to be at least $r+1$ for the new permutation to avoid 34-21. Finally, if $r=1$, the appended entry has to be $n+1$ for the permutation to avoid 3-12, unless $s=0$, which means that π is the decreasing permutation. Combining these conditions we get the four possible cases and the new labels in each case. □

The four cases in the succession rule above suggest dividing the set Θ of values that the pair (s,r) can take into four disjoint sets: $\Theta_1 = \{(s,r) : s<r\neq 1\}$, $\Theta_2 = \{(0,1)\}$, $\Theta_3 = \{(s,r) : s>r=1\}$,

$\Theta_4 = \{(s,r) : s > r > 1\}$. For $i = 1, 2, 3, 4$, let

$$R_i(t,u,v) := \sum_{n\geq 1} \sum_{\substack{\pi\in\mathcal{S}_n(1\text{-}23,3\text{-}12,34\text{-}21)\\ \text{with } (s(\pi),r(\pi))\in\Theta_i}} u^{s(\pi)} v^{r(\pi)}\ t^n,$$

and let $R(t,u,v) = R_1(t,u,v) + R_2(t,u,v) + R_3(t,u,v) + R_4(t,u,v)$.

Proposition 4.2. *The generating function for* $\{1\text{-}23, 3\text{-}12, 34\text{-}21\}$*-avoiding permutations where* u *marks the parameter* s *defined above is*

$$1 + R(t,u,1) = \sum_{k\geq 0} \frac{t^{2k}u^k(1+ktu)}{(1-(k+1)t)\prod_{j=1}^{k-1}(1-jt)}. \tag{29}$$

Proof. Lemma 4.1 translates into the following equations for the generating functions R_i:

$$\begin{aligned}
R_1(t,u,v) &= tuvR_2(tv,1,1) + tvR_3(tv,u,1) + \frac{tv}{v-1}[R_1(t,u,v) - R_1(t,uv,1)],\\
R_2(t,u,v) &= \frac{tv}{1-t},\\
R_3(t,u,v) &= tuvR_1(t,u,1), \qquad (30)\\
R_4(t,u,v) &= \frac{tuv}{v-1}[R_1(t,uv,1) - vR_1(t,u,1)]. \qquad (31)
\end{aligned}$$

Combining them we get an equation involving only R_1:

$$R_1(t,u,v) = \frac{t^2uv^2}{1-tv} + t^2uv^2R_1(tv,u,1) + \frac{tv}{v-1}[R_1(t,u,v) - R_1(t,uv,1)].$$

If we collect on one side the terms with $R_1(t,u,v)$, the kernel of the equation is $1 - \frac{tv}{v-1}$. Introducing a new variable $w = uv$ and canceling the kernel with $v = \frac{1}{1-t}$, we obtain an expression involving $R_1(t,w,1)$ and $R_1(\frac{t}{1-t}, (1-t)w, 1)$, which can be simplified to

$$R_1(t,w,1) = tw^2\left[\frac{1}{1-2t} + \frac{1}{1-t}R_1\left(\frac{t}{1-t}, (1-t)w, 1\right)\right].$$

By iterated application of this formula,

$$\begin{aligned}
R_1(t,u,1) &= t^2u\left(\frac{1}{1-2t} + \frac{t^2u}{1-t}\left(\frac{1}{1-3t} + \frac{t^2u}{1-2t}\left(\frac{1}{1-4t} + \frac{t^2u}{1-3t}\left(\frac{1}{1-5t} + \cdots\right)\right)\right)\right)\\
&= \frac{t^2u}{1-2t} + \frac{(t^2u)^2}{(1-t)(1-3t)} + \frac{(t^2u)^3}{(1-t)(1-2t)(1-4t)} + \frac{(t^2u)^4}{(1-t)(1-2t)(1-3t)(1-5t)} + \cdots\\
&= \sum_{k\geq 1} \frac{t^{2k}u^k}{(1-(k+1)t)\prod_{j=1}^{k-1}(1-jt)}.
\end{aligned}$$

Equation (30) gives now an expression for $R_3(t,u,1)$, and (31) implies that

$$R_4(t,u,v) = \sum_{k\geq 1} \frac{t^{2k+1}u^{k+1}(v^2+v^3+\cdots+v^k)}{(1-(k+1)t)\prod_{j=1}^{k-1}(1-jt)}.$$

Adding up the four generating functions $R(t,u,1) = R_1(t,u,1) + R_2(t,u,1) + R_3(t,u,1) + R_4(t,u,1)$ we get (29). □

The first coefficients of $R(t,1,1)$, which are the values of $|\mathcal{S}_n(\{1\text{-}23, 3\text{-}12, 34\text{-}21\})|$ for $n = 1, 2, \ldots$, are $1, 2, 4, 8, 19, 47, 125, \ldots$. This sequence does not appear in [16] at the moment.

4.2. **$\{$1-23, 34-21$\}$-avoiding permutations.** The derivation of the generating function for this class of permutations is very similar to the previous subsection. The labels that we associate to a permutation are again (s,r,n). The proof of the next lemma is analogous to that of Lemma 4.1.

Lemma 4.3. *The rightward generating tree for $\{$1-23, 34-21$\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$
\begin{array}{l}
(0,1,1) \\
(s,r,n) \longrightarrow \begin{cases}
(s+1,1,n+1)\ (s+1,2,n+1)\ \cdots\ (s+1,s,n+1) & \\
\qquad (s,s+1,n+1)\ (s,s+2,n+1)\ \cdots\ (s,r,n+1) & \text{if } s<r\neq 1, \\
(0,1,n+1)\ (1,2,n+1)\ (1,3,n+1)\ \cdots\ (1,n+1,n+1) & \text{if } (s,r)=(0,1), \\
(s+1,2,n+1)\ (s+1,3,n+1)\ \cdots\ (s+1,s,n+1) & \\
\qquad (s,s+1,n+1)\ (s,s+2,n+1)\ \cdots\ (s,n+1,n+1) & \text{if } s>r=1, \\
\emptyset & \text{if } s>r>1.
\end{cases}
\end{array}
$$

Divide the set Θ of values that the pair (s,r) can take into four disjoint sets Θ_i, $i=1,2,3,4$ as before, and let

$$T_i(t,u,v) := \sum_{n\geq 1} \sum_{\substack{\pi\in\mathcal{S}_n(\text{1-23,34-21}) \\ \text{with } (s(\pi),r(\pi))\in\Theta_i}} u^{s(\pi)} v^{r(\pi)}\ t^n$$

and $T(t,u,v)=T_1(t,u,v)+T_2(t,u,v)+T_3(t,u,v)+T_4(t,u,v)$.

Proposition 4.4. *The generating function for $\{$1-23, 34-21$\}$-avoiding permutations where u marks the parameter s defined above is*

$$T(t,u,1)=\sum_{k\geq 0}\frac{t^{k+1}u^k(1+ktu)}{(1+tu)^k(1-kt)(1-(k+1)t)}. \tag{32}$$

Proof. The equations that follow from Lemma 4.3 are now

$$
\begin{aligned}
T_1(t,u,v) &= \frac{tuv}{v-1}[T_2(tv,1,1)-T_2(t,1,1)]+\frac{tv}{v-1}[vT_3(tv,u,1)-T_3(t,uv,1)] \\
&\quad +\frac{tv}{v-1}[T_1(t,u,v)-T_1(t,uv,1)], \\
T_2(t,u,v) &= \frac{tv}{1-t}, \\
T_3(t,u,v) &= tuvT_1(t,u,1), \qquad (33)\\
T_4(t,u,v) &= \frac{tuv}{v-1}[T_1(t,uv,1)+T_3(t,uv,1)-vT_1(t,u,1)-vT_3(t,u,1)]. \qquad (34)
\end{aligned}
$$

From them we can get an equation involving only T_1:

$$T_1(t,u,v)=\frac{t^2uv^2}{v-1}\left(\frac{v}{1-tv}-\frac{1}{1-t}\right)+\frac{t^2uv^2}{v-1}[vT_1(tv,u,1)-T_1(t,uv,1)]+\frac{tv}{v-1}[T_1(t,u,v)-T_1(t,uv,1)].$$

Letting $w=uv$ and canceling the kernel with $v=\frac{1}{1-t}$, we get that

$$T_1(t,w,1)=\frac{tw}{(1+tw)(1-t)}\left[\frac{t}{1-2t}+T_1\left(\frac{t}{1-t},(1-t)w,1\right)\right].$$

Iterating this formula, we see that

$$T_1(t,u,1)=\sum_{k\geq 1}\frac{t^{k+1}u^k}{(1+tu)^k(1-kt)(1-(k+1)t)}.$$

Using (33) and (34) we get expressions for $T_3(t,u,1)$ and $T_4(t,u,v)$. Finally, the sum $T(t,u,1)=T_1(t,u,1)+T_2(t,u,1)+T_3(t,u,1)+T_4(t,u,1)$ gives the formula (32). □

The first coefficients of $T(t,1,1)$ are $1,2,5,14,42,138,492,\ldots$, which teaches us not to judge a sequence by looking only at its first five terms. This sequence gives the number of $\{1\text{-}23, 34\text{-}21\}$-avoiding permutations of size $n=1,2,\ldots$, and does not currently appear in [16].

5. Concluding remarks

The main results in the paper have been obtained by constructing rightward generating trees with up to three labels for several families of pattern-avoiding permutations, and solving the functional equations for the generating functions that the succession rule produces. This is a useful method for enumerating permutations avoiding generalized patterns. There is nothing special about the sets of patterns studied in this paper, except that this method happens to work out nicely on them.

We expect that this technique of rightward generating trees with several labels, together with the kernel method and other ad-hoc tools for solving the functional equations that are obtained, will lead to many more enumerative results for classes of permutations avoiding generalized patterns.

Acknowledgements. I am grateful to Mireille Bousquet-Mélou for many helpful ideas that have made this paper possible, and to two anonymous referees for useful suggestions to improve its presentation.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, D. Gouyou-Beauchamps, Generating functions of generating trees, *Discrete Math.* 246 (2002), 29-55.

[2] C. Banderier and P. Flajolet, Basic analytic combinatorics of directed lattice paths, *Theoret. Comput. Sci.* 281 (2002), 37-80.

[3] A. Bernini, L. Ferrari, R. Pinzani, Enumerating permutations avoiding three Babson-Steingrímsson patterns, *Ann. Combin.* 9 (2005), 137–162.

[4] M. Bousquet-Mélou, Four classes of pattern-avoiding permutations under one roof: generating trees with two labels, *Electron. J. Combin.* 9 (2003), #R19.

[5] M. Bousquet-Mélou and Marko Petkovšek, Linear recurrences with constant coefficients: the multivariate case, *Discrete Math.* 225 (2000), 51-75.

[6] A. Burstein, S. Elizalde, T. Mansour, Restricted Dumont permutations, Dyck paths, and noncrossing partitions, *Discrete Math.* 306 (2006), 2851–2869.

[7] D. Callan, Two bijections for Dyck path parameters, preprint, arxiv:math.CO/0406381v2.

[8] A. Claesson, Generalised pattern avoidance, *Europ. J. Combin.* 22 (2001), 961–973.

[9] A. Claesson, T. Mansour, Enumerating permutations avoiding a pair of Babson-Steingrímsson patterns, *Ars Combinatorica* 77 (2005).

[10] S. Elizalde, Asymptotic enumeration of permutations avoiding generalized patterns, *Adv. in Appl. Math.* 36 (2006), 138–155.

[11] S. Elizalde, T. Mansour, Restricted Motzkin permutations, Motzkin paths, continued fractions, and Chebyshev polynomials, *Discrete Math.* 305 (2005), 170–189.

[12] S. Elizalde, M. Noy, Consecutive subwords in permutations, *Adv. in Appl. Math.* 30 (2003), 110–125.

[13] G. Firro, T. Mansour, Three-letter-pattern-avoiding permutations and functional equations, *Electron. J. Combin.* 13 (2006), #R51.

[14] C. Krattenthaler, Permutations with restricted patterns and Dyck paths, *Adv. in Appl. Math.* 27 (2001), 510–530.

[15] T. Mansour, Restricted 1-3-2 permutations and generalized patterns, *Ann. Combin.* 6 (2002), 65-76.

[16] N.J.A. Sloane, S. Plouffe, The Encyclopedia of Integer Sequences, Academic Press, San Diego, 1995, `http://www.research.att.com/~njas/sequences`.

[17] J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Math.* 146 (1995), 247–262.

[18] J. West, Generating trees and forbidden subsequences, *Discrete Math.* 157 (1996), 363–374.

Department of Mathematics, Dartmouth College, Hanover, NH 03755 / Centre de Recerca Matemàtica, E-08193 Bellaterra, Spain

E-mail address: `sergi.elizalde@dartmouth.edu`