Connection between ordinary multinomials, generalized Fibonacci numbers, partial Bell partition polynomials and convolution powers of discrete uniform distribution

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Abstract

Using an explicit computable expression of ordinary multinomials, we establish three remarkable connections, with the q-generalized Fibonacci sequence, the exponential partial Bell partition polynomials and the density of convolution powers of the discrete uniform distribution. Identities and various combinatorial relations are derived.

Keywords. Ordinary multinomials, Exponential partial Bell partition polynomials, Generalized Fibonacci sequence, Convolution powers of discrete uniform distribution.

1 Introduction

Ordinary multinomials are a natural extension of binomial coefficients, for an appropriate introduction of these numbers see Smith and Hogatt (1979), Bollinger (1986) and Andrews and Baxter (1987). These coefficients are defined as follows:
Let $q \geq 1$ and $L \geq 0$ be integers. For an integer $a = 0, 1, \ldots, qL$, the ordinary multinomial $\binom{L}{a}_q$ is the coefficient of the a-th term of the following multinomial expansion

$$\left(1 + x + x^2 + \cdots + x^q\right)^L = \sum_{a \geq 0} \binom{L}{a}_q x^a, \tag{1}$$

with $\binom{L}{a}_1 = \binom{L}{a}$ (being the usual binomial coefficient) and $\binom{L}{a}_q = 0$ for $a > qL$. Using the classical binomial coefficient, one has

$$\binom{L}{a}_q = \sum_{j_1+j_2+\cdots+j_q=a} \binom{L}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{q-1}}{j_q}. \tag{2}$$

Some readily well known established properties are

the symmetry relation

$$\binom{L}{a}_q = \binom{L}{qL-a}_q, \tag{3}$$

the longitudinal recurrence relation

$$\binom{L}{a}_q = \sum_{m=0}^{q} \binom{L-1}{a-m}_q, \tag{4}$$

and the diagonal recurrence relation

$$\binom{L}{a}_q = \sum_{m=0}^{L} \binom{L}{m}\binom{m}{a-m}_{q-1}. \tag{5}$$

These coefficients, as for usual binomial coefficients, are built trough the Pascal triangle, known as "Pascal Pyramid", see tables: 1, 2 and 3. One can find the first values of the pyramid in SLOANE [16] as A027907 for $q=2$, A008287 for $q=3$ and A035343 for $q=4$.

As an illustration of recurrence relation, we give the triangles of trinomial, quadrinomial and pentanomial coefficients:

Table 1: Triangle of *trinomial* coefficients: $\binom{L}{k}_2$

$L\backslash k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	9	*10*
0	1										
1	1	1	1								
2	1	2	3	2	1						
3	1	3	6	7	6	3	1				
4	1	4	10	16	19	16	10	4	1		
5	1	5	15	30	45	51	45	30	15	5	1

Table 2: Triangle of *quadrinomial* coefficients: $\binom{L}{k}_3$

$L\backslash k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	7	*8*	*9*	*10*	*11*	
0	1												
1	1	1	1	1									
2	1	2	3	4	3	2	1						
3	1	3	6	10	12	12	10	6	3	1			
4	1	4	10	20	31	40	44	40	31	20	10	4	1

Table 3: Triangle of *pentanomial* coefficients: $\binom{L}{k}_4$

$L\backslash k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	9	*10*	*11*	*12*	*13*	
0	1														
1	1	1	1	1	1										
2	1	2	3	4	5	4	3	2	1						
3	1	3	6	10	15	18	19	18+	15+	10+	6+	3+	1		
4	1	4	10	20	35	52	68	80	85	80	68	=52	35	20	$\cdots$

Several extensions and commentaries about these numbers have been investigated in the literature:

Brondarenko [6, 1993] gives a combinatorial interpretation of ordinary multinomials $\binom{L}{a}_q$ as *the number of different ways of distributing "a" balls among "L" cells where each cell contains at most "q" balls.*

Using this combinatorial argument, one can easily establish the following relation

$$\begin{aligned}\binom{L}{a}_q &= \sum_{L_1+2L_2+\cdots+qL_q=a} \binom{L}{L_1}\binom{L-L_1}{L_2}\cdots\binom{L-L_1-\cdots-L_{q-1}}{L_q} \\ &= \sum_{L_1+2L_2+\cdots+qL_q=a} \binom{L}{L_0, L_1, L_2 \cdots, L_{q-1}}. \qquad (6)\end{aligned}$$

For a computational view of the relation (6) see Bollinger [5, 1986]. Andrews and Baxter [2, 1987] have considered the q-analog generalization of ordinary multinomials (see also [18, 1997] for an exhaustive bibliography). They have defined the q-multinomial coefficients as follows

$$\begin{bmatrix} n \\ a \end{bmatrix}_q^{(p)} = \sum_{j_1+j_2+\cdots+j_q=a} q^{\sum_{l=1}^{q-1}(L-j_l)j_{l+1}-\sum_{l=q-p}^{q-1} j_{l+1}} \begin{bmatrix} L \\ j_1 \end{bmatrix}\begin{bmatrix} j_1 \\ j_2 \end{bmatrix}\cdots\begin{bmatrix} j_{q-1} \\ j_q \end{bmatrix},$$

where

$$\begin{bmatrix} L \\ a \end{bmatrix} = \begin{bmatrix} L \\ a \end{bmatrix}_q = \begin{cases} (q)_L \,/\, (q)_a\, (q)_{L-a} & \text{if } 0 \le a \le L \\ 0 & \text{otherwise} \end{cases}$$

is the usual q-binomial coefficient, and where $(q)_k = \prod_{m=1}^{\infty} (1-q^m) \,/\, \left(1-q^{k+m}\right)$, is called q-series. This definition is motivated by the relation (2).

Another extension, the supernomials, has also been considered by Schilling and Warnaar [15, 1998]. These coefficients are defined to be the coefficients of x^a in the expression of $\prod_{j=1}^{N}\left(1+x+\cdots+x^j\right)^{L_j}$

A refinement of the q-multinomial coefficient is also considered for the trinomial case by Warnaar [19, 2001].

Barry [3, 2006] gives a generalized Pascal triangle as $\binom{n}{k}_{a(n)} := \prod_{j=1}^{k} a(n-j+1)/a(j)$, where $a(n)$ is a suitably chosen sequence of integers.

Kallas [10, 2006] and Noe [13, 2006] give a generalization of Pascal's triangle by considering the coefficient of x^a in the expression of $(a_0 + a_1 x + \cdots + a_q x^q)^L$.

The main goal of this paper is to give some connections of the ordinary multinomials with the generalized Fibonacci sequence, the exponential Bell polynomials, and the density of convolution powers of discrete uniform distribution. We will give also some intersting combinatorial identities

2 A simple expression of ordinary multinomials

If we denote x_i the number of balls in a cell, the previous combinatorial interpretation given by Brondarenko is equivalent to evaluate the number of solutions of the system

$$\left\{ \begin{array}{l} x_1 + \cdots + x_L = a, \\ 0 \leq x_1, \cdots, x_L \leq q. \end{array} \right. \tag{7}$$

Now, let us consider the system (7). For $t \in]-1, 1[$, we have (see also Comtet [7, Vol.1, p. 92 (pb 16).])

$$\sum_{a \geq 0} \binom{L}{a}_q t^a = (1 + t + \cdots + t^q)^L = \sum_{0 \leq x_1, \cdots, x_L \leq q} t^{x_1 + \cdots + x_L},$$

and

$$\begin{aligned} (1 + t + \cdots + t^q)^L &= \left(1 - t^{q+1}\right)^L (1 - t)^{-L} \\ &= \left(\sum_{j=0}^{L} (-1)^j \binom{L}{j}_q t^{j(q+1)} \right) \left(\sum_{j \geq 0} \binom{j + L - 1}{L - 1} t^j \right). \end{aligned}$$

By identification, we obtain the following theorem.

Theorem 1 *The following identity holds*

$$\binom{L}{a}_q = \sum_{j=0}^{\lfloor a/(q+1) \rfloor} (-1)^j \binom{L}{j} \binom{a - j(q+1) + L - 1}{L - 1}. \tag{8}$$

This explicite relation seems to be important since in contrast to relations (2), (3) and (5), it allows to compute the ordinary multinomials with one summation symbol.
In 1711, de Moivre (see [12, 1731] or [11, 1756 3rd ed. p.39]) solves the system (7) as the right hand side of (8).

Corollary 2 *We have the following identity*

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j} \binom{n - j}{j} = \sum_{j=0}^{\lfloor n/3 \rfloor} (-1)^j \binom{n}{j} \binom{2n - 3j - 1}{n - 1}.$$

Proof. It suffices to use relation (4) in Theorem 1 for $q = 2$ and $a = L = n$. □

3 Generalized Fibonacci sequences

Now, let us consider for $q \geq 1$, the "multibonacci" sequence $(\Phi_n^{(q)})_{n\geq -q}$ defined by

$$\begin{cases} \Phi_{-q}^{(q)} = \cdots = \Phi_{-2}^{(q)} = \Phi_{-1}^{(q)} = 0, \\ \Phi_0^{(q)} = 1, \\ \Phi_n^{(q)} = \Phi_{n-1}^{(q)} + \Phi_{n-2}^{(q)} + \cdots + \Phi_{n-q-1}^{(q)} \text{ for } n \geq 1. \end{cases}$$

In [4, 2006], Belbachir and Bencherif proved that

$$\Phi_n^{(q-1)} = \sum_{k_1+2k_2+\cdots+qk_q=n} \binom{k_1+k_2+\cdots+k_q}{k_1,k_2,\cdots,k_q},$$

and, for $n \geq 1$

$$\Phi_n^{(q-1)} = \sum_{k=0}^{\lfloor n/(q+1)\rfloor} (-1)^k \, \frac{n-k\,(q-1)}{n-kq} \binom{n-kq}{k} 2^{n-1-k(q+1)},$$

leading to

$$\sum_{k_1+\cdots+qk_q=n} \binom{k_1+\cdots+k_q}{k_1,\cdots,k_q} = \sum_{k=0}^{\lfloor n/(q+1)\rfloor} (-1)^k \, \frac{n-k\,(q-1)}{n-kq} \binom{n-kq}{k} 2^{n-1-k(q+1)}.$$

This is an analogous situation in writing above a multiple summation with one symbol of summation. On the other hand, we establish a connection between the ordinary multinomials and the generalized Fibonacci sequence:

Theorem 3 *We have the following identity*

$$\Phi_n^{(q)} = \sum_{l=0}^{qm-r} \binom{n-l}{l}_q, \tag{9}$$

where m is given by the extended euclidean algorithm for division: $n = m\,(q+1) - r$, $0 \leq r \leq q$.

Proof. We have

$$\begin{aligned} \Phi_n^{(q)} &= \sum_{k_1+2k_2+\cdots+(q+1)k_{q+1}=n} \binom{k_1+k_2+\cdots+k_{q+1}}{k_1,k_2,\cdots,k_{q+1}} \\ &= \sum_{L\geq 0} \sum_{k_1+2k_2+\cdots+(q+1)k_{q+1}=n} \binom{L}{k_1,k_2,\cdots,k_{q+1}} \\ &= \sum_{L\geq 0} \sum_{k_2+2k_3+\cdots+qk_{q+1}=n-L} \binom{L}{k_1,k_2,\cdots,k_{q+1}} \\ &= \sum_{L\geq \frac{n}{q+1}}^{n} \binom{L}{n-L}_q . \end{aligned}$$

Now consider the unique writing of n given by the extended euclidean algorithm for division: $n = m\,(q+1) - r$, $0 \le r < q+1 \to \frac{n}{q+1} = m - \frac{r}{q+1}$, which gives

$$\Phi_n^{(q)} = \sum_{k=0}^{qm-r} \binom{m+k}{qm-r-k}_q = \sum_{l=0}^{qm-r} \binom{m+k}{(q+1)\,k+r}_q = \sum_{l=0}^{qm-r} \binom{n-l}{l}_q.$$

□

As an immediate consequence of Theorem 3, we obtain the following identities

$$\begin{aligned}
\Phi_{(q+1)m}^{(q)} &= \sum_{l=0}^{qm} \binom{(q+1)\,m-l}{l}_q = \sum_{k=0}^{qm} \binom{m+k}{(q+1)\,k}_q,\\
\Phi_{(q+1)m-1}^{(q)} &= \sum_{l=0}^{qm-1} \binom{(q+1)\,m-l-1}{l}_q = \sum_{k=0}^{qm} \binom{m+k}{(q+1)\,k+1}_q,\\
&\cdots\cdots\cdots\cdots\\
\Phi_{(q+1)m-r}^{(q)} &= \sum_{l=0}^{qm-r} \binom{(q+1)\,m-l-r}{l}_q = \sum_{k=0}^{qm} \binom{m+k}{(q+1)\,k+r}_q.
\end{aligned}$$

For $q = 1$, we find the classical Fibonacci sequence:

$$F_{-1} = 0,\; F_0 = 1,\; F_{n+1} = F_n + F_{n-1},\; \text{for } n \ge 0.$$

Thus, we obtain the well known identity

$$F_n = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l}.$$

4 Exponential partial Bell partition polynomials

In this section, we establish a connection of the ordinary multinomials with exponential partial Bell partition polynomials $B_{n,L}\,(t_1, t_2, \ldots)$ which are defined (see Comtet [7, 1970, p. 144]) as follows

$$\frac{1}{L!}\left(\sum_{m\ge1} \frac{t_m}{m!} x^m\right)^L = \sum_{n\ge L} B_{n,L} \frac{x^n}{n!},\; L = 0, 1, 2, \ldots. \tag{10}$$

An exact expression of such polynomials is given by

$$B_{n,L}\,(t_1, t_2, \ldots) = \sum_{\substack{k_1+2k_2+\cdots=n\\ k_1+k_2+\cdots=L}} \frac{n!}{k_1!k_2!\cdots(1!)^{k_1}\,(2!)^{k_2}\cdots} t_1^{k_1} t_2^{k_2} \cdots.$$

In this expression, the number of variables is finite according to $k_1 + 2k_2 + \cdots = n$.

Next, we give some particular values of $B_{n,L}$:

$$\begin{aligned}
B_{n,L}(1,1,1,\ldots) &= \left\{ \begin{matrix} n \\ L \end{matrix} \right\} \text{ Stirling numbers of second kind,} \\
B_{n,L}(0!,1!,2!,\ldots) &= \left[\begin{matrix} n \\ L \end{matrix} \right] \text{ Stirling numbers of first kind,} \\
B_{n,L}(1!,2!,3!,\ldots) &= \frac{n!}{L!}\binom{n-1}{n-L}. \qquad (11)
\end{aligned}$$

In [1, 2005], Abbas and Bouroubi give several extended values of $B_{n,L}$.

The connection with ordinary multinomials is given by the following result:

Theorem 4 *We have the following identity*

$$B_{n,L}(1!,2!,\ldots,(q+1)!,0,\ldots) = \frac{n!}{L!}\binom{L}{n-L}_q. \qquad (12)$$

Proof. Taking in (10) $t_m = m!$ for $1 \le m \le q+1$ and zero otherwise, we obtain

$$\left(x + \cdots + x^{q+1}\right)^L = L! \sum_{n-L \ge 0} B_{n,L}(1!,2!,\ldots,(q+1)!,0,\ldots)\frac{x^n}{n!},$$

from which it follows

$$\sum_{a \ge 0}\binom{L}{a}_q x^a = \sum_{n-L\ge 0}\frac{L!}{n!}B_{n,L}(1!,2!,\ldots,(q+1)!,0,\ldots)\,x^{n-L}.$$

□

Corollary 5 *Let $q \ge 1$, $L \ge 0$ be integers, and $a \in \{0,1,\ldots,qL\}$. For $q \ge a$, we have the following identity*

$$\binom{L}{a}_q = \binom{L+a-1}{a}.$$

Proof. Using the fact that $B_{n,L}(1!,2!,\ldots,(q+1)!,0,\ldots) = B_{n,L}(1!,2!,3!,\ldots)$ for $q+1 \ge n-L+1$, we obtain $\binom{L}{n-L}_q = \binom{n-1}{n-L}$ for $q \ge n-L$. We conclude with $a = n-L$. □

5 Convolution powers of discrete uniform distribution

This section gives a connection between the ordinary multinomials and the convolution power of the discrete uniform distribution. The right hand side of identity (8) is a very well known expression. Indeed for $q, L \in \mathbb{N}$, let us denote by U_q the L^{th} convolution power of the discrete uniform distribution

$$U_q := \frac{1}{q+1}(\delta_0 + \delta_1 + \cdots + \delta_q) \qquad (\delta_a \text{ is the Dirac measure}),$$

then for $a \in \mathbb{N}$ (see de Moivre [12, 1731] or [9, 1998]), with respect to the counting measure, its density is given by

$$P\left(U_q^{\star L} = a\right) = \frac{1}{(q+1)^L} \sum_{j=0}^{\lfloor a/(q+1) \rfloor} (-1)^j \binom{L}{j} \binom{a + L - (q+1)\,j - 1}{L-1}. \tag{13}$$

Combining Theorem 1 and relation (13), we have the following result:

Corollary 6 *Using the above notations, we obtain the following identity*

$$P\left(U_q^{\star L} = a\right) = \frac{\binom{L}{a}_q}{(q+1)^L}.$$

It should be noted that the multinomials may be seen as the number of favorable cases to the realization of the elementary event $\{a\}$.
It is easy to show that the distribution of $U_q^{\star L}$ is symmetric by relation (3).

Corollary 7 *We have the following identity*

$$\begin{aligned}
\sum_{k=0}^{qL} k \binom{L}{k}_q &= (q+1)^L \frac{qL}{2}, \\
\sum_{k=0}^{qL} k_q^2 \binom{L}{k} &= (q+1)^L \frac{qL}{2} \left(\frac{qL}{2} + \frac{q+2}{6} \right), \\
\sum_{k=0}^{qL} k_q^3 \binom{L}{k} &= (q+1)^L \left(\frac{qL}{2} \right)^2 \left(\frac{qL}{2} + \frac{q+2}{2} \right),
\end{aligned}$$

More generally, for $m \geq 1$, the following identity holds

$$\sum_{k=0}^{qL} k_q^m \binom{L}{k} = (q+1)^L \sum_{i_1 + i_2 + \cdots + i_L = m} \binom{m}{i_1, i_2, \cdots, i_L} u_{i_1} u_{i_2} \cdots u_{i_L},$$

where u_i is the i-th moment of the random variable U_q.

Proof. It suffices to compute the expectation of $U_q^{\star L}$ using, first the density distribution and second the summation of uniform distributions. □

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