On Kalai's conjectures concerning centrally symmetric polytopes

Raman Sanyal Axel Werner Günter M. Ziegler

Institute of Mathematics, MA 6-2
TU Berlin
D-10623 Berlin, Germany
{sanyal,awerner,ziegler}@math.tu-berlin.de

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Abstract

In 1989 Kalai stated the three conjectures **A**, **B**, **C** of increasing strength concerning face numbers of centrally symmetric convex polytopes. The weakest conjecture, **A**, became known as the "3^d-conjecture". It is well-known that the three conjectures hold in dimensions $d \leq 3$. We show that in dimension 4 only conjectures **A** and **B** are valid, while conjecture **C** fails. Furthermore, we show that both conjectures **B** and **C** fail in all dimensions $d \geq 5$.

1 Introduction

A convex d-polytope P is *centrally symmetric*, or *cs* for short, if $P = -P$. Concerning face numbers, this implies that for $0 \leq i \leq d-1$ the number of i-faces $f_i(P)$ is even and, since P is full-dimensional, that $\mathsf{min}\{f_0(P), f_{d-1}(P)\} \geq 2d$. Beyond this, only very little is known for the general case. That is to say, the extra (structural) information of a central symmetry yields no substantial additional constraints for the face numbers on the restricted class of polytopes.

Not uncommon to the f-vector business, the knowledge about face numbers is concentrated on the class of centrally symmetric *simplicial*, or dually *simple*, polytopes. In 1982, Bárány and Lovász [3] proved a lower bound on the number of vertices of simple cs polytopes with prescribed number of facets, using a generalization of the Borsuk–Ulam theorem. Moreover, they conjectured lower bounds for all face numbers of this class of polytopes with respect to the number of facets. In 1987 Stanley [24] proved a conjecture of Björner concerning the h-vectors of simplicial cs polytopes that implies the one by Bárány and Lovász. The proof uses Stanley-Reisner rings and toric varieties plus a pinch of representation theory. The result of Stanley [24] for cs polytopes was reproved in a more geometric setting by Novik [18] by using "symmetric flips" in McMullen's weight algebra [16]. For general polytopes, lower bounds on the *toric* h-vector were recently obtained by A'Campo-Neuen [2] by using combinatorial intersection cohomology. Unfortunately, the toric h-vector contains only limited information about the face numbers of general (cs) polytopes and thus the applicability of the result is limited (see Section 2.1).

In [14], Kalai stated three conjectures about the face numbers of general cs polytopes. Let P

be a (cs) d-polytope with f-vector $f(P) = (f_0, f_1, \dots, f_{d-1})$. Define the function $s(P)$ by

$$s(P) := 1 + \sum_{i=0}^{d-1} f_i(P) = f_P(1)$$

where $f_P(t) := f_{d-1}(P) + f_{d-2}(P)t + \cdots + f_0(P)t^{d-1} + t^d$ is the f-polynomial. Thus, $s(P)$ measures the total number of non-empty faces of P. Here is Kalai's first conjecture from [14], the "3^d-conjecture".

Conjecture A. Every centrally-symmetric d-polytope has at least 3^d non-empty faces, i.e. $s(P) \geq 3^d$.

Is is easy to see that the bound is attained for the d-dimensional cube C_d and for its dual, the d-dimensional crosspolytope $C_d^{\triangle}$. It takes a moment's thought to see that in dimensions $d \geq 4$ these are not the only polytopes with 3^d non-empty faces. An important class that attains the bound is the class of *Hanner polytopes* [11]. These are defined recursively: As a start, every cs 1-dimensional polytope is a Hanner polytope. For dimensions $d \geq 2$, a d-polytope H is a Hanner polytope if it is the direct sum or the direct product of two (lower dimensional) Hanner polytopes H' and H''.

The number of Hanner polytopes grows exponentially in the dimension d, with a Catalan-type recursion. It is given by the number of two-terminal networks with d edges, $n(d) = 1, 1, 2, 4, 8, 18, 40, 94, 224, 548, 1356, \dots$, for $d = 1, 2, \dots$, as counted by Moon [17]; see also [22].

Conjecture B. For every centrally-symmetric d-polytope P there is a d-dimensional Hanner polytope H such that $f_i(P) \geq f_i(H)$ for all $i = 0, \dots, d-1$.

For a d-polytope P and $S = \{i_1, i_2, \dots, i_k\} \subseteq [d] = \{0, 1, \dots, d-1\}$ let $f_S(P) \in \mathbb{Z}^{2^{[d]}}$ be the number of chains of faces $F_1 \subset F_2 \subset \cdots \subset F_k \subset P$ with $\mathsf{dim}\, F_j = i_j$ for all $j = 1, \dots, k$. Identifying $\mathbb{R}^{2^{[d]}}$ with its dual space via the standard inner product, we write $\alpha(P) := \sum_S \alpha_S f_S(P)$ for $(\alpha_S)_{S \subseteq [d]} \in \mathbb{R}^{2^{[d]}}$. The set

$$\mathcal{P}_d = \left\{ (\alpha_S)_{S \subseteq [d]} \in \mathbb{R}^{2^{[d]}} : \alpha(P) = \sum_S \alpha_S f_S(P) \geq 0 \text{ for all } d\text{-polytopes } P \right\}$$

is the polar to the set of flag-vectors of d-polytopes, that is, the cone of all linear functionals that are non-negative on all flag-vectors of (not necessarily cs) d-polytopes.

Conjecture C. For every centrally-symmetric d-polytope P there is a d-dimensional Hanner polytope H such that $\alpha(P) \geq \alpha(H)$ for all $\alpha \in \mathcal{P}_d$.

It is easy to see that $\mathbf{C} \Rightarrow \mathbf{B} \Rightarrow \mathbf{A}$: Define $\alpha^i(P) := f_i(P)$, then $\alpha^i \in \mathcal{P}_d$ and the validity of **C** on the functionals α^i implies **B**; the remaining implication follows since $s(P)$ is a non-negative combination of the $f_i(P)$.

In this paper we investigate the validity of these three conjectures in various dimensions. Our main results are as follows.

Theorem 1.1. *The conjectures* **A** *and* **B** *hold for centrally symmetric polytopes of dimension* $d \leq 4$.

Theorem 1.2. *Conjecture* **C** *is false in dimension* $d = 4$.

Theorem 1.3. *For all* $d \geq 5$ *both conjectures* **B** *and* **C** *fail.*

The paper is organized as follows. In Section 2 we establish a lower bound on the flag-vector functional g_2^{tor} on the class of cs 4-polytopes. Together with some combinatorial and geometric reasoning this leads to a proof of Theorem 1.1. In Section 3, we exhibit a centrally symmetric 4-polytope and a flag vector functional that disprove conjecture **C**. In Section 4 we consider centrally symmetric hypersimplices in odd dimensions; combined with basic properties of Hanner polytopes, this gives a proof of Theorem 1.3. We close with two further interesting examples of centrally symmetric polytopes in Section 5.

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2 Conjectures A and B in dimensions $d \leq 4$

In this section we prove Theorem 1.1, that is, the conjectures **A** and **B** for polytopes in dimensions $d \leq 4$. The work of Stanley [24] implies **A** and **B** for simplicial and thus also for simple polytopes. Furthermore, if $f_0(P) = 2d$, then P is linearly isomorphic to a crosspolytope. *Therefore, we assume throughout this section that all cs d-polytopes P are neither simple nor simplicial, and that $f_{d-1}(P) \geq f_0(P) \geq 2d + 2$.*

The main work will be in dimension 4. The claims for dimensions one, two, and three are vacuous, clear, and easy to prove, in that order. In particular, the case $d = 3$ can be obtained from an easy f-vector calculation. But, to get in the right mood, let us sketch a geometric argument. Let P be a cs 3-polytope. Since P is not simplicial, P has a non-triangle facet. Let F be a facet of P with $f_0(F) \geq 4$ vertices. Let $F_0 = P \cap H$ with H being the hyperplane parallel to the affine hulls of F and of $-F$ that contains the origin. Now, F_0 is a cs 2-polytope and it is clear that every face G of P that has a nontrivial intersection with H is neither a face of F nor of $-F$. We get

$$s(P) \quad \geq \quad s(F) + s(F_0) + s(-F) \geq 3 \cdot 3^2.$$

This type of argument fails in dimensions $d \geq 4$. Applying small (symmetric) perturbations to the vertices of a prism over an octahedron yields a cs 4-polytope with the following two types of facets: prisms over a triangle and square pyramids. Every such facet has less than 3^3 faces, which shows that less than a third of the alleged 81 faces are concentrated in any facet.

Let's come back to dimension 4. The proof of the conjectures **A** and **B** splits into a combinatorial part (*f-vector yoga*) and a geometric argument. We partition the class of cs 4-polytopes into *large* and (few) *small* polytopes, where "large" means that

$$f_0(P) + f_3(P) \quad \geq \quad 24. \tag{1}$$

We will reconsider an argument of Kalai [13] that proves a lower bound theorem for polytopes and, in combination with flag-vector identities, leads to a tight flag-vector inequality for cs 4-polytopes. With this new tool, we prove that (1) implies conjectures **A** and **B** for dimension 4.

We show that the *small* cs 4-polytopes, i.e. those not satisfying (1), are *twisted prisms*, to be introduced in Section 2.3, over 3-polytopes. We then establish basic properties of twisted prisms that imply the validity of conjectures **A** and **B** for small cs 4-polytopes.

2.1 Rigidity with symmetry and flag-vector inequalities

For a general simplicial d-polytope P the *h-vector* $h(P)$ is the ordered collection of the coefficients of the polynomial $h_P(t) := f_P(t-1)$, the *h-polynomial* of P. Clearly, $h_P(t)$ encodes the same information as the f-polynomial, but additionally $h_P(t)$ is a unimodal, palindromic polynomial with non-negative, integral coefficients (see e.g. [28, Sect. 8.3]). This gives more insight in the nature of face numbers of simplicial polytopes and, in a compressed form, this numerical information is carried by its *g-vector* $g(P)$ with $g_i(P) = h_i(P) - h_{i-1}(P)$ for $i = 1, \ldots, \lfloor \frac{d}{2} \rfloor$. There are various interpretations for the h- and g-numbers and, via the g-Theorem, they carry a complete characterization of the f-vectors of simplicial d-polytopes.

For general d-polytopes a much weaker invariant is given by the *generalized* or *toric h*-vector $h^{\mathsf{tor}}(P)$ introduced by Stanley [23]. In contrast to the ordinary h-vector, the toric h-numbers $h_i^{\mathsf{tor}}(P)$ are not determined by the f-vector: They are linear combinations of the face numbers and of other entries of the flag-vector of P. For example,

$$g_2^{\mathsf{tor}} \;=\; h_2^{\mathsf{tor}} - h_1^{\mathsf{tor}} \;=\; f_1 + f_{02} - 3f_2 - df_0 + \binom{d+1}{2}.$$

The corresponding toric h-polynomial shares the same properties as its simplicial relative but, unfortunately, carries quite incomplete information about the f-vector.

For example, in the case of P being a *quasi-simplicial* polytope, i.e. if every facet of P is simplicial, the toric h-vector depends only on the f-numbers $f_i(P)$ for $0 \le i \le \lfloor \frac{d}{2} \rfloor$ and, therefore, does not carry enough information to determine a lower bound on $s(P)$ for $d \ge 5$. However, the information gained in dimension 4 will be a major step in the direction of a proof of Theorem 1.1. To be more precise, for the class of centrally symmetric d-polytopes there is a refinement of the flag-vector inequality $g_2^{\mathsf{tor}} = h_2^{\mathsf{tor}} - h_1^{\mathsf{tor}} \ge 0$.

Theorem 2.1. *Let P be a centrally symmetric d-polytope. Then*

$$g_2^{\mathsf{tor}}(P) \;=\; f_1(P) + f_{02}(P) - 3f_2(P) - df_0(P) + \binom{d+1}{2} \;\ge\; \binom{d}{2} - d.$$

With Euler's equation and the Generalized Dehn-Sommerville equations [5] it is routine to derive the following inequality for the class of cs 4-polytopes.

Corollary 2.2. *If P is a centrally symmetric 4-polytope, then*

$$f_{03}(P) \;\ge\; 3f_0(P) + 3f_3(P) - 8. \tag{2}$$

We will prove Theorem 2.1 using the theory of *infinitesimally rigid frameworks*. For information about rigidity beyond our needs we refer the reader to Roth [20] for a very readable introduction and to Whiteley [27] and Kalai [14] for rigidity in connection with polytopes.

Let $d \ge 1$ and let $G = (V, E)$ be an abstract simple undirected graph. The *edge function* associated to G and d is the map

$$\begin{aligned} \Phi : (\mathbb{R}^d)^V &\to \mathbb{R}^E \\ (p_v)_{v\in V} &\mapsto \left(\|p_u - p_v\|^2\right)_{uv\in E}, \end{aligned}$$

which measures the (squared) lengths of the edges of G for any choice of coordinates $\mathbf{p} = (p_v)_{v\in V} \in (\mathbb{R}^d)^V$. The pair $(G, \mathbf{p})$ is called a *framework* in $\mathbb{R}^d$ and the points of $\Phi_{\mathbf{p}} := \Phi^{-1}(\Phi(\mathbf{p}))$ give the possible frameworks in $\mathbb{R}^d$ with constant edge lengths $\Phi(\mathbf{p})$.

Let $v = |V| \geq d+1$ and let $\mathbf{p}$ be a generic embedding. Then the set $\Phi_{\mathbf{p}} \subset (\mathbb{R}^d)^V$ is a smooth submanifold on which the group of *Euclidean/rigid motions* $E(\mathbb{R}^d)$ acts smoothly and faithfully. Therefore the dimension of $\Phi_{\mathbf{p}}$ is $\operatorname{dim} \Phi_{\mathbf{p}} \geq \binom{d+1}{2}$ and in case of equality the framework $(G, \mathbf{p})$ is *infinitesimally rigid*.

The *rigidity matrix* $R = R(G, \mathbf{p}) \in (\mathbb{R}^d)^{E \times V}$ of $(G, \mathbf{p})$ is the Jacobian matrix of Φ evaluated at $\mathbf{p}$. Invoking the Implicit Function Theorem, it is easy to see that $(G, \mathbf{p})$ is infinitesimally rigid if and only if $\operatorname{rank} R = dv - \binom{d+1}{2}$. A *stress* on the framework $(G, \mathbf{p})$ is an assignment $\omega = (\omega_e)_{e \in E} \in \mathbb{R}^E$ of weights $\omega_e \in \mathbb{R}$ to the edges $e \in E$ such that there is an equilibrium $\sum_{u:uv \in E} \omega_{uv}(p_v - p_u) = 0$ at every vertex $v \in V$. We denote by $S(G, \mathbf{p}) = \{\omega \in \mathbb{R}^E : \omega R = 0\}$ the kernel of $R^\top$, called the *space of stresses* on $(G, \mathbf{p})$.

Theorem 2.3 (Whiteley [27, Thm. 8.6 with Thm. 2.9]). *Let $P \subset \mathbb{R}^d$ be a d-polytope. Let $G = G(P) = (V, E)$ be the graph obtained from a triangulation of the 2-skeleton of P without new vertices and let $\mathbf{p} = \mathbf{p}(P)$ be the vertex coordinates. Then the resulting framework $(G, \mathbf{p})$ is infinitesimally rigid.*

The above theorem makes no reference to the triangulation of the 2-skeleton. The important fact to note is that the graph G of Theorem 2.3 will have exactly $e := |E| = f_1(P) + f_{02}(P) - 3f_2(P)$ edges: In addition to the $f_1(P)$ edges of P, $k - 3$ edges are needed for every 2-face with k vertices.

For the dimension of the space of stresses $S(G, \mathbf{p})$ we get

$$\begin{aligned} 0 \leq \operatorname{dim} S(G, \mathbf{p}) &= e - \operatorname{rank} R \\ &= e - dv + \binom{d+1}{2} \\ &= f_1(P) + f_{02}(P) - 3f_2(P) - df_0(P) + \binom{d+1}{2} \\ &= g_2^{\mathsf{tor}}(P). \end{aligned}$$

Now let P be a centrally symmetric d-polytope, $d \geq 3$. Let $G = G(P) = (V, E)$ be the graph in Theorem 2.3 obtained from a triangulation that respects the central symmetry of the 2-skeleton and let $\mathbf{p} = \mathbf{p}(P)$ be the vertex coordinates of P. The antipodal map $\mathbf{x} \mapsto -\mathbf{x}$ induces a free action of the group $\mathbb{Z}_2$ on the graph G. We denote by $\overline{V} = V/\mathbb{Z}_2$ and $\overline{E} = E/\mathbb{Z}_2$ the respective quotients and, after choosing representatives, we denote by $V = V^+ \uplus V^-$ and $E = E^+ \uplus E^-$ the decompositions of the set of vertices and edges according to the action. Since the action is free we have $|\overline{V}| = |V^\pm| = \frac{v}{2}$ and $|\overline{E}| = |E^\pm| = \frac{e}{2}$.

Concerning the rigidity matrix, it is easy to see that

$$R \;=\; \begin{array}{c} \\ E^+ \\ E^- \end{array} \!\!\begin{array}{c} \begin{array}{cc} V^+ & V^- \end{array} \\ \begin{pmatrix} R_1 & R_2 \\ -R_2 & -R_1 \end{pmatrix} \end{array} \;\in\; (\mathbb{R}^d)^{V \times E}$$

with labels above and to the left of the matrix. The embedding $\mathbf{p} = \mathbf{p}(P)$ respects the central symmetry of G and we can augment the edge function by a second component that takes the symmetry information into account:

$$\begin{aligned} \Phi^{\mathsf{sym}} : \quad (\mathbb{R}^d)^{V^+} \times (\mathbb{R}^d)^{V^-} &\longrightarrow \mathbb{R}^E \times (\mathbb{R}^d)^{\overline{V}} \\ \mathbf{p} = (\mathbf{p}_{V^+}, \mathbf{p}_{V^-}) &\longmapsto (\Phi(\mathbf{p}), \mathbf{p}_{V^+} + \mathbf{p}_{V^-}). \end{aligned}$$

Thus Φ^{sym} additionally measures the degree of asymmetry of the embedding. By the symmetry of P, $\Phi^{\mathsf{sym}}(\mathbf{p}) = (\Phi(\mathbf{p}), 0)$ for $\mathbf{p} = \mathbf{p}(P)$. The preimage of this point under Φ^{sym} is $\Phi^{\mathsf{sym}}_{\mathbf{p}} \subset \Phi_{\mathbf{p}}$, the set of all centrally symmetric embeddings with edge lengths $\Phi(\mathbf{p})$. Any small (close to identity) rigid motion that fixes the origin takes $\mathbf{p} \in \Phi_{\mathbf{p}}^{\mathsf{sym}}$ to a *distinct* centrally symmetric realization $\mathbf{p}' \in \Phi_{\mathbf{p}}^{\mathsf{sym}}$. Thus the action of the subgroup $O(\mathbb{R}^d)$, the group of orthogonal transformations, on $\Phi_{\mathbf{p}}^{\mathsf{sym}}$ locally gives a smooth embedding. It follows that $\dim \Phi_{\mathbf{p}}^{\mathsf{sym}} \geq \dim O(\mathbb{R}^d) = \binom{d}{2}$ and thus

$$\operatorname{rank} R^{\mathsf{sym}} \quad \leq \quad dv - \tbinom{d}{2}, \tag{3}$$

where we can compute the rank of R^{sym}, the Jacobian of Φ^{sym} at $\mathbf{p}$, as

$$\operatorname{rank} R^{\mathsf{sym}} \quad = \quad \operatorname{rank} \begin{pmatrix} R_1 & R_2 \\ -R_2 & -R_1 \\ I_{V^+} & I_{V^-} \end{pmatrix} \quad = \quad \frac{dv}{2} + \operatorname{rank}\,(R_1 - R_2)\,. \tag{4}$$

Proof of Theorem 2.1. Consider the space of *symmetric stresses*, that is, the linear subspace

$$S^{\mathsf{sym}}(G,\mathbf{p}) = \{\omega = (\omega_{E^+}, \omega_{E^-}) \in S(G,\mathbf{p}) : \omega_{E^+} = \omega_{E^-}\} \cong \{\overline{\omega} \in \mathbb{R}^{\overline{E}} : \overline{\omega}\,(R_1 - R_2) = 0\}.$$

From (3) and (4) it follows that

$$\dim S^{\mathsf{sym}}(G,\mathbf{p}) \quad = \quad \frac{e}{2} - \operatorname{rank}(R_1 - R_2) \quad \geq \quad \frac{e}{2} - \frac{dv}{2} + \binom{d}{2}.$$

The theorem follows from noting that $S^{\mathsf{sym}}(G,\mathbf{p}) \subseteq S(G,\mathbf{p})$ and therefore

$$e - dv + \binom{d+1}{2} \quad \geq \quad \frac{1}{2}(e - dv) + \binom{d}{2}.$$

□

Theorem 2.1 can also be deduced from the following result of A'Campo-Neuen [2]; see also [1].

Theorem 2.4 ([2, Theorem 2])**.** *Let P be a centrally symmetric d-polytope and let $h_P^{\mathsf{tor}}(t) = \sum_{i=0}^{d} h_i^{\mathsf{tor}}(P)\,t^i$ be its toric h-polynomial. Then the polynomial*

$$h_P^{\mathsf{tor}}(t) - h_{C_d^\triangle}^{\mathsf{tor}}(t) = h_P^{\mathsf{tor}}(t) - (1+t)^d \in \mathbb{Z}[t]$$

is palindromic and unimodal with non-negative, even coefficients. In particular,

$$g_i^{\mathsf{tor}}(P) = h_i^{\mathsf{tor}}(P) - h_{i-1}^{\mathsf{tor}}(P) \geq \tbinom{d}{i} - \tbinom{d}{i-1} \text{ for all } 1 \leq i \leq \left\lfloor \tfrac{d}{2} \right\rfloor.$$

The proof of Theorem 2.4 relies on the (heavy) machinery of combinatorial intersection cohomology for fans. Theorem 2.1 concerns the special case of the coefficient of the quadratic term. In light of McMullen's *weight algebra* [16], it would be interesting to know whether/how Theorem 2.4 can be deduced by considering (generalized) stresses. A connection between the combinatorial intersection cohomology set-up for fans and rigidity was established by Braden [6, Sect. 2.9].

2.2 Large centrally symmetric 4-polytopes

In order to prove conjectures **A** and **B** for large polytopes, we need one more ingredient.

Proposition 2.5. *Let P be a 4-polytope. Then*

$$\begin{aligned} f_{03}(P) &\leq 4f_2(P) - 4f_3(P) \\ &= 4f_1(P) - 4f_0(P). \end{aligned} \tag{5}$$

Equality holds if and only if P is center-boolean, *i.e. if every facet is simple.*

Proof. The inequality was first proved by Bayer [4]. Every facet F of P is a 3-polytope satisfying $3f_0(F) \leq 2f_1(F)$. By summing up over all facets of P we get

$$3f_{03}(P) = \sum_{F \text{ facet}} 3f_0(F) \leq \sum_{F \text{ facet}} 2f_1(F) = 2f_{13}(P).$$

By one of the Generalized Dehn-Sommerville Equations [5] we have

$$f_{03} - f_{13} + f_{23} = 2f_3,$$

which, together with $f_{23} = 2f_2$ immediately implies the asserted inequality. Equality holds if the above inequality for 3-polytopes holds with equality for all facets of P, which means that all facets are simple 3-polytopes. The equality in the assertion is Euler's equation. □

Combining the inequalities (2) and (5), we obtain

$$\begin{aligned} f_2 &\geq \tfrac{1}{4}(3f_0 + 7f_3) - 2 &= f_3 + \tfrac{3}{4}(f_0 + f_3) - 2 \\ f_1 &\geq \tfrac{1}{4}(7f_0 + 3f_3) - 2 &= f_0 + \tfrac{3}{4}(f_0 + f_3) - 2. \end{aligned} \tag{6}$$

In terms of f_0 and f_3 this gives

$$s(P) \geq \tfrac{14}{4}(f_0 + f_3) - 3 \geq 81$$

where the last inequality holds if P is large.

To prove conjecture **B** for large polytopes, we have to show that the f-vector of every large polytope is component-wise larger than the f-vector of one of the following four Hanner polytopes:

	(f_0, f_1, f_2, f_3)
C_4	$(16, 32, 24, 8)$
$C_4^{\triangle}$	$(8, 24, 32, 16)$
$\mathsf{bip}\, C_3$	$(10, 28, 30, 12)$
$\mathsf{prism}\, C_3^{\triangle}$	$(12, 30, 28, 10)$

It suffices to treat the case $f_0 + f_3 = 24$. Indeed, for $f_0 + f_3 \geq 26$ and $f_3 \geq f_0 \geq 10$ we get from (6) that

$$\begin{aligned} f_1 &\geq f_0 + 18 \geq 28 \\ f_2 &\geq f_3 + 18 \geq 30 \end{aligned}$$

and thus $f(\mathsf{bip}\, C_3)$ is componentwise smaller.

We claim that the same bounds hold for $f_0 + f_3 = 24$. Otherwise, if $f_1 \leq 26$ or $f_2 \leq 28$, then by using (5) together with $f_0 \geq 10$ and $f_3 \geq 12$ we get in both cases that $f_{03} \leq 64$. In fact, we now get $f_{03} = 64$ from (2), which tells us that P is *center boolean*, i.e. every facet is simple. Granted that every facet of P is simple and has at most 6 vertices, the possible facet types are the 3-simplex Δ_3 and the triangular prism $\mathsf{prism}\, \Delta_2$. Using the assumption that P is not simplicial, there is a facet $F \cong \mathsf{prism}\,\Delta_2$. The three quad faces of F give rise to three more prism facets and, due to the number of vertices, no two of them are antipodes. For the same reason, any two prism facets cannot intersect in a triangle face. In total, we note that P has exactly eight prism facets and four tetrahedra. Since every antipodal pair of prism facets give a partition of the vertices, it follows that every vertex is contained in a simplex and exactly 4 prism facets. Therefore, every vertex has degree ≥ 6 and thus $2f_1 \geq 6 \cdot 12$. By Euler's equation, the same holds for f_2.

2.3 Twisted prisms and the small polytopes

The class of small cs 4-polytopes consists of all cs 4-polytopes P with $12 \geq f_3(P) \geq f_0(P) = 10$. Since P is not simplicial, P has a facet F that has $5 = d+1 = f_0(F)$ vertices, and $P = \mathsf{conv}\,(F \cup -F)$. In particular, F is a 3-polytope with $3+2$ vertices, which does not leave much diversity in terms of combinatorial types. The facet F is combinatorially equivalent to

- a pyramid over a quadrilateral, or
- a bipyramid over a triangle.

Definition 2.6 (Twisted prism)**.** Let $Q \subset \mathbb{R}^{d-1}$ be a $(d-1)$-polytope. The centrally symmetric d-polytope

$$P = \mathsf{tprism}\,Q = \mathsf{conv}\ (Q \times \{1\} \quad \cup \quad -Q \times \{-1\}) \subset \mathbb{R}^d$$

is called the *twisted prism* over the base Q.

The following basic properties of twisted prisms will be of good service.

Proposition 2.7. *Let $Q \subset \mathbb{R}^{d-1}$ be a $(d-1)$-polytope and $\mathsf{tprism}\,Q$ the twisted prism over Q.*

1. *If $\mathsf{T} : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ is a non-singular affine transformation, then $\mathsf{tprism}\,Q$ and $\mathsf{tprism}\,\mathsf{T}Q$ are affinely isomorphic.*
2. *If $Q = \mathsf{pyr}\,Q'$ is a pyramid with base Q', then $\mathsf{tprism}\,Q$ is combinatorially equivalent to $\mathsf{bip}\,\mathsf{tprism}\,Q'$, a bipyramid over the twisted prism over Q'.* □

The second statement of Proposition 2.7 actually proves the conjectures **A** and **B** for half of the small cs 4-polytopes: Let $P = \mathsf{tprism}\,Q$ and Q a pyramid over a quadrilateral. By the second statement P is combinatorially equivalent to $\mathsf{bip}\,P'$, where P' is a cs 3-polytope. In terms of f-polynomials, it is easy to show that for a bipyramid $f_{\mathsf{bip}\,Q}(t) = (2+t)f_Q(t)$. Thus

$$s(P) = f_{\mathsf{bip}\,P'}(1) = 3f_{P'}(1) \geq 3^4.$$

Since **B** is true in dimension 3 there is a 3-dimensional Hanner polytope H such that $f_i(P') \geq f_i(H)$ for $i = 0, 1, 2$. From the above identity of f-polynomials it follows that $f_i(\mathsf{bip}\,P') \geq f_i(\mathsf{bip}\,H)$ for $1 \leq i \leq 3$, where $\mathsf{bip}\,H = I \oplus H$ is a Hanner polytope.

The next lemma shows that the above class already contains all small polytopes, which finally settles **A** and **B** for dimension 4.

Lemma 2.8. *Let $d \geq 4$ and let $P = \mathsf{tprism}\,F \subset \mathbb{R}^d$ be a cs d-polytope with F combinatorially equivalent to $\Delta_i \oplus \Delta_{d-i-1}$ and $1 \leq i \leq \frac{d-1}{2}$. Then*

$$f_{d-1}(P) \quad \geq \quad 2(1 + (i+1)(d-i)) \quad \geq \quad 2(2d-1).$$

Proof. The facet F in P has $(i+1)(d-i)$ ridges and thus F and its neighbors account for $1+(i+1)(d-i)$ facets. The result now follows by considering $-F$ as soon as we have checked that no facet G shares a ridge with F and with $-F$. This, however, is impossible, since G would have to have two vertex disjoint $(d-2)$-simplices as maximal faces and, therefore, at least $f_0(G) \geq 2d-2$ vertices. Thus $2d+2 = f_0(P) \geq f_0(G) + f_0(-G) \geq 4d-4$. $\square$

Corollary 2.9. *If $P = \mathsf{tprism}\, Q$ with $Q \cong \mathsf{bip}\, \Delta_2$, then P is large.*

3 Conjecture C in dimension 4

We will refute conjecture **C** *strongly* for dimension 4: We exhibit a flag-functional $\alpha \in \mathcal{P}_4$ and a cs 4-polytope P such that $\alpha(P) < \alpha(H)$ for every 4-dimensional Hanner polytope H.

Geometrically, this means that there is an oriented hyperplane in the vector space $\mathbb{R}^{2^{[d]}}$ that has the flag vector $(f_S(P))_S$ on its negative side, but all the flag-vectors of Hanner polytopes on its positive side, while some parallel hyperplane has the flag-vectors of *all* (not-necessarily cs) 4-polytopes on its positive side.

For this, consider the two functionals

$$\begin{aligned} \ell_1(P) &= f_{02}(P) - 3f_2(P) \\ \ell_2(P) &= f_{13}(P) - 3f_1(P) \\ &= f_{02}(P) - 3f_1(P). \end{aligned}$$

Let $F_k(P)$ be the number of 2-faces with exactly k vertices. Then $f_{02}(P) = \sum_{k\geq 3} k \cdot F_k(P)$. Thus $\ell_1(P) = \sum_{k\geq 4}(k-3)\cdot F_k(P)$, which is clearly non-negative for every 4-polytope. In case of equality the polytope is 2-*simplicial.* For the second functional note that $\ell_2(P) = \ell_1(P^\triangle) \geq 0$ and the bound is attained by the 2-*simple* polytopes. Thus, the functional

$$\alpha(P) \quad := \quad \frac{1}{2}(\ell_1 + \ell_2) \quad = \quad f_{02} - \frac{3}{2}(f_1 + f_2)$$

is non-negative for all 4-polytopes; it vanishes exactly for 2-simple 2-simplicial polytopes. (See [19] for examples of such polytopes.)

Consider the cs 4-polytope

$$P_4 \quad := \quad [-1,+1]^4 \cap \{\mathbf{x} \in \mathbb{R}^4 : -2 \leq x_1 + \cdots + x_4 \leq 2\}$$

which arises from the 4-cube C_4 by *chopping off* the vertices $\pm\mathbb{1}$ by hyperplanes that pass through the respective neighbors. It is straightforward to verify that the f-vector of P_4 is

$$f(P_4) = (10, 32, 36, 14).$$

Indeed, the only faces that go missing are the $2 \cdot 4$ edges incident to the two vertices; the added faces are the faces of strictly positive dimension of the vertex figures at $\mathbb{1}$ and $-\mathbb{1}$. Concerning the number of vertex–2-face incidences: there are only triangles and quadrilaterals. The number of triangles is twice the number of 2-faces and facets incident to any given vertex. Thus, $f_{02} = 3 \cdot 20 + 4 \cdot 12 = 108$ and $\alpha(P_4) = 6$.

Theorem 1.2 now follows from inspecting the following table, which lists in its first row the data for P_4, and then (extended) data for the 4-dimensional Hanner polytopes:

	$(f_0, f_1, f_2\ f_3)$	f_{02}	α
P_4	$(10, 32, 36\ 14)$	108	6
C_4	$(16, 32, 24\ \ 8)$	96	12
$C_4^{\triangle}$	$(\ 8, 24, 32\ 16)$	96	12
bip C_3	$(10, 28, 30\ 12)$	96	9
prism $C_3^{\triangle}$	$(12, 30, 28\ 10)$	96	9

4 The central hypersimplices $\tilde{\Delta}_k = \Delta(k, 2k)$

For natural numbers $d > k > 0$, the (k, d)-*hypersimplex* is the $(d-1)$-dimensional polytope

$$\Delta(k, d) = \mathsf{conv}\left\{\mathbf{x} \in \{0,1\}^d : x_1 + x_2 + \cdots + x_d = k\right\} \subset \mathbb{R}^d.$$

Hypersimplices were considered as (regular) polytopes in [7, §11.8] (see also [19, Sect. 3.3.2] and [10, Exercise 4.8.16]), as well as in connection with algebraic geometry in [8], [9], and [25].

One rather simple observation is that $\Delta(k, d)$ and $\Delta(d-k, d)$ are affinely isomorphic under the map $\mathbf{x} \mapsto \mathbb{1} - \mathbf{x}$. In particular, the hypersimplex $\tilde{\Delta}_k := \Delta(k, 2k)$ is a centrally symmetric $(2k-1)$-polytope with $f_0(\tilde{\Delta}_k) = \binom{2k}{k}$ vertices.

In a different, full-dimensional realization, the central hypersimplex is given by

$$\tilde{\Delta}_k \quad \cong \quad \mathsf{conv}\left\{\mathbf{x} \in \{+1,-1\}^{2k-1} : -1 \leq x_1 + x_2 + \cdots + x_{2k-1} \leq 1\right\}.$$

From this realization it is easy to see that for $k \geq 2$ the hypersimplex $\tilde{\Delta}_k$ is a twisted prism over $\Delta(k, 2k-1)$ with $f_{2k-2}(\tilde{\Delta}_k) = 4k = 2(2k-1)+2$ facets: Since the above realization lives in an odd-dimensional space, the sum of the coordinates for any vertex is either $+1$ or -1. The points satisfying $\sum_i x_i = 1$ form a face that is affinely isomorphic to $\Delta(k, 2k-1)$. To verify the number of facets, observe that $\tilde{\Delta}_k$ is the intersection of the $2k$-cube with a hyperplane that cuts all its $4k$ facets.

We will show that in odd dimensions $d = 2k-1 \geq 5$ a d-dimensional Hanner polytope that has no more facets than $\tilde{\Delta}_k$ has way too many vertices for conjecture **B**. In even dimensions $d \geq 6$ Theorem 1.3 follows then by taking a prism over $\tilde{\Delta}_k$. The following proposition gathers the information needed about Hanner polytopes.

Proposition 4.1. *Let H be a d-dimensional Hanner polytope. Then*

(a) $f_{d-1}(H) \geq 2d$.
(b) *If $f_{d-1}(H) = 2d$, then H is a d-cube.*
(c) *If $f_{d-1}(H) = 2d+2$, then $H = C_{d-3} \times C_3^{\triangle}$.*

Proof. Since all three claims are certainly true for Hanner polytopes of dimension $d \leq 3$, let us assume that $d \geq 4$. By definition, H is the direct sum or product of two Hanner polytopes H' and H'' of dimensions i and $d-i$ with $1 \leq i \leq \frac{d}{2}$.

If $H = H' \oplus H''$, then, by induction on d, we get

$$f_{d-1}(H) = f_{i-1}(H') \cdot f_{d-i-1}(H'') \geq 4i(d-i) \geq 2d+4.$$

Therefore, we can assume that $H = H' \times H''$ and $f_{d-1}(H) = f_{i-1}(H') + f_{d-i-1}(H'') \geq 2d$ which proves (a). The condition in (b) is satisfied if and only if it is satisfied for each of the two factors. Therefore, by induction, both factors are cubes and so is their product.

Similarly, the condition in (c) is satisfied iff it is satified for one of the two factors. By using (a) we see that the remaining factor is a cube, which proves (c). $\square$

Proof of Theorem 1.3. Let $d = 2k - 1 \geq 5$ and let H be a d-dimensional Hanner polytope with $f_i(H) \leq f_i(\tilde{\Delta}_k)$ for all $i = 0, \dots, d-1$. Since the hypersimplex $\tilde{\Delta}_k$ has $2d + 2$ facets, it follows from Proposition 4.1 that H is either C_{2k-1} or $C_{2k-4} \times C_3^{\triangle}$. In either case, the Hanner polytope satisfies $f_0(H) \geq 3 \cdot 2^{2k-3} > \binom{2k}{k}$, where the last inequality holds for $k \geq 3$.

For even dimensions $d = 2k$ consider $\mathsf{prism}\, \tilde{\Delta}_k = I \times \tilde{\Delta}_k$, which has $2(2k-1)+4 = 2d+2$ facets. Again by Proposition 4.1, a Hanner polytope H with componentwise smaller f-vector is of the form $I \times H'$ and the result follows from the odd case. $\square$

5 Two more examples

We wish to discuss two examples of centrally symmetric polytopes that exhibit some remarkable properties, two of which are being *self-dual* and being counter-examples to conjecture **C**. Both polytopes are instances of *Hansen polytopes* [12], for which we sketch the construction.

Let $G = (V, E)$ be a *perfect* graph on the vertex set $V = \{1, \dots, d-1\}$, that is, a simple, undirected graph without induced odd cycles of length ≥ 5 (cf. Schrijver [21, Chap. 65]). Let $\mathsf{Ind}(G) \subseteq 2^V$ be the *independence complex* of G. So $\mathsf{Ind}(G)$ is the simplicial complex on the vertices V defined by the relation that $S \subseteq V$ is contained in $\mathsf{Ind}(G)$ if and only if the vertex induced subgraph $G[S]$ has no edges. To every independent set $S \in \mathsf{Ind}(G)$ associate the (characteristic) vector $\tilde{\chi}_S \in \{+1, -1\}^{d-1}$ with $(\tilde{\chi}_S)_i = +1$ if and only if $i \in S$. The collection of vectors is a subset of the vertex set of the $(d-1)$-cube. Let $P_{\mathsf{Ind}(G)} = \mathsf{conv}\,\{\tilde{\chi}_S : S \in \mathsf{Ind}(G)\} \subset [-1, +1]^{d-1}$ be the vertex induced subpolytope. The *Hansen polytope* $H(G)$ associated to G is the twisted prism over $P_{\mathsf{Ind}(G)}$. In particular, $H(G)$ is a centrally symmetric d-polytope with $f_0(H(G)) = 2\,|\mathsf{Ind}(G)|$ vertices. A graph $G = (V, E)$ is *self-complementary* if G is isomorphic to its complementary graph $\overline{G} = (V, \binom{V}{2} \backslash E)$.

Proposition 5.1. *If $G = (V, E)$ is a self-complementary, perfect graph on $d-1$ vertices, then $H(G)$ is a centrally symmetric, self-dual d-polytope.*

Proof. By [12, Thm. 4], the polytope $H(G)^{\triangle}$ is isomorphic to $H(\overline{G}) = H(G)$. $\square$

Example 5.2. Let G_4 the path on four vertices v_1, v_2, v_3, v_4. This is a self-complementary perfect graph, so $H(G_4)$ is a 5-dimensional self-dual cs polytope. We compute its f-vector, and compare it to the f-vectors of the 5-dimensional hypersimplex $\tilde{\Delta}_3$ and of the eight 5-dimensional Hanner polytopes. This results in the following table (the four Hanner polytopes not listed are the duals of the ones given here, with the corresponding reversed f-vectors):

	$(f_0, f_1, f_2, f_3, f_4)$	$f_0 + f_4$	s
$H(G_4)$	$(16, 64, 98, 64, 16)$	32	259
$\tilde{\Delta}_3$	$(20, 90, 120, 60, 12)$	32	303
$C_5^{\triangle}$	$(10, 40, 80, 80, 32)$	42	243
bip bip C_3	$(12, 48, 86, 72, 24)$	36	243
bip prism $C_3^{\triangle}$	$(14, 54, 88, 66, 20)$	34	243
prism $C_4^{\triangle}$	$(16, 56, 88, 64, 18)$	34	243

Thus $H(G_4)$ refutes conjecture **B** in dimension 5 *strongly*: its value for $f_0 + f_4$ is smaller than for any Hanner polytope. Furthermore, $H(G_4)$ has a smaller face number sum s than the hypersimplex, so in that sense it is even a better example to look at in view of conjecture **A**.

Example 5.3. Let G_5 be the path on five vertices v_1, v_2, v_3, v_4, v_5 (in this order), with an additional edge connecting the second vertex v_2 to the fourth vertex v_4 on the path. This is a self-complementary perfect graph, so we obtain a 6-dimensional self-dual cs polytope $H(G_5)$. Again its f-vector can be computed and compared to those of the prism over the 5-dimensional hypersimplex, $I \times \tilde{\Delta}_3$, which we had used for Theorem 1.3 as well as the eighteen Hanner polytopes in dimension 6 (again we do not list the duals explicitly):

	$(f_0, f_1, f_2, f_3, f_4, f_5)$	$f_0 + f_5$	s
$H(G_5)$	$(24, 116, 232, 232, 116, 24)$	48	745
prism $\tilde{\Delta}_3$	$(40, 200, 330, 240, 84, 14)$	54	908
$C_6^{\triangle}$	$(12, 60, 160, 240, 192, 64)$	76	729
bip bip bip C_3	$(14, 72, 182, 244, 168, 48)$	62	729
bip bip prism $C_3^{\triangle}$	$(16, 82, 196, 242, 152, 40)$	56	729
bip prism $C_4^{\triangle}$	$(18, 88, 200, 240, 146, 36)$	54	729
bip bip C_4	$(20, 100, 216, 232, 128, 32)$	52	729
prism $C_5^{\triangle}$	$(20, 90, 200, 240, 144, 34)$	54	729
bip prism bip C_3	$(22, 106, 220, 230, 122, 28)$	50	729
prism bip bip C_3	$(24, 108, 220, 230, 120, 26)$	50	729
$C_3 \oplus C_3$	$(16, 88, 204, 240, 144, 36)$	52	729

Thus $H(G_5)$ is a self-dual cs polytope that also refutes conjecture **B** in dimension 6 *strongly.* Moreover, also looking at the pair (f_1, f_4) suffices to derive a contradiction to conjecture **B**. In these respects, $H(G_5)$ is the nicest and strongest counter-example that we currently have for conjecture **B** in dimension 6.

Note that there are no self-complementary (perfect) graphs on 6 or on 7 vertices, since $\binom{6}{2} = 15$ and $\binom{7}{2} = 21$ are odd. Thus, we cannot derive self-dual polytopes in dimensions 7 or 8 from Hansen's construction.

The Hansen polytopes, derived from perfect graphs, are subject to further research. For example, $H(G_4)$ and $H(G_5)$ are interesting examples in view of the Mahler conjecture, since they exhibit only a small deviation from the Mahler volume of the d-cube, which is conjectured to be minimal (see Kuperberg [15] and Tao [26]).

The Hansen polytopes in turn are special cases of *weak Hanner polytopes*, as defined by Hansen [12], which are twisted prisms over any of their facets. Greg Kuperberg has observed that all of these are equivalent to ±1-polytopes.

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