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# ON SOME NEW CONGRUENCES FOR BINOMIAL COEFFICIENTS

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Abstract. In this paper we establish some new congruences involving central binomial coefficients as well as Catalan numbers. Let $p$ be a prime and let $a$ be any positive integer. We determine $\sum_{k=0}^{p^a-1}\binom{2k}{k+d} \bmod p^2$ for $d=0,\ldots,p^a$ and $\sum_{k=0}^{p^a-1}\binom{2k}{k+\delta} \bmod p^3$ for $\delta=0,1$. We also show that

$$\frac{1}{C_n}\sum_{k=0}^{p^a-1} C_{p^a n+k} \equiv 1-3(n+1)\left(\frac{p^a-1}{3}\right) \pmod{p^2}$$

for every $n=0,1,2,\ldots$, where $C_m$ is the Catalan number $\binom{2m}{m}/(m+1)$, and $(\frac{\cdot}{3})$ is the Legendre symbol.

## 1. Introduction

For $n\in\mathbb{N}=\{0,1,2,\ldots\}$, the $n$th Catalan number is given by

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n+1}.$$

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Here is an alternate definition:

$$C_0 = 1 \quad\text{and}\quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad (n = 0, 1, 2, \ldots).$$

The Catalan numbers play important roles in combinatorics; they arise naturally in many enumeration problems (see, e.g., [St, pp. 219–229]). For example, $C_n$ is the number of binary parenthesizations of a string of $n+1$ letters, and it is also the number of ways to triangulate a convex $(n+2)$-gon into $n$ triangles by $n-1$ diagonals that do not intersect in their interiors.

In 2006 H. Pan and Z. W. Sun [PS] employed a useful identity to deduce many congruences on Catalan numbers, in particular they determined the partial sum $\sum_{k=0}^{p-1} C_k$ modulo a prime $p$ in terms of the Legendre symbol $(\frac{\cdot}{3})$. For any $a \in \mathbb{Z}$, $(\frac{a}{3}) \in \{0, \pm 1\}$ satisfies the congruence $a \equiv (\frac{a}{3}) \pmod 3$.

In this paper we establish some further congruences involving Catalan numbers and related central binomial coefficients.

For an assertion $A$, we adopt Iverson's notation:

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

For $d \in \mathbb{N}$ we set

$$S_d = \sum_{0<k<d} \frac{(-1)^{k-1}}{k} \left(\frac{d-k}{3}\right). \tag{1.0}$$

Note that

$$S_0 = S_1 = 0,\ S_2 = 1,\ S_3 = -\frac{3}{2},\ S_4 = \frac{5}{6},\ S_5 = \frac{5}{12},\ S_6 = -\frac{21}{20}.$$

Here is our first theorem.

**Theorem 1.1.** *Let $p$ be a prime, and let $d \in \{0, \ldots, p^a\}$ with $a \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Set*

$$\begin{aligned} F(d) =& \frac{1}{2}\left(\sum_{k=0}^{p^a-1} \binom{2k}{k+d} - \left(\frac{p^a-d}{3}\right)\right) \\ &+ (-1)^p p^a S_d - p[p=3]\left(\frac{d}{3}\right). \end{aligned} \tag{1.1}$$

*Then*

$$F(d) \equiv p[p = 2\ \&\ 3 \nmid (-1)^a + d] \pmod{p^2} \tag{1.2}$$

*and*

$$F(p^a - d) \equiv F(d) \pmod{p^3}. \tag{1.3}$$

*Remark* 1.1. Let $p$ be any prime, and let $a\in\mathbb{Z}^+$ and $\delta\in\{0,1\}$. (1.3) in the case $d=\delta$ yields the congruence

$$\begin{aligned}&\frac{1}{2}\left(\sum_{k=0}^{p^a-1}\binom{2k}{k+\delta}-\left(\frac{p^a-\delta}{3}\right)\right)\\&\equiv 2\delta p[p=3]+(-1)^{p-1}p^a\sum_{k=1}^{p^a-1}\frac{(-1)^k}{k}\left(\frac{p^a-\delta-k}{3}\right)\pmod{p^3}.\end{aligned}\tag{1.4}$$

**Corollary 1.1.** *Let $p$ be any prime and let $a\in\mathbb{Z}^+$. For $d\in\{0,1,\ldots,p^a\}$, we have*

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+d}\equiv\left(\frac{p^a-d}{3}\right)-p[p=3]\left(\frac{d}{3}\right)+2p^aS_d\pmod{p^2}.\tag{1.5}$$

*Also,*

$$\sum_{k=0}^{p^a-1}C_k\equiv 1-3\left(\frac{p^a-1}{3}\right)\equiv\frac{3(\frac{p^a}{3})-1}{2}\pmod{p^2}\tag{1.6}$$

*and*

$$\sum_{k=0}^{p^a-1}kC_k\equiv\left(\frac{p^a-1}{3}\right)-p[p=3]\equiv\frac{1-(\frac{p^a}{3})}{2}\pmod{p^2}.\tag{1.7}$$

*Proof.* (1.5) holds since $2F(d)\equiv 0\pmod{p^2}$. As $C_k=\binom{2k}{k}-\binom{2k}{k+1}$ and $kC_k=\binom{2k}{k+1}$, (1.6) and (1.7) follow from (1.5) with $d=0,1$. $\square$

*Remark* 1.2. Let $p$ be a prime and let $a\in\mathbb{Z}^+$. For $d=0,1,\ldots,p^a$, (1.5) implies that

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+d}\equiv\left(\frac{p^a-d}{3}\right)\pmod{p},$$

which was proved by Pan and Sun [PS, Theorem 1.2] in the case $a=1$ via a sophisticated combinatorial identity. (1.5) in the case $d=0,1$ yields that

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\pmod{p^2}\tag{1.8}$$

and

$$\sum_{k=1}^{p^a-1}\binom{2k}{k+1}\equiv\left(\frac{p^a-1}{3}\right)-p[p=3]\pmod{p^2}.\tag{1.9}$$

(1.8) in the case $a = 1$ implies the following observation of A. Adamchuk [A] (who told the second author that he could not find a proof): If $p > 3$ then

$$\sum_{k=1}^{p+(\frac{p+1}{3})} \binom{2k}{k} \equiv 0 \pmod{p^2}.$$

(Recall the Wolstenholme congruence $\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ for $p > 3$ (see, e.g., [HT]).) Recently, (1.8) in the case $a = 2$ was posed by D. Callan as a problem in [C]; in fact, (1.8) in the case $a \in \{2, 3, 4\}$ was also observed by A. Adamchuk [A] slightly earlier who could not provide a proof.

Now we state our second theorem.

**Theorem 1.2.** *Let $p$ be a prime. Let $d \in \{0, 1, \ldots, p^a\}$ with $a \in \mathbb{Z}^+$, and let $m, n \in \mathbb{N}$ with $m \geqslant n$. Then*

$$\begin{aligned}&\frac{n+1}{\binom{m}{n}} \sum_{k=0}^{p^a-1} \binom{p^a m + 2k}{p^a n + k + d} - (n+1)\left(\frac{p^a - d}{3}\right) - (m-n)\left(\frac{d}{3}\right) \\ \equiv&((m-n)^2 + (m+1)(n+2))p^a S_d - [p=3]\left(\frac{d}{3}\right) p(n+1)(m+n+1) \\ &+ [p = 2 \ \&\ 3 \nmid d - (-1)^a] pm(n+1) \pmod{p^2}.\end{aligned} \tag{1.10}$$

*In particular,*

$$\begin{aligned}&\frac{1}{C_n} \sum_{k=0}^{p^a-1} \binom{2p^a n + 2k}{p^a n + k + d} - n\left(\frac{d}{3}\right) - (n+1)\left(\frac{p^a - d}{3}\right) \\ \equiv&p^a(n+1)(3n+2)S_d - [p=3]p(n+1)\left(\frac{d}{3}\right) \pmod{p^2}.\end{aligned} \tag{1.11}$$

**Corollary 1.2.** *Let $p$ be a prime and let $a, n \in \mathbb{N}$ with $a > 0$. Then*

$$\frac{1}{C_n} \sum_{k=0}^{p^a-1} C_{p^a n + k} \equiv 1 - 3(n+1)\left(\frac{p^a - 1}{3}\right) \pmod{p^2} \tag{1.12}$$

*and*

$$\begin{aligned}&\frac{1}{C_n} \sum_{k=0}^{p^a-1} k C_{p^a n + k} + [p=3]p(n+1) \\ \equiv&(1 - p^a)n + (3p^a n + 1)(n+1)\left(\frac{p^a - 1}{3}\right) \pmod{p^2}.\end{aligned} \tag{1.13}$$

*Remark* 1.3. Note that (1.12) and (1.13) are extensions of (1.6) and (1.7).

(1.6) and (1.8) in the case $a = 1$ suggest the following open problem.

**Problem 1.1.** *Are there any composite numbers* $n \not\equiv 0 \pmod 3$ *such that*

$$\sum_{k=0}^{n-1}\binom{2k}{k} \equiv \left(\frac{n}{3}\right) \pmod{n^2}\ ?$$

*Are there any composite numbers* $n \not\equiv 0 \pmod 3$ *satisfying*

$$\sum_{k=0}^{n-1} C_k \equiv 1 - 3\left(\frac{n-1}{3}\right) \pmod{n^2}\ ?$$

*Remark* 1.4. It seems that the answers to Problem 1.1 are negative. We have confirmed this for $n \leqslant 10^4$ via Maple.

We are going to do some preparations in the next section. We will show Theorem 1.1 in Sections 3, and prove Theorem 1.2 and Corollary 1.2 in Section 4.

## 2. Some lemmas

As usual, for a prime $p$ and an integer $m$, we define

$$\mathrm{ord}_p(m) = \sup\{a \in \mathbb{N} :\ p^a \mid m\}$$

(and thus $\mathrm{ord}_p(0) = +\infty$). A congruence modulo $+\infty$ refers to the corresponding equality.

**Lemma 2.1.** *Let $p$ be any prime, and let $a \in \mathbb{Z}^+$ and $m, n \in \mathbb{N}$ with $m \geqslant n$. Then*

$$\binom{p^a m}{p^a n}\Big/\binom{m}{n} \equiv 1 + [p = 2]pn(m-n) \pmod{p^{2+\mathrm{ord}_p(n)}}. \tag{2.1}$$

*Proof.* (2.1) holds trivially when $n$ is 0 or $m$.

Below we assume $0 < n < m$. Observe that

$$\binom{p^a m}{p^a n} = \prod_{j=0}^{p^a n-1}\frac{p^a m - j}{p^a n - j} = \prod_{i=0}^{n-1}\frac{p^a m - p^a i}{p^a n - p^a i} \times \prod_{\substack{0 \leqslant j < p^a n\\ p^a \nmid j}}\left(1 + \frac{p^a(m-n)}{p^a n - j}\right).$$

Thus

$$\begin{aligned}
\frac{\binom{p^a m}{p^a n}}{\binom{m}{n}} \equiv& 1 + (m-n)\sum_{\substack{0 \leqslant j < p^a n\\ p^a \nmid j}}\frac{p^a}{p^a n - j} = 1 + (m-n)\sum_{\substack{0 < i < p^a n\\ p^a \nmid i}}\frac{p^a}{i}\\
\equiv& 1 + (m-n)\sum_{q=0}^{n-1}\sum_{k=1}^{p^a-1}\frac{p^a}{p^a q + k} \equiv 1 + (m-n)n\sum_{k=1}^{p^a-1}\frac{p^a}{k}\\
\equiv& 1 + [p = 2]p(m-n)n \pmod{p^{2+\mathrm{ord}_p(m-n)}}
\end{aligned}$$

since

$$2\sum_{k=1}^{p^a-1}\frac{p^a}{k}=\sum_{k=1}^{p^a-1}\left(\frac{p^a}{k}+\frac{p^a}{p^a-k}\right)=\sum_{k=1}^{p^a-1}\frac{p^a}{k}\cdot\frac{p^a}{p^a-k}\equiv 0 \pmod{p^2}$$

and

$$\sum_{k=1}^{2^a-1}\frac{2^a}{k}-\sum_{j=1}^{2^{a-1}-1}\frac{2^{a-1}}{j}=\sum_{\substack{k=1\\2\nmid k}}^{2^a-1}\frac{2^a}{k}\equiv 0 \pmod{2^a}.$$

Similarly,

$$\begin{aligned}\binom{pm}{pn}\Big/\binom{m}{n}=&\binom{pm}{p(m-n)}\Big/\binom{m}{m-n}\\ \equiv&1+[p=2]pn(m-n) \pmod{p^{2+\mathrm{ord}_p(n)}}.\end{aligned}$$

We are done. $\square$

*Remark* 2.1. By a deep result of Jacobsthal (see, e.g., [Gr]), if $p>3$ is a prime and $m\geqslant n\geqslant 0$ are integers, then $\binom{pm}{pn}=\binom{m}{n}\left(1+p^3mn(m-n)v\right)$ for some $p$-adic integer $v$. However, the proof of Jacobsthal's result is complicated and not easily found in modern literature.

**Lemma 2.2.** *Let $p$ be a prime and let $a\in\mathbb{Z}^+$. Then*

$$\begin{aligned}&\frac{1}{2}\sum_{k=1}^{p^a-1}\binom{2k}{k}+\sum_{k=1}^{p^a-1}\binom{2k}{k+1}\\ =&\binom{2p^a-1}{p^a-1}-1\equiv p[p=2]+p^2[p=3] \pmod{p^3}.\end{aligned}\tag{2.2}$$

*Proof.* Clearly

$$\binom{2k}{k}+\binom{2k}{k+1}=\binom{2k+1}{k+1}=\frac{1}{2}\binom{2k+2}{k+1}$$

for all $k\in\mathbb{N}$. Thus

$$2\sum_{k=0}^{p^a-1}\binom{2k}{k}+2\sum_{k=0}^{p^a-1}\binom{2k}{k+1}=\sum_{k=0}^{p^a-1}\binom{2k+2}{k+1}=\sum_{k=1}^{p^a}\binom{2k}{k}$$

and hence

$$\frac{1}{2}\sum_{k=1}^{p^a-1}\binom{2k}{k}+\sum_{k=1}^{p^a-1}\binom{2k}{k+1}=\frac{1}{2}\binom{2p^a}{p^a}-1=\binom{2p^a-1}{p^a-1}-1.$$

By Lemma 2.1,

$$\frac{\binom{2p^i}{p^i}/2}{\binom{2p^{i-1}}{p^{i-1}}/2} \equiv 1+[p=2]p^{2i-1} \equiv 1 \pmod{p^{i+1}} \quad \text{for } i=2,3,\ldots.$$

So we have

$$\frac{1}{2}\binom{2p^a}{p^a} \equiv \frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1+p[p=2]+p^2[p=3] \pmod{p^3}$$

by applying the well-known Wolstenholme congruence in the last step.

Clearly (2.2) follows from the above. $\square$

**Lemma 2.3.** *Let $n>1$ and $d$ be integers. Then*

$$\sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{d+k}{3}\right) = (1+(-1)^n-3[3\mid n])\left(\frac{d-n}{3}\right). \tag{2.3}$$

*Proof.* Let $\omega$ denote the cubic root $(-1+\sqrt{-3})/2$ of unity. As observed by E. Lehmer [L1] in 1938, for any $r\in\mathbb{Z}$ we have

$$\begin{aligned}3\sum_{k\equiv r \,(\mathrm{mod}\ 3)}\binom{n}{k} =&\sum_{k=0}^{n}\binom{n}{k}(1+\omega^{k-r}+\omega^{2(k-r)})\\ =&2^n+\omega^{-r}(1+\omega)^n+\omega^{-2r}(1+\omega^2)^n\\ =&2^n+\omega^{-r}(-\omega^2)^n+\omega^{r}(-\omega)^n\\ =&2^n+(-1)^n(\omega^{n+r}+\omega^{-n-r})\\ =&\begin{cases}2^n+2(-1)^n & \text{if } 3\mid n+r,\\ 2^n-(-1)^n & \text{if } 3\nmid n+r.\end{cases}\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{d+k}{3}\right)+\left(\frac{d}{3}\right)+\left(\frac{d+n}{3}\right)\\ =&\sum_{k=0}^{n}\binom{n}{k}\left(\frac{d+k}{3}\right)=\sum_{k\equiv 1-d\,(\mathrm{mod}\ 3)}\binom{n}{k}-\sum_{k\equiv 2-d\,(\mathrm{mod}\ 3)}\binom{n}{k}\\ =&\frac{2^n+(-1)^n(3[3\mid n+1-d]-1)}{3}-\frac{2^n+(-1)^n(3[3\mid n+2-d]-1)}{3}\\ =&(-1)^n([3\mid n+1-d]-[3\mid n+2-d])=(-1)^n\left(\frac{d-n}{3}\right).\end{aligned}$$

Since

$$\left(\frac{d-n}{3}\right)+\left(\frac{d}{3}\right)+\left(\frac{d+n}{3}\right)=3[3\mid n]\left(\frac{d}{3}\right),$$

we finally obtain the desired (2.3). $\square$

*Remark* 2.2. The evaluation of $\sum_{k\equiv r\,(\mathrm{mod}\ 12)}\binom{n}{k}$ with $n\in\mathbb{N}$ and $r\in\mathbb{Z}$, can be found in [Su].

**Lemma 2.4.** *Let $p$ be a prime and let $a \in \mathbb{Z}^+$. Then*

$$p^a\bigg(3\sum_{\substack{k=1\\ 3\mid k-p^a}}^{p^a-1}\frac{(-1)^k}{k}-\sum_{k=1}^{p^a-1}\frac{(-1)^k}{k}\bigg)\equiv p[p=2]+\frac{p}{2}[p=3] \pmod{p^3}. \quad (2.4)$$

*Proof.* By Lehmer's result mentioned in the proof of Lemma 2.3,

$$\begin{aligned}3\sum_{\substack{0<k<p^a\\ 3\mid k-p^a}}\binom{p^a}{k}+3[3\mid p^a]+3&=3\sum_{k\equiv p^a \pmod 3}\binom{p^a}{k}\\ =&2^{p^a}+(-1)^{p^a}(3[3\mid p^a+p^a]-1)=2^{p^a}-3[3\mid p^a]-(-1)^{p^a}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{k=1}^{p^a-1}\binom{p^a}{k}-3\sum_{\substack{k=1\\ 3\mid k-p^a}}^{p^a-1}\binom{p^a}{k}\\ =&(1+1)^{p^a}-2-\Big(2^{p^a}-6[p=3]-3-(-1)^p\Big)=6[p=3]+2[p=2].\end{aligned}$$

For $k=1,\ldots,p^a-1$, since $p^a/k\equiv 0 \pmod p$ and

$$\binom{p^a-1}{k-1}(-1)^{k-1}=\prod_{0<j<k}\Big(1-\frac{p^a}{j}\Big)\equiv 1-\sum_{0<j<k}\frac{p^a}{j} \pmod{p^2},$$

we have

$$\binom{p^a}{k}+p^a\frac{(-1)^k}{k}=\frac{p^a}{k}\bigg(\binom{p^a-1}{k-1}+(-1)^k\bigg)\equiv(-1)^k\frac{p^a}{k}\sum_{0<j<k}\frac{p^a}{j} \pmod{p^3}.$$

If $1\leqslant k\leqslant p^a-1$ and $p^{a-1}\nmid k$, then $p^a/k\equiv 0 \pmod{p^2}$. Note also that $-1\equiv 1 \pmod 2$. Thus

$$\begin{aligned}&\sum_{k=1}^{p^a-1}(3[3\mid k-p^a]-1)(-1)^k\frac{p^a}{k}\sum_{0<j<k}\frac{p^a}{j}\\ \equiv&\sum_{k=1}^{p-1}(3[3\mid p^{a-1}k-p^a]-1)(-1)^{p^{a-1}k}\frac{p^a}{p^{a-1}k}\sum_{0<j<p^{a-1}k}\frac{p^a}{j}\\ \equiv&\sum_{k=1}^{p-1}(3[3\mid p^{a-1}(k-p)]-1)(-1)^k\frac{p}{k}\sum_{0<j<k}\frac{p^a}{p^{a-1}j}\\ \equiv&\sum_{k=1}^{p-1}(3[p\neq 3\ \&\ 3\mid k-p]-1)(-1)^k\frac{p}{k}\sum_{0<j<k}\frac{p}{j} \pmod{p^3}.\end{aligned}$$

Combining the above,

$$p^a\sum_{k=1}^{p^a-1}(3[3\mid k-p^a]-1)\frac{(-1)^k}{k}\ \mathrm{mod}\ p^3$$

does not depend on $a$. So it suffices to prove (2.4) for $a=1$.

When $a=1$ and $p\in\{2,3\}$, (2.4) can be verified directly.

Below we assume $p>3$. With the help of Wolstenholme's result $\sum_{k=1}^{p-1}1/k\equiv 0 \pmod{p^2}$ (cf. [Gr] or [HT]), we see that

$$\begin{aligned}&3\sum_{\substack{k=1\\3\mid k-p}}^{p-1}\frac{(-1)^k}{k}-\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\\ \equiv&-\sum_{j=1}^{(p-1)/2}\frac{1}{j}+3\sum_{j=1}^{\lfloor p/3\rfloor}\frac{1}{p-3j}-6\sum_{j=1}^{\lfloor p/6\rfloor}\frac{1}{p-6j}\pmod{p^2}.\end{aligned}$$

Recall the following congruences of E. Lehmer [L2]:

$$\begin{aligned}\sum_{j=1}^{(p-1)/2}\frac{1}{j}\equiv&-2q_p(2)+pq_p^2(2)\pmod{p^2},\\ \sum_{j=1}^{\lfloor p/3\rfloor}\frac{1}{p-3j}\equiv&\frac{q_p(3)}{2}-\frac{p}{4}q_p^2(3)\pmod{p^2},\\ \sum_{j=1}^{\lfloor p/6\rfloor}\frac{1}{p-6j}\equiv&\frac{q_p(2)}{3}+\frac{q_p(3)}{4}-\frac{p}{6}q_p^2(2)-\frac{p}{8}q_p^2(3)\pmod{p^2},\end{aligned}$$

where $q_p(2)=(2^{p-1}-1)/p$ and $q_p(3)=(3^{p-1}-1)/p$ are Fermat quotients. Consequently,

$$\sum_{k=1}^{p-1}(3[3\mid k-p]-1)\frac{(-1)^k}{k}\equiv 0\pmod{p^2}$$

and hence (2.4) holds for $a=1$.

In view of the above, we have completed the proof. □

## 3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 on the basis of Section 2.

**Lemma 3.1.** *Let $p$ be a prime and let $d \in \{1, \ldots, p^a - 1\}$ with $a \in \mathbb{Z}^+$. For the function $F$ given by (1.1) we have*

$$\begin{aligned}&F(d-1)+F(d)+F(d+1)\\ \equiv&2\binom{p^a}{d} - p^a\left(\frac{(-1)^{p^a-d}}{d} + (-1)^{p^a-1}\frac{(-1)^d}{p^a-d}\right) \pmod{p^3} \qquad (3.1)\\ \equiv&0 \pmod{p^2}.\end{aligned}$$

*Proof.* Observe that

$$\begin{aligned}&\binom{2k}{k+d-1}+\binom{2k}{k+d}+\binom{2k}{k+d+1}\\ =&\binom{2k+1}{k+d}+\left(\binom{2k+1}{k+d+1}-\binom{2k}{k+d}\right)=\binom{2k+2}{k+d+1}-\binom{2k}{k+d}\end{aligned}$$

for every $k = 0, 1, 2, \ldots$. Thus

$$\begin{aligned}&\sum_{k=0}^{p^a-1}\left(\binom{2k}{k+d-1}+\binom{2k}{k+d}+\binom{2k}{k+d+1}\right)\\ =&\sum_{k=0}^{p^a-1}\left(\binom{2(k+1)}{(k+1)+d}-\binom{2k}{k+d}\right)=\binom{2p^a}{p^a+d}-\binom{0}{d}=\binom{2p^a}{p^a-d}.\end{aligned}$$

Note also that

$$\begin{aligned}&\sum_{\varepsilon=-1}^{1}\sum_{0<k<d+\varepsilon}\frac{(-1)^k}{k}\left(\frac{d+\varepsilon-k}{3}\right)\\ =&\sum_{0<k<d}\frac{(-1)^k}{k}\left(\left(\frac{d-1-k}{3}\right)+\left(\frac{d-k}{3}\right)+\left(\frac{d+1-k}{3}\right)\right)\\ &+\frac{(-1)^d}{d}\left(\frac{d+1-d}{3}\right)\\ =&\frac{(-1)^d}{d}.\end{aligned}$$

Therefore

$$\begin{aligned}&F(d-1)+F(d)+F(d+1)\\ =&\frac{1}{2}\left(\binom{2p^a}{p^a-d}-\left(\frac{p^a-d+1}{3}\right)-\left(\frac{p^a-d}{3}\right)-\left(\frac{p^a-d-1}{3}\right)\right)\\ &+(-1)^{p-1}p^a\frac{(-1)^d}{d}-p[p=3]\left(\left(\frac{d-1}{3}\right)+\left(\frac{d}{3}\right)+\left(\frac{d+1}{3}\right)\right)\\ =&\frac{p^a}{p^a-d}\binom{2p^a-1}{p^a-d-1}+(-1)^{p^a-d-1}\frac{p^a}{d}.\end{aligned}$$

Clearly

$$\begin{aligned}(-1)^{p^a-d-1}\binom{2p^a-1}{p^a-d-1} &= \prod_{0<r<p^a-d}\left(1-\frac{2p^a}{r}\right)\\ \equiv 1-\sum_{0<r<p^a-d}\frac{2p^a}{r} &= 2\left(1-\sum_{0<r<p^a-d}\frac{p^a}{r}\right)-1\\ \equiv 2(-1)^{p^a-d-1}\binom{p^a-1}{p^a-d-1}&-1 \pmod{p^2}.\end{aligned}$$

So, by the above we have

$$\begin{aligned}&F(d-1)+F(d)+F(d+1)\\ \equiv&\frac{p^a}{p^a-d}(-1)^{p^a-d-1}+(-1)^{p^a-d-1}\frac{p^a}{d}\\ \equiv&(-1)^{p^a-d-1}\frac{p^a}{d}\cdot\frac{p^a}{p^a-d}\equiv 0 \pmod{p^2}\end{aligned}$$

and

$$\begin{aligned}&F(d-1)+F(d)+F(d+1)\\ \equiv&2\binom{p^a}{p^a-d}-(-1)^{p^a-d-1}\frac{p^a}{p^a-d}+(-1)^{p^a-d-1}\frac{p^a}{d}\\ \equiv&2\binom{p^a}{d}-p^a\left(\frac{(-1)^{p^a-d}}{d}+(-1)^{p^a-1}\frac{(-1)^d}{p^a-d}\right) \pmod{p^3}.\end{aligned}$$

This concludes the proof. □

*Proof of Theorem 1.1.* (i) For $k=1,\ldots,p^a-1$, clearly

$$\binom{p^a}{k}=\frac{p^a}{k}\prod_{0<j<k}\left(\frac{p^a}{j}-1\right)\equiv p^a\frac{(-1)^k}{k} \pmod{p^2}.$$

Thus, with the help of Lemma 2.3, for $d\in\{p^a,p^a-1\}$ we have

$$\begin{aligned}F(d)=&(-1)^{p-1}p^a\sum_{k=1}^{p^a-1}\frac{(-1)^k}{k}\left(\frac{d-k}{3}\right)-p[p=3]\left(\frac{d}{3}\right)\\ \equiv&(-1)^{p-1}\sum_{k=1}^{p^a-1}\binom{p^a}{k}\left(\frac{k-d}{3}\right)-p[p=3]\left(\frac{d}{3}\right)\\ \equiv&(-1)^{p-1}\left(1+(-1)^{p^a}-3[3\mid p^a]\right)\left(\frac{-d-p^a}{3}\right)-p[p=3]\left(\frac{d}{3}\right)\\ \equiv&p[p=2]\left(\frac{d+p^a}{3}\right)\equiv p[p=2\ \&\ 3\nmid d+2^a] \pmod{p^2}.\end{aligned}$$

This proves (1.2) for $d = p^a, p^a - 1$.

Assume that $0 < d < p^a$. If

$$F(d) \equiv p[p = 2 \ \&\ 3 \nmid (-1)^a + d] \pmod{p^2}$$

and

$$F(d+1) \equiv p[p = 2 \ \&\ 3 \nmid (-1)^a + d + 1] \pmod{p^2},$$

then by Lemma 3.1 we have

$$\begin{aligned} F(d-1) \equiv & -F(d) - F(d+1) \\ \equiv & -p[p=2]([3 \nmid (-1)^a + d] + [3 \nmid (-1)^a + d + 1]) \\ \equiv & -p[p=2](2 - [3 \nmid (-1)^a + d - 1]) \\ \equiv & p[p = 2 \ \&\ 3 \nmid (-1)^a + d - 1] \pmod{p^2}. \end{aligned}$$

By induction. we obtain from the above that (1.2) holds for any $d = p^a, p^a - 1, \ldots, 0$.

(ii) By Lemma 2.2,

$$\begin{aligned} & F(0) + 2F(1) \\ = & \frac{1}{2}\left(\sum_{k=0}^{p^a-1}\binom{2k}{k} - \left(\frac{p^a}{3}\right)\right) + \sum_{k=0}^{p^a-1}\binom{2k}{k+1} - \left(\frac{p^a-1}{3}\right) - 2p[p=3] \\ \equiv & p[p=2] + p^2[p=3] + \frac{1}{2}\left(1 - \left(\frac{p^a}{3}\right)\right) - \left(\frac{p^a-1}{3}\right) - 2p[p=3] \\ \equiv & p[p=2] + \frac{p^2}{2}[p=3] \pmod{p^3}. \end{aligned}$$

With the help of Lemma 2.4,

$$\begin{aligned} & F(p^a) + 2F(p^a-1) \\ = & (-1)^{p-1}p^a\sum_{k=1}^{p^a-1}\frac{(-1)^k}{k}\left(\left(\frac{p^a-k}{3}\right) + 2\left(\frac{p^a-1-k}{3}\right)\right) \\ & - p[p=3]\left(\left(\frac{p^a}{3}\right) + 2\left(\frac{p^a-1}{3}\right)\right) \\ = & (-1)^{p-1}p^a\sum_{k=1}^{p^a-1}\frac{(-1)^k}{k}(1 - 3[3 \mid p^a - k]) + 2p[p=3] \\ \equiv & (-1)^p\left(p[p=2] + \frac{p}{2}[p=3]\right) + 2p[p=3] \\ \equiv & p[p=2] + \frac{p^2}{2}[p=3] \pmod{p^3}. \end{aligned}$$

Therefore

$$F(0)-F(p^a)\equiv -2(F(1)-F(p^a-1)) \pmod{p^3}.$$

When $p=2$ and $d\in\{1,\ldots,p^a-1\}$, we clearly have

$$\frac{p^a}{d}-\frac{p^a}{p^a-d}\equiv -\frac{p^a}{d}-\frac{p^a}{p^a-d}=-\frac{p^a}{d}\cdot\frac{p^a}{p^a-d}\equiv 0 \pmod{p^2}$$

and hence

$$\frac{p^a}{d}-\frac{p^a}{p^a-d}\equiv \frac{p^a}{p^a-d}-\frac{p^a}{d} \pmod{p^3}.$$

Thus, whether $p=2$ or not, by Lemma 3.1 we always have

$$F(d-1)+F(d)+F(d+1)\equiv F(p^a-d-1)+F(p^a-d)+F(p^a-d-1) \pmod{p^3}$$

whenever $d\in\{1,\ldots,p^a-1\}$.

Set $D(i)=F(i)-F(p^a-i)$ for $i=0,1,\ldots,p^a$. If $0\leqslant i\leqslant p^a-3$ then

$$D(i)+D(i+1)+D(i+2)\equiv 0\equiv D(i+1)+D(i+2)+D(i+3) \pmod{p^3}$$

and hence $D(i+3)\equiv D(i) \pmod{p^3}$. If $p=3$ then $-D(0)=D(p^a)\equiv D(0) \pmod{p^3}$ and hence $D(0)\equiv 0 \pmod{p^3}$. If $p^a\equiv 1 \pmod 3$ then

$$-D(0)=D(p^a)\equiv D(1) \pmod{p^3}.$$

If $p^a\equiv 2 \pmod 3$ then

$$-D(0)=D(p^a)\equiv D(2)\equiv -D(0)-D(1) \pmod{p^3}.$$

As $D(0)+2D(1)\equiv 0 \pmod{p^3}$, we always have $D(i)\equiv 0 \pmod{p^3}$ for $i=0,1$. Therefore

$$D(i)\equiv 0 \pmod{p^3}\quad \text{for all } i=0,1,2,\ldots,p^a.$$

So (1.3) is valid and we are done. □

## 4. Proofs of Theorem 1.2 and Corollary 1.2

*Proof of Theorem 1.2.* In the case $m=n=0$, (1.10) reduces to (1.5).

Below we assume $m>0$. By the Chu-Vandermonde convolution identity (cf. [GKP, (5.22)]),

$$\binom{p^am+2k}{p^an+k+d}=\sum_{j\in\mathbb{Z}}\binom{p^am}{p^an-j}\binom{2k}{k+j+d}$$

for any $k \in \mathbb{N}$. Thus we have

$$\begin{aligned}
&\sum_{k=0}^{p^a-1}\binom{p^am+2k}{p^an+k+d}-\binom{p^am}{p^an}\sum_{k=0}^{p^a-1}\binom{2k}{k+d}\\
=&\sum_{j>0}\binom{p^am}{p^an-j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j+d}+\sum_{j>0}\binom{p^am}{p^an+j}\sum_{k=0}^{p^a-1}\binom{2k}{k-j+d}\\
=&\sum_{j=1}^{p^a-1}\binom{p^am}{p^an-j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j+d}+\sum_{j=1}^{p^a-1}\binom{p^am}{p^an+j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j-d}\\
&+\binom{p^am}{p^an+p^a}\sum_{k=0}^{p^a-1}\binom{2k}{k+p^a-d}+R_d,
\end{aligned}$$

where

$$\begin{aligned}
R_d=&\sum_{p^a<j<p^a+d}\binom{p^am}{p^an+j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j-d}\\
=&\sum_{0<j<d}\binom{p^am}{p^a(n+1)+j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j+p^a-d}.
\end{aligned}$$

Note that $R_d=0$ if $d\in\{0,1\}$.

By Lemma 2.1, there are $p$-adic integers $u$ and $v$ such that

$$\binom{p^am}{p^an}=\binom{m}{n}\left(1+[p=2]p(m-n)n+p^2(m-n)u\right)$$

and

$$\begin{aligned}
\binom{p^am}{p^a(n+1)}&=\binom{m}{n+1}\left(1+[p=2]p(n+1)(m-n-1)+p^2(n+1)v\right)\\
&=\binom{m}{n}\left(\frac{m-n}{n+1}+[p=2]p(m-n)(m-n-1)+p^2(m-n)v\right).
\end{aligned}$$

Let $j\in\{1,\ldots,p^a-1\}$. When $n\neq m$, we have

$$\begin{aligned}
\binom{p^am}{p^an+j}=&\frac{(p^am)!}{(p^an)!(p^am-p^an)!}\times\frac{\prod_{0\leqslant i<j}(p^am-p^an-i)}{(p^an+1)\cdots(p^an+j)}\\
=&\binom{p^am}{p^an}\frac{p^a(m-n)}{p^an+j}\prod_{0<i<j}\frac{p^a(m-n)-i}{p^an+i}
\end{aligned}$$

and hence

$$\frac{\binom{p^am}{p^an+j}}{(m-n)\binom{m}{n}} \equiv \frac{\binom{p^am}{p^an}}{\binom{m}{n}} \cdot \frac{p^a}{j} \prod_{0<i<j} \frac{p^a-i}{i} \equiv \binom{p^a}{j} \pmod{p^2}$$

since

$$\frac{p^a}{j} - \frac{p^a}{p^an+j} = \frac{p^a}{j} \cdot \frac{p^an}{p^an+j} \equiv 0 \pmod{p^2}$$

and

$$\frac{p^a(m-n)-i}{p^an+i} \equiv \frac{p^a-i}{p^an+i} \equiv \frac{p^a-i}{i} \pmod{p} \quad \text{for} \quad 0<i<p^a.$$

Similarly, if $n \neq 0$ then

$$\frac{\binom{p^am}{p^an-j}}{n\binom{m}{n}} = \frac{\binom{p^am}{p^a(m-n)+j}}{(m-(m-n))\binom{m}{m-n}} \equiv \binom{p^a}{j} \pmod{p^2}.$$

Also,

$$\begin{aligned}\frac{n+1}{\binom{m}{n}}\binom{p^am}{p^a(n+1)+j} =&(m-n)\frac{\binom{pm}{p^a(n+1)+j}}{\binom{m}{n+1}}\\ \equiv&(m-n)(m-n-1)\binom{p^a}{j} \pmod{p^{2+\mathrm{ord}_p(m-n)}}.\end{aligned}$$

Combining the above, we find that

$$\begin{aligned}&\binom{m}{n}^{-1}\sum_{k=0}^{p^a-1}\binom{p^am+2k}{p^an+k+d} - (1+[p=2]pn(m-n))\sum_{k=0}^{p^a-1}\binom{2k}{k+d}\\ &-\frac{m-n}{n+1}\sum_{k=0}^{p^a-1}\binom{2k}{k+p^a-d} - \frac{R_d}{\binom{m}{n}}\\ \equiv&\sum_{j=1}^{p^a-1}\binom{p^a}{j}\sum_{k=0}^{p^a-1}\left(n\binom{2k}{k+j+d} + (m-n)\binom{2k}{k+j-d}\right) \pmod{p^2}\end{aligned} \tag{4.1}$$

and

$$\frac{n+1}{\binom{m}{n}}R_d \equiv (m-n)(m-n-1)\sum_{0<j<d}\binom{p^a}{j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j+p^a-d} \pmod{p^2}. \tag{4.2}$$

By (1.5),

$$\begin{aligned}\sum_{k=0}^{p^a-1}\binom{2k}{k+d}&\equiv\left(\frac{p^a-d}{3}\right)-p[p=3]\left(\frac{d}{3}\right)+2p^aS_d\pmod{p^2},\\ \sum_{k=0}^{p^a-1}\binom{2k}{k+p^a-d}&\equiv\left(\frac{d}{3}\right)-p[p=3]\left(\frac{p^a-d}{3}\right)+2p^aS_{p^a-d}\pmod{p^2}.\end{aligned}\tag{4.3}$$

Clearly $p^aS_d$ is congruent to

$$\sum_{0<k<d}\left(\frac{p^a}{k}\prod_{0<j<k}\frac{p^a-j}{j}\right)\left(\frac{d-k}{3}\right)=\sum_{0<k<d}\binom{p^a}{k}\left(\frac{d-k}{3}\right)$$

modulo $p^2$. Observe that both $p^aS_{p^a-d}$ and

$$\sum_{j=1}^{p^a-1}\binom{p^a}{j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j+d}$$

are congruent to

$$\sum_{0<j<p^a-d}\binom{p^a}{j}\left(\frac{p^a-j-d}{3}\right)=\sum_{d<k<p^a}\binom{p^a}{k}\left(\frac{k-d}{3}\right)$$

modulo $p^2$. Also,

$$\begin{aligned}\sum_{d<k<p^a}\binom{p^a}{k}\left(\frac{k-d}{3}\right)&=\sum_{k=1}^{p^a-1}\binom{p^a}{k}\left(\frac{k-d}{3}\right)-\sum_{0<k\leqslant d}\binom{p^a}{k}\left(\frac{k-d}{3}\right)\\ &=\left(1+(-1)^{p^a}-3[3\mid p^a]\right)\left(\frac{-d-p^a}{3}\right)+\sum_{0<k\leqslant d}\binom{p^a}{k}\left(\frac{d-k}{3}\right)\end{aligned}$$

with the help of Lemma 2.3. Thus,

$$\begin{aligned}p^aS_{p^a-d}\equiv&\sum_{j=1}^{p^a-1}\binom{p^a}{j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j+d}\\ \equiv&p[p=2\ \&\ 3\nmid d+(-1)^a]+p[p=3]\left(\frac{d}{3}\right)+p^aS_d\pmod{p^2}.\end{aligned}\tag{4.4}$$

Note that

$$\begin{aligned}&\sum_{j=1}^{p^a-1}\binom{p^a}{j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j-d}\\ =&\sum_{0<j<d}\binom{p^a}{j}\sum_{k=0}^{p^a-1}\binom{2k}{k+d-j}+\sum_{d\leqslant j<p^a}\binom{p^a}{j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j-d}\end{aligned}$$

is congruent to

$$\begin{aligned}&\sum_{0<j<d}\binom{p^a}{j}\left(\frac{p^a-d+j}{3}\right)+\sum_{d\leqslant j<p^a}\binom{p^a}{j}\left(\frac{p^a-j+d}{3}\right)\\=&\sum_{0<j<d}\binom{p^a}{j}\left(\left(\frac{p^a-d+j}{3}\right)-\left(\frac{p^a-j+d}{3}\right)\right)+\sum_{k=1}^{p^a-1}\binom{p^a}{k}\left(\frac{k+d}{3}\right)\\=&(1-p[p=3])\sum_{0<j<d}\binom{p^a}{j}\left(\frac{d-j}{3}\right)+(1+(-1)^{p^a}-3[3\mid p^a])\left(\frac{d-p^a}{3}\right)\end{aligned}$$

modulo $p^2$ in light of (1.5) and Lemma 2.3. Therefore

$$\begin{aligned}&\sum_{j=1}^{p^a-1}\binom{p^a}{j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j-d}\\\equiv&p^aS_d+p[p=2\ \&\ 3\nmid d-(-1)^a]-p[p=3]\left(\frac{d}{3}\right)\pmod{p^2}.\end{aligned}\tag{4.5}$$

As

$$\begin{aligned}&\sum_{0<j<d}\binom{p^a}{j}\sum_{k=0}^{p^a-1}\binom{2k}{k+j+p^a-d}\\\equiv&\sum_{0<j<d}\binom{p^a}{j}\left(\frac{p^a-(j+p^a-d)}{3}\right)\equiv p^aS_d\pmod{p^2},\end{aligned}$$

(4.2) yields that

$$\frac{n+1}{\binom{m}{n}}R_d\equiv(m-n)(m-n-1)p^aS_d\pmod{p^2}.\tag{4.6}$$

Combining (4.1) with (4.3)–(4.6), we finally obtain (1.10). Note that (1.11) follows from (1.10) in the case $m=2n$. This ends the proof. □

*Proof of Corollary 1.2.* By (1.11) in the case $d\in\{0,1\}$, we have

$$\begin{aligned}&\frac{1}{C_n}\sum_{k=0}^{p^a-1}C_{p^an+k}=\frac{1}{C_n}\sum_{k=0}^{p^a-1}\left(\binom{2(p^an+k)}{p^an+k}-\binom{2(p^an+k)}{p^an+k+1}\right)\\\equiv&(n+1)\left(\frac{p^a}{3}\right)-\left(n+(n+1)\left(\frac{p^a-1}{3}\right)-[p=3]p(n+1)\right)\\\equiv&1-3(n+1)\left(\frac{p^a-1}{3}\right)\pmod{p^2}.\end{aligned}$$

This proves (1.12). On the other hand, (1.11) in the case $d=0$ yields

$$\frac{1}{C_n}\sum_{k=0}^{p^a-1}(p^an+k+1)C_{p^an+k}\equiv(n+1)\binom{p^a}{3}\pmod{p^2}.$$

So we have

$$\begin{aligned}\frac{1}{C_n}\sum_{k=0}^{p^a-1}kC_{p^an+k}\equiv&(n+1)\binom{p^a}{3}-\frac{p^an+1}{C_n}\sum_{k=0}^{p^a-1}C_{p^an+k}\\ \equiv&(n+1)\binom{p^a}{3}-(p^an+1)\left(1-3(n+1)\binom{p^a-1}{3}\right)\\ \equiv&(3p^an+1)(n+1)\binom{p^a-1}{3}-p^an-1\\ &+(n+1)\left(2\binom{p^a-1}{3}+\binom{p^a}{3}\right)\pmod{p^2}.\end{aligned}$$

Since

$$2\binom{p^a-1}{3}+\binom{p^a}{3}=1-p[p=3],$$

(1.13) follows at once. We are done. □

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