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CONGRUENCES INVOLVING CATALAN NUMBERS

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Abstract. In this paper we establish some new congruences involving Catalan numbers as well as central binomial coefficients. Let p be a prime and let a be any positive integer. We show that

$$\frac{1}{C_n}\sum_{k=0}^{p^a-1} C_{p^a n+k} \equiv 1-3(n+1)\left(\frac{p^a-1}{3}\right) \pmod{p^2}$$

for every $n=0,1,2,\ldots$, where C_m is the Catalan number $\binom{2m}{m}/(m+1)$, and $(\frac{\cdot}{3})$ is the Legendre symbol. We also determine $\sum_{k=0}^{p^a-1}\binom{2k}{k+d}$ and $\sum_{k=0}^{p^a-1}k\binom{2k}{k+d}$ modulo p^2 for all $d=0,1,\ldots,p$.

1. Introduction

For $n\in\mathbb{N}=\{0,1,\ldots\}$, the nth Catalan number is given by

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n+1}.$$

Here is an alternate definition:

$$C_0=1 \text{ and } C_{n+1}=\sum_{k=0}^{n}C_kC_{n-k} \quad (n=0,1,2,\ldots).$$

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The Catalan numbers play important roles in combinatorics; they arise naturally in many enumeration problems (see, e.g., [St, pp. 219–229]). For example, C_n is the number of binary parenthesizations of a string of $n+1$ letters, and it is also the number of ways to triangulate a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors.

In 2006 H. Pan and Z. W. Sun [PS] employed a useful identity to deduce many congruences on Catalan numbers. For example, by [PS, (1.16)–(1.17)], for any prime p we have

$$\sum_{k=0}^{p-1} C_k \equiv \frac{3(\frac{p}{3})-1}{2} \pmod p \quad\text{and}\quad \sum_{k=0}^{p-1} kC_k \equiv \frac{1-(\frac{p}{3})}{2} \pmod p,$$

where the Legendre symbol $(\frac{a}{3}) \in \{0, \pm1\}$ satisfies $a \equiv (\frac{a}{3}) \pmod 3$.

In this paper we establish some further congruences for Catalan numbers and related central binomial coefficients. For an assertion A, we adopt Iverson's notation:

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

Here is our first theorem.

Theorem 1.1. *Let p be a prime and let $a, n \in \mathbb{N}$ with $a > 0$. Then*

$$\frac{1}{C_n}\sum_{k=0}^{p^a-1} C_{p^a n+k} \equiv 1 - 3(n+1)\binom{p^a-1}{3} \pmod{p^2} \tag{1.1}$$

and

$$\begin{aligned}&\frac{1}{C_n}\sum_{k=0}^{p^a-1} kC_{p^a n+k} + [p=3]p(n+1) \\ \equiv&(1-p^a)n + (3p^a n+1)(n+1)\binom{p^a-1}{3} \pmod{p^2}.\end{aligned} \tag{1.2}$$

Also,

$$\begin{aligned}&\frac{3}{C_n}\sum_{k=0}^{p^a-1} k^2 C_{p^a n+k} \\ \equiv&\begin{cases} -p^a(n-2)(3n+1) - 5n - 2 \pmod{p^2} & \text{if } p^a \equiv 1 \pmod 3, \\ -p^a(9n^2+n-2) - 4n - 1 \pmod{p^2} & \text{if } p^a \equiv 2 \pmod 3. \end{cases}\end{aligned} \tag{1.3}$$

Remark 1.1. Note that (1.1)-(1.3) are congruences modulo p^2, different from those congruences obtained by Pan and Sun [PS].

Theorem 1.1 in the case $n = 0$ yields the following consequence.

Corollary 1.1. *Let p be any prime, and let $a\in\mathbb{Z}^+=\{1,2,3,\ldots\}$. Then*

$$\sum_{k=0}^{p^a-1}C_k\equiv 1-3\left(\frac{p^a-1}{3}\right)\equiv\frac{3(\frac{p^a}{3})-1}{2}\pmod{p^2}$$

and

$$\sum_{k=0}^{p^a-1}kC_k\equiv\left(\frac{p^a-1}{3}\right)-p[p=3]\equiv\frac{1-(\frac{p^a}{3})}{2}\pmod{p^2}.$$

Provided $p>3$, we also have

$$\sum_{k=0}^{p^a-1}k^2C_k\equiv\frac{(\frac{p^a-1}{3})-2}{3}\equiv-\frac{1}{6}\left(\frac{p^a}{3}\right)-\frac{1}{2}\pmod{p^2}.$$

Actually we deduce Theorem 1.1 from the following more general result.

Theorem 1.2. *Let p be a prime and let $a\in\mathbb{Z}^+$ and $m,n\in\mathbb{N}$ with $m\geqslant n$.*

(i) *If $p\neq 2$ and $d\in\{0,1\}$, then*

$$\begin{aligned}&\binom{m}{n}^{-1}\sum_{k=0}^{p^a-1}\binom{p^am+2k}{p^an+k+d}-d\frac{m-n}{n+1}\\&\equiv\left(\frac{p^a-d}{3}\right)-[p=3]pd(m+n+1)\pmod{p^2}.\end{aligned}\tag{1.4}$$

If $p>3$ and $d\in\{0,1\}$, then

$$\begin{aligned}&\binom{m}{n}^{-1}\sum_{k=0}^{p^a-1}k\binom{p^am+2k}{p^an+k+d}-d(p^a-1)\frac{m-n}{n+1}\\&\equiv\frac{2}{3}\left(p^a-\left(\frac{p^a}{3}\right)\right)+\left(\frac{p^a-1}{3}\right)(p^a((-1)^dm+dn)-d)\pmod{p^2}.\end{aligned}\tag{1.5}$$

(ii) *For every $d=0,\ldots,p$ we have*

$$\begin{aligned}&\frac{n+1}{\binom{m}{n}}\sum_{k=0}^{p^a-1}\binom{p^am+2k}{p^an+k+d}-(n+1)\left(\frac{p^a-d}{3}\right)-(m-n)\left(\frac{d}{3}\right)\\&\equiv((m-n)^2+(m+1)(n+2))p^aS_d-[p=3]\left(\frac{d}{3}\right)p(n+1)(m+n+1)\\&\quad+[p=2\ \&\ 3\nmid d-(-1)^a]pm(n+1)\pmod{p^2},\end{aligned}\tag{1.6}$$

where S_d is a rational p-adic integer given by

$$S_d := \sum_{0<k<d} \frac{(-1)^{k-1}}{k} \left(\frac{d-k}{3}\right); \tag{1.7}$$

in particular,

$$\begin{aligned}&\frac{1}{C_n}\sum_{k=0}^{p^a-1}\binom{2p^an+2k}{p^an+k+d} - n\left(\frac{d}{3}\right) - (n+1)\left(\frac{p^a-d}{3}\right)\\ \equiv& p^a(n+1)(3n+2)S_d - [p=3]p(n+1)\left(\frac{d}{3}\right) \pmod{p^2}.\end{aligned} \tag{1.8}$$

Remark 1.2. If $p=2$ and $d\in\{0,1\}$, then we should add

$$p[p=2](d[2\nmid n-a \ \&\ 2\mid a(m-1)] + (1-d)[2\nmid m \text{ or } 2\nmid n])$$

to the right-hand side of the congruence (1.4). For $p\in\{2,3\}$ and $d\in\{0,1\}$, we also find a suitable term that should be added to the right-hand side of the congruence (1.5).

Let p be a prime, and let $a\in\mathbb{Z}^+$ and $d\in\{0,\ldots,p\}$. In 2006 Pan and Sun [PS, Theorem 1.2] showed that

$$\sum_{k=0}^{p-1}\binom{2k}{k+d} \equiv \left(\frac{p-d}{3}\right) \pmod{p}. \tag{1.9}$$

By (1.8) in the case $n=0$, we have

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+d} \equiv \left(\frac{p^a-d}{3}\right) - p[p=3]\left(\frac{d}{3}\right) + 2p^aS_d \pmod{p^2}. \tag{1.10}$$

Note that $2p^aS_d\equiv 0 \pmod{p^2}$ if $a\geqslant 2$ or $p=2$. Also,

$$S_0=S_1=0,\ S_2=1,\ S_3=-\frac{3}{2},\ S_4=\frac{5}{6},\ S_5=\frac{5}{12},\ S_6=-\frac{21}{20}.$$

(1.10) in the case $a=1$ and $d=0$ implies the following observation of A. Adamchuk [A] (who told the second author that he could not find a proof): If $p>3$ then

$$\sum_{k=1}^{p+(\frac{p+1}{3})}\binom{2k}{k} \equiv 0 \pmod{p^2}.$$

(Recall the Wolstenholme congruence $\frac{1}{2}\binom{2p}{p}=\binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$ for $p>3$.) Recently, (1.10) with $p>3$, $a=2$ and $d=0$ was posed by D. Callan as a problem in [C]; in fact, (1.10) with $p>3$, $a\in\{2,3,4\}$ and $d=0$ was also observed by A. Adamchuk [A] slightly earlier who could not provide a proof.

(1.10) in the case $a=1$ and $d=0$, together with the first congruence in Corollary 1.1, suggests the following open problem.

Problem 1.1. *Are there any composite numbers* $n \not\equiv 0 \pmod 3$ *such that*

$$\sum_{k=0}^{n-1} \binom{2k}{k} \equiv \left(\frac{n}{3}\right) \pmod{n^2}\ ?$$

Are there any composite numbers $n \not\equiv 0 \pmod 3$ *satisfying*

$$\sum_{k=0}^{n-1} C_k \equiv 1 - 3\left(\frac{n-1}{3}\right) \pmod{n^2}\ ?$$

Remark 1.3. It seems that the answers to Problem 1.1 are negative. We have confirmed this for $n \leqslant 5,000$ via Maple.

Here is one more theorem.

Theorem 1.3. *Let* p *be a prime and let* $a > 1$ *be an integer. For any* $d \in \{0, 1, \ldots, p\}$, *if* $p \neq 3$ *then*

$$\begin{aligned}&\sum_{k=0}^{p^a-1} k\binom{2k}{k+d} + \frac{2}{3}\left(\frac{p^a-d}{3}\right)\\ \equiv&\frac{d}{3}\left(\left(\frac{p^a-d-1}{3}\right) - \left(\frac{p^a-d+1}{3}\right)\right) \pmod{p^2};\end{aligned} \tag{1.11}$$

when $p = 3$ *we have*

$$\sum_{k=0}^{3^a-1} k\binom{2k}{k+d} \equiv -(4d+1)\left(\frac{d}{3}\right) - 3[a=2] + [d=3] \pmod{3^2}. \tag{1.12}$$

Remark 1.4. Let $p > 3$ be a prime. For $a \in \mathbb{Z}^+$ and $d \in \{0, 1\}$ we have

$$\sum_{k=0}^{p^a-1} k\binom{2k}{k+d} \equiv \frac{2}{3}\left(p^a - \left(\frac{p^a}{3}\right)\right) - \left(\frac{p^a-1}{3}\right)d \pmod{p^2}$$

by applying (1.5) with $m = n = 0$. This coincides with (1.11) when $a \geqslant 2$. As a supplement to Theorem 1.3, we can also determine $\sum_{k=0}^{p-1} k\binom{2k}{k+d} \bmod p^2$ for all $d = 0, \ldots, p$ (see Remark 4.1) but the result is very complicated.

We will give an auxiliary theorem in the next section, and establish a result close to (1.10) and Theorem 1.3 in Section 3, and then prove Theorems 1.1-1.3 in Section 4. The last section contains three conjectures and an announcement of some new results.

2. An auxiliary theorem

Now we introduce some basic notations throughout this section. For $m,n\in\mathbb{N}$, by (m,n) we mean the greatest common divisor of m and n. For a prime p and an integer m, we define

$$\mathrm{ord}_p(m)=\sup\{a\in\mathbb{N}:\ p^a\mid m\}$$

(and thus $\mathrm{ord}_p(0)=+\infty$). A congruence modulo $+\infty$ refers to the corresponding equality.

Lemma 2.1. *Let p be any prime, and let $m,n\in\mathbb{N}$ with $m\geqslant n$. Then*

$$\binom{pm}{pn}\Big/\binom{m}{n}\equiv 1+[p=2]pn(m-n)\ (\mathrm{mod}\ p^{2+\mathrm{ord}_p(n)}).\tag{2.1}$$

Proof. (2.1) holds trivially when n is 0 or m. Below we assume $0<n<m$. Observe that

$$\binom{pm}{pn}=\prod_{j=0}^{pn-1}\frac{pm-j}{pn-j}=\prod_{i=0}^{n-1}\frac{pm-pi}{pn-pi}\times\prod_{\substack{0\leqslant j<pn\\ p\nmid j}}\left(1+\frac{p(m-n)}{pn-j}\right).$$

Thus

$$\begin{aligned}\binom{pm}{pn}\Big/\binom{m}{n}&\equiv 1+p(m-n)\sum_{\substack{0\leqslant j<pn\\ p\nmid j}}\frac{1}{pn-j}=1+p(m-n)\sum_{\substack{0<i<pn\\ p\nmid i}}\frac{1}{i}\\&\equiv 1+p(m-n)\sum_{q=0}^{n-1}\sum_{k=1}^{p-1}\frac{1}{pq+k}\equiv 1+p(m-n)n\sum_{k=1}^{p-1}\frac{1}{k}\\&\equiv 1+[p=2]p(m-n)n\ (\mathrm{mod}\ p^{2+\mathrm{ord}_p(m-n)})\end{aligned}$$

since

$$2\sum_{k=1}^{p-1}\frac{1}{k}=\sum_{k=1}^{p-1}\left(\frac{1}{k}+\frac{1}{p-k}\right)\equiv 0\ (\mathrm{mod}\ p).$$

Similarly,

$$\begin{aligned}\binom{pm}{pn}\Big/\binom{m}{n}&=\binom{pm}{p(m-n)}\Big/\binom{m}{m-n}\\&\equiv 1+[p=2]pn(m-n)\ (\mathrm{mod}\ p^{2+\mathrm{ord}_p(n)}).\end{aligned}$$

We are done. □

Remark 2.1. Lemma 2.1 can be further strengthened. By a deep result of Jacobsthal (see, e.g., [Gr]), if $p>3$ is a prime and $m\geqslant n\geqslant 0$ are integers, then $\binom{pm}{pn}=\binom{m}{n}\left(1+p^3mn(m-n)u\right)$ for some p-adic integer u.

Lemma 2.2. (i) *If $n>1$ is an integer relatively prime to 6, then for any $a\in\mathbb{Z}$ we have*

$$\sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{k+a}{3}\right)=0. \tag{2.2}$$

(ii) *If $p>3$ is a prime, then for $\varepsilon\in\{0,\pm1\}$ we have*

$$\sum_{j=1}^{p-1}\binom{p}{j}\sum_{k=1}^{p-1}k\binom{2k}{k+j+\varepsilon}\equiv p[3\nmid p-1+\varepsilon]-|\varepsilon|p \pmod{p^2}.$$

Proof. Let $n\in\mathbb{N}$ and $r\in\mathbb{Z}$, and let ω denote the cubic root $(-1+\sqrt{-3})/2$ of unity. As observed by E. Lehmer [L] in 1938,

$$\begin{aligned}3\sum_{k\equiv r\,(\mathrm{mod}\ 3)}\binom{n}{k}=&\sum_{k=0}^{n}\binom{n}{k}(1+\omega^{k-r}+\omega^{2(k-r)})\\ =&2^n+\omega^{-r}(1+\omega)^n+\omega^{-2r}(1+\omega^2)^n\\ =&2^n+\omega^{-r}(-\omega^2)^n+\omega^{r}(-\omega)^n\\ =&2^n+(-1)^n(\omega^{n+r}+\omega^{-n-r})\\ =&\begin{cases}2^n+2(-1)^n & \text{if } 3\mid n+r,\\ 2^n-(-1)^n & \text{if } 3\nmid n+r.\end{cases}\end{aligned}$$

We will use this basic result in our following proofs of parts (i) and (ii).

(i) Concerning the first part, as

$$\left(\frac{k-1}{3}\right)+\left(\frac{k}{3}\right)+\left(\frac{k+1}{3}\right)=0,$$

it suffices to show (2.2) for $a=0,1$. Since $(6,n)=1$, we have

$$\begin{aligned}\left(\frac{n}{3}\right)\sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{k}{3}\right)=&\sum_{k\equiv n\,(\mathrm{mod}\ 3)}\binom{n}{k}-1-\sum_{k\equiv -n\,(\mathrm{mod}\ 3)}\binom{n}{k}\\ =&\frac{2^n+1}{3}-1-\frac{2^n-2}{3}=0\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{k+1}{3}\right)=&\sum_{k\equiv 0\,(\mathrm{mod}\ 3)}\binom{n}{k}-1-\sum_{k\equiv 1\,(\mathrm{mod}\ 3)}\binom{n}{k}+[3\mid n-1]\\ =&\frac{2^n+1}{3}-\frac{2^n+1}{3}=0.\end{aligned}$$

This proves part (i).

(ii) Clearly $p \mid \binom{p}{j}$ for all $j = 1, \dots, p-1$. In view of [PS, (1.5)],

$$\sum_{j=1}^{p-1}\binom{p}{j}\sum_{k=1}^{p-1}k\binom{2k}{k+j+\varepsilon} \equiv T_\varepsilon \pmod{p^2},$$

where

$$T_\varepsilon = \sum_{j=1}^{p-1}\binom{p}{j}\bigg(-\frac{2}{3}\left(\frac{p-j-\varepsilon}{3}\right)+\frac{j+\varepsilon}{3}-[3 \mid p-j-\varepsilon](j+\varepsilon)\bigg).$$

By part (i),

$$\sum_{j=1}^{p-1}\binom{p}{j}\left(\frac{p-j-\varepsilon}{3}\right)=\sum_{k=1}^{p-1}\left(\frac{k-\varepsilon}{3}\right)=0.$$

Therefore

$$\begin{aligned}T_\varepsilon =&\frac{p}{3}\sum_{j=1}^{p-1}\binom{p-1}{j-1}+\frac{\varepsilon}{3}\sum_{j=1}^{p-1}\binom{p}{j}-p\sum_{\substack{0<j<p\\ 3\mid p-j-\varepsilon}}\binom{p-1}{j-1}-\varepsilon\sum_{\substack{0<j<p\\ 3\mid p-j-\varepsilon}}\binom{p}{j}\\ =&\frac{p}{3}(2^{p-1}-1)+\frac{\varepsilon}{3}(2^p-2)-p\sum_{\substack{0<k<p\\ 3\mid k-\varepsilon}}\binom{p-1}{k}-\varepsilon\sum_{\substack{0<k<p\\ 3\mid k-\varepsilon}}\binom{p}{k}.\end{aligned}$$

Note that

$$\begin{aligned}T_0 =&\frac{p}{3}(2^{p-1}-1)-p\bigg(\sum_{k\equiv 0 \pmod 3}\binom{p-1}{k}-1\bigg)\\ =&\frac{p}{3}(2^{p-1}-1)-p\left(\frac{2^{p-1}-1}{3}-[3\nmid p-1]\right)=[3\nmid p-1]p.\end{aligned}$$

For $\varepsilon=\pm1$ we have

$$\begin{aligned}T_\varepsilon =&\frac{p}{3}(2^{p-1}-1)+\varepsilon\frac{2^p-2}{3}\\ &-p\sum_{k\equiv\varepsilon \pmod 3}\binom{p-1}{k}-\varepsilon\bigg(\sum_{k\equiv\varepsilon \pmod 3}\binom{p}{k}-[3\mid p-\varepsilon]\bigg)\\ =&\frac{p}{3}(2^{p-1}-1)+\varepsilon\frac{2^p-2}{3}-p\left(\frac{2^{p-1}-1}{3}+[3\mid p-1+\varepsilon]\right)\\ &-\varepsilon\bigg(\frac{2^p-2}{3}+[3\nmid p+\varepsilon]-[3\mid p-\varepsilon]\bigg)\\ =&-p[3\mid p-1+\varepsilon].\end{aligned}$$

This concludes the proof. □

Remark 2.2. The evaluation of $\sum_{k\equiv r \pmod{12}}\binom{n}{k}$ with $n\in\mathbb{N}$ and $r\in\mathbb{Z}$, can be found in [Su].

Theorem 2.1. *Let p be any prime, and let $m \geqslant n \geqslant 0$ be integers. Provided $d \in \{0,1\}$, we have*

$$\begin{aligned}&\binom{m}{n}^{-1}\sum_{k=0}^{p-1}\binom{pm+2k}{pn+k+d} - d\left(\frac{m-n}{n+1}+[p=2]p(m-n)(m-n-1)\right)\\ \equiv&(1+[p=2]pn(m-n))\sum_{k=0}^{p-1}\binom{2k}{k+d}\\ &+[p\leqslant 3]p\left(n\left(\frac{p+d}{3}\right)+(m-n)\left(\frac{p-d}{3}\right)\right) \pmod{p^{2+\mathrm{ord}_p(m,n)}},\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}&\binom{m}{n}^{-1}\sum_{k=0}^{p-1}k\binom{pm+2k}{pn+k+d} - d(p-1)\frac{m-n}{n+1}\\ &+[p=2]pd(m-n)(m-n-1)\\ \equiv&(1+[p=2]pn(m-n))\sum_{k=0}^{p-1}k\binom{2k}{k+d}\\ &+\begin{cases}(\frac{p-1}{3})p(m(-1)^d+dn) \pmod{p^{2+\mathrm{ord}_p(m,n)}} & \text{if } p>3,\\ pm([p=2](1-d)-[p=3]) \pmod{p^{2+\mathrm{ord}_p(m,n)}} & \text{if } p\in\{2,3\}.\end{cases}\end{aligned}\tag{2.4}$$

Moreover, for any $d=0,1,\ldots,p$ we have

$$\begin{aligned}&\frac{n+1}{\binom{m}{n}}\sum_{k=0}^{p-1}\binom{pm+2k}{pn+k+d} - (n+1)\sum_{k=0}^{p-1}\binom{2k}{k+d}\\ \equiv&(m-n)(1+[p=2]p(n+1)(m-n-1))\sum_{k=0}^{p-1}\binom{2k}{k+p-d}\\ &+\left(m^2-(m-1)n+n^2\right)\sum_{0<k<d}\binom{p}{k}\left(\frac{d-k}{3}\right)\\ &+p[p\leqslant 3](n+1)\left(n\left(\frac{p+d}{3}\right)+(m-n)\left(\frac{p-d}{3}\right)\right)\\ &\pmod{p^{2+\mathrm{ord}_p(m,n)}}.\end{aligned}\tag{2.5}$$

Proof. Fix $d \in \{0,1\ldots,p\}$. Clearly (2.3)-(2.5) hold trivially when $m=n=0$. Below we assume $m>0$.

By the Chu-Vandermonde convolution identity (cf. [GKP, (5.22)]),

$$\binom{pm+2k}{pn+k+d} = \sum_{j\in\mathbb{Z}}\binom{pm}{pn-j}\binom{2k}{k+j+d}$$

for any $k \in \mathbb{N}$. Thus, for $r = 0, 1$ we have

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{k}{r}\binom{pm+2k}{pn+k+d} - \binom{pm}{pn}\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k+d}\\
=&\sum_{j>0}\binom{pm}{pn-j}\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k+j+d}\\
&+\sum_{j>0}\binom{pm}{pn+j}\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k-j+d}\\
=&\sum_{j=1}^{p-1}\binom{pm}{pn-j}\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k+j+d}\\
&+\sum_{j=1}^{p-1}\binom{pm}{pn+j}\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k+j-d}\\
&+\binom{pm}{pn+p}\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k+p-d} + R_d(r),
\end{aligned}$$

where

$$\begin{aligned}
R_d(r) =& \sum_{p<j<p+d}\binom{pm}{pn+j}\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k+j-d}\\
=& \sum_{0<j<d}\binom{pm}{p(n+1)+j}\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k+j+p-d}.
\end{aligned}$$

Note that $R_d(r) = 0$ if $d \in \{0, 1\}$.

By Lemma 2.1, there are p-adic integers u and v such that

$$\binom{pm}{pn} = \binom{m}{n}\left(1 + [p=2]p(m-n)n + p^2(m-n)u\right)$$

and

$$\begin{aligned}
\binom{pm}{p(n+1)} =& \binom{m}{n+1}\left(1 + [p=2]p(n+1)(m-n-1) + p^2(n+1)v\right)\\
=& \binom{m}{n}\left(\frac{m-n}{n+1} + [p=2]p(m-n)(m-n-1) + p^2(m-n)v\right).
\end{aligned}$$

If $n \neq m$, then for $j = 1, \ldots, p-1$ we have

$$\begin{aligned}
\binom{pm}{pn+j} =& \frac{(pm)!}{(pn)!(pm-pn)!} \times \frac{\prod_{0\leqslant i<j}(pm-pn-i)}{(pn+1)\cdots(pn+j)}\\
=& \binom{pm}{pn}\frac{p(m-n)}{pn+j}\prod_{0<i<j}\frac{p(m-n)-i}{pn+i}
\end{aligned}$$

and hence

$$\frac{\binom{pm}{pn+j}}{(m-n)\binom{m}{n}} \equiv \frac{\binom{pm}{pn}}{\binom{m}{n}} \cdot \frac{p}{j} \prod_{0<i<j} \frac{p-i}{i} \equiv \binom{p}{j} \pmod{p^2};$$

also,

$$\begin{aligned}\frac{(n+1)\binom{pm}{p(n+1)+j}}{\binom{m}{n}} =& (m-n)\frac{\binom{pm}{p(n+1)+j}}{\binom{m}{n+1}} \\ \equiv& (m-n)(m-n-1)\binom{p}{j} \pmod{p^{2+\mathrm{ord}_p(m-n)}}.\end{aligned}$$

Similarly, if $n \neq 0$ then

$$\frac{\binom{pm}{pn-j}}{n\binom{m}{n}} = \frac{\binom{pm}{p(m-n)+j}}{(m-(m-n))\binom{m}{m-n}} \equiv \binom{p}{j} \pmod{p^2}$$

for every $j = 1, \ldots, p-1$.

Combining the above, we find that $(n+1)R_d(r)/\binom{m}{n}$ is congruent to

$$(m-n)(m-n-1)\sum_{0<j<d}\binom{p}{j}\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k+j+p-d}$$

modulo $p^{2+\mathrm{ord}_p(m-n)}$, and

$$\begin{aligned}&\binom{m}{n}^{-1}\sum_{k=0}^{p-1}\binom{k}{r}\binom{pm+2k}{pn+k+d} \\ &-(1+[p=2]pn(m-n))\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k+d} - \frac{R_d(r)}{\binom{m}{n}} \\ &-\left(\frac{m-n}{n+1}+[p=2]p(m-n)(m-n-1)\right)\sum_{k=0}^{p-1}\binom{k}{r}\binom{2k}{k+p-d} \\ \equiv&\sum_{j=1}^{p-1}\binom{p}{j}\sum_{k=0}^{p-1}\binom{k}{r}\left(n\binom{2k}{k+j+d}+(m-n)\binom{2k}{k+j-d}\right) \\ &\pmod{p^{2+\mathrm{ord}_p(m,n)}}.\end{aligned} \tag{2.6}$$

In light of (1.9) and Lemma 2.2(i),

$$\begin{aligned}&\sum_{j=1}^{p-1}\binom{p}{j}\sum_{k=0}^{p-1}\binom{2k}{k+j+d} = \sum_{0<j<p-d}\binom{p}{j}\sum_{k=0}^{p-1}\binom{2k}{k+j+d} \\ \equiv&\sum_{0<j<p-d}\binom{p}{j}\left(\frac{p-j-d}{3}\right) = \sum_{d<k<p}\binom{p}{k}\left(\frac{k-d}{3}\right) \pmod{p^2}\end{aligned}$$

and

$$\begin{aligned}\sum_{d<k<p}\binom{p}{k}\left(\frac{k-d}{3}\right)=&[p\leqslant 3]\sum_{k=1}^{p-1}\binom{p}{k}\left(\frac{k-d}{3}\right)-\sum_{0<k\leqslant d}\binom{p}{k}\left(\frac{k-d}{3}\right)\\ =&p[p\leqslant 3](-1)^{p-1}\left(\frac{p+d}{3}\right)+\sum_{0<k<d}\binom{p}{k}\left(\frac{d-k}{3}\right).\end{aligned}$$

Also,

$$\begin{aligned}&\sum_{j=1}^{p-1}\binom{p}{j}\sum_{k=0}^{p-1}\binom{2k}{k+j-d}\\ =&\sum_{0<j<d}\binom{p}{j}\sum_{k=0}^{p-1}\binom{2k}{k+d-j}+\sum_{d\leqslant j<p}\binom{p}{j}\sum_{k=0}^{p-1}\binom{2k}{k+j-d}\end{aligned}$$

is congruent to

$$\begin{aligned}&\sum_{0<j<d}\binom{p}{j}\left(\frac{p-d+j}{3}\right)+\sum_{d\leqslant j<p}\binom{p}{j}\left(\frac{p-j+d}{3}\right)\\ =&\sum_{0<j<d}\binom{p}{j}\left(\left(\frac{p-d+j}{3}\right)-\left(\frac{p-j+d}{3}\right)\right)+\sum_{j=1}^{p-1}\binom{p}{j}\left(\frac{p-j+d}{3}\right)\\ =&(1-p[p=3])\sum_{0<j<d}\binom{p}{j}\left(\frac{d-j}{3}\right)+p[p\leqslant 3](-1)^{p-1}\left(\frac{p-d}{3}\right)\end{aligned}$$

modulo p^2, and $(n+1)R_d(0)/\binom{m}{n}$ is congruent to

$$(m-n)(m-n-1)\sum_{0<j<d}\binom{p}{j}\left(\frac{p-(j+p-d)}{3}\right)$$

modulo $p^{2+\mathrm{ord}_p(m-n)}$. In view of these and (2.6), (2.5) follows, and (2.3) holds for $d=0,1$.

Now suppose $d\in\{0,1\}$. By (2.6) in the case $r=1$,

$$\begin{aligned}&\frac{1}{\binom{m}{n}}\sum_{k=0}^{p-1}k\binom{pm+2k}{pn+k+d}-(1+[p=2]pn(m-n))\sum_{k=0}^{p-1}k\binom{2k}{k+d}\\ &-\left(\frac{m-n}{n+1}+[p=2]p(m-n)(m-n-1)\right)(p-1)\binom{2(p-1)}{p-1+p-d}\end{aligned}$$

is congruent to

$$\Sigma=\sum_{j=1}^{p-1}\binom{p}{j}\left(n\sum_{k=0}^{p-1}k\binom{2k}{k+j+d}+(m-n)\sum_{k=0}^{p-1}k\binom{2k}{k+j-d}\right)$$

modulo $p^{2+\mathrm{ord}_p(m,n)}$. Note that $\binom{2p-2}{2p-1-d}=d$. If $p>3$, then by Lemma 2.2(ii) we have

$$\begin{aligned}\Sigma &\equiv np([3\nmid p-1+d]-d)+(m-n)p([3\nmid p-1-d]-d)\\ &\equiv\begin{cases} pm(\frac{p-1}{3}) \pmod{p^{2+\mathrm{ord}_p(m,n)}} & \text{if } d=0,\\ -p(m-n)(\frac{p-1}{3}) \pmod{p^{2+\mathrm{ord}_p(m,n)}} & \text{if } d=1.\end{cases}\end{aligned}$$

For $p\in\{2,3\}$, it is easy to verify that

$$\Sigma\equiv pm([p=2](1-d)-[p=3]) \pmod{p^{2+\mathrm{ord}_p(m,n)}}.$$

Therefore (2.4) is valid, and we are done. □

3. A theorem close to (1.10) and Theorem 1.3

Lemma 3.1. *Let p be a prime, and let $a\in\mathbb{Z}^+$. Then, for any $d=0,1,\dots,p$ we have*

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+d}\equiv\left(\frac{p^a-d}{3}\right) \pmod p. \tag{3.1}$$

Proof. We use induction on a.

By (1.9), we have (3.1) for $a=1$.

Now let $a>1$ and assume the corresponding result for $a-1$. In view of (2.5), (1.9) and the induction hypothesis,

$$\begin{aligned}\sum_{k=0}^{p^a-1}\binom{2k}{k+d} &= \sum_{n=0}^{p^{a-1}-1}\sum_{k=0}^{p-1}\binom{2pn+2k}{pn+k+d}\\ &\equiv\sum_{n=0}^{p^{a-1}-1}C_n\bigg((n+1)\sum_{k=0}^{p-1}\binom{2k}{k+d}+n\sum_{k=0}^{p-1}\binom{2k}{k+p-d}\bigg)\\ &\equiv\sum_{n=0}^{p^{a-1}-1}\binom{2n}{n}\left(\frac{p-d}{3}\right)+\sum_{n=0}^{p^{a-1}-1}\binom{2n}{n+1}\left(\frac{d}{3}\right)\\ &\equiv\left(\frac{p^{a-1}}{3}\right)\left(\frac{p-d}{3}\right)+\left(\frac{p^{a-1}-1}{3}\right)\left(\frac{d}{3}\right) \pmod p.\end{aligned}$$

Thus, it suffices to show the equality

$$\left(\frac{p^{a-1}}{3}\right)\left(\frac{p-d}{3}\right)+\left(\frac{p^{a-1}-1}{3}\right)\left(\frac{d}{3}\right)=\left(\frac{p^a-d}{3}\right). \tag{3.2}$$

Clearly (3.2) holds when $p = 3$.

Now we consider the case $p \neq 3$. Write $p^a - d \equiv \varepsilon p \pmod 3$ with $\varepsilon \in \{0, \pm 1\}$. Then

$$\begin{aligned}&\left(\frac{p^{a-1}}{3}\right)\left(\frac{p-d}{3}\right)+\left(\frac{p^{a-1}-1}{3}\right)\left(\frac{d}{3}\right)\\ =&\left(\frac{p^{a-1}}{3}\right)\left(\frac{p+\varepsilon p-p^a}{3}\right)+\left(\frac{p^{a-1}-1}{3}\right)\left(\frac{p^a-\varepsilon p}{3}\right)\\ =&-\left(\frac{p^a}{3}\right)\left(\frac{p^{a-1}-1-\varepsilon}{3}\right)+\left(\frac{p^{a-1}-1}{3}\right)\left(\frac{p^{a-1}-\varepsilon}{3}\right)\left(\frac{p}{3}\right)\\ =&\varepsilon\left(\frac{p}{3}\right)=\left(\frac{p^a-d}{3}\right).\end{aligned}$$

So (3.2) is valid, and we are done. □

Lemma 3.2. *Let p be a prime, and let $a \in \mathbb{Z}^+$. Then, for $d \in \{2, \dots, p\}$ and $P(x) \in \mathbb{Z}[x]$, we have*

$$\begin{aligned}&\sum_{\substack{k=0\\ \{k\}_p\leqslant d-2}}^{p^a-1}\frac{p^{a-1}}{p^{a-1}-\lfloor k/p\rfloor}P(k)\binom{2k}{k+d}\\ \equiv&[a=2\ \&\ 3\mid p+1]3p\sum_{k=0}^{d-2}P(k)\binom{2k}{k+p-d}\pmod{p^2},\end{aligned}\tag{3.3}$$

where $\{k\}_p$ denotes the least nonnegative residue of k modulo p.

Proof. Let L denote the left-hand side of the congruence (3.3). In the case $a = 1$, clearly

$$L=\sum_{k=0}^{d-2}P(k)\binom{2k}{k+d}=0.$$

Below we let $a > 1$. Observe that

$$\begin{aligned}L=&\sum_{n=0}^{p^{a-1}-1}\frac{p^{a-1}}{p^{a-1}-n}\sum_{k=0}^{d-2}P(pn+k)\binom{2pn+2k}{pn+k+d}\\ \equiv&-\sum_{0<n<p^{a-1}}\frac{p^{a-1}}{n}\sum_{k=0}^{d-2}P(k)\binom{2pn+2k}{pn+k+d}\pmod{p^2}.\end{aligned}$$

By the Chu-Vandermonde identity, for each $k=0,\dots,d-2$ we have

$$\begin{aligned}\binom{2pn+2k}{pn+k+d} &= \sum_{j\geqslant 0}\binom{2pn}{pn-j}\binom{2k}{k+j+d}+\sum_{j>0}\binom{2pn}{pn+j}\binom{2k}{k-j+d}\\ &= \sum_{j=1}^{2p-1}\binom{2pn}{pn+j}\binom{2k}{k-j+d}\\ &= \sum_{j=1}^{2p-1}\frac{2pn}{pn+j}\binom{2pn-1}{pn+j-1}\binom{2k}{k+j-d}\\ &\equiv \binom{2pn}{pn+p}\binom{2k}{k+p-d}\equiv\binom{2n}{n+1}\binom{2k}{k+p-d} \pmod p.\end{aligned}$$

Therefore

$$L\equiv-\sum_{0<n<p^{a-1}}\frac{p^{a-1}}{n}\binom{2n}{n+1}\sum_{k=0}^{d-2}P(k)\binom{2k}{k+p-d} \pmod{p^2}.$$

With the help of Lemma 3.1,

$$\begin{aligned}\sum_{0<n<p^{a-1}}\frac{1}{n}\binom{2n}{n+1} &= \sum_{0<n<p^{a-1}}\binom{2n}{n}-\sum_{0<n<p^{a-1}}\binom{2n}{n+1}\\ &= \sum_{n=0}^{p^{a-1}-1}\binom{2n}{n}-1-\sum_{n=0}^{p^{a-1}-1}\binom{2n}{n+1}\\ &\equiv \left(\frac{p^{a-1}}{3}\right)-1-\left(\frac{p^{a-1}-1}{3}\right)=-3[3\mid p^{a-1}+1] \pmod p.\end{aligned}$$

Thus

$$L\equiv 3p^{a-1}[3\mid p^{a-1}+1]\sum_{k=0}^{d-2}P(k)\binom{2k}{k+p-d} \pmod{p^2}$$

and hence (3.3) holds. □

Theorem 3.1. *Let p be any prime, and let $a\in\{2,3,\dots\}$ and $d\in\{0,1,\dots,p\}$. Then*

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+d}\equiv\left(\frac{p^a-d}{3}\right)-\left(\frac{d}{3}\right)[p=3]p-[\![a,d,p]\!]3pD \pmod{p^2}, \quad (3.4)$$

where

$$[\![a,d,p]\!]:=[a=2\ \&\ 3\mid p+1\ \&\ 2d>p+(-1)^{p-1}]$$

and

$$D:=\sum_{0\leqslant k\leqslant d-2}\binom{2k}{k+p-d}+\left(\frac{p+d}{3}\right).$$

Also,

$$\begin{aligned}&\sum_{k=0}^{p^a-1}(d-2-k)\binom{2k}{k+d}+[\![a,d,p]\!]3pD'\\ \equiv&2(1-3[3\mid p^a-d+1])\frac{d-1+(\frac{p^a-d+1}{3})}{3}\\ &+p[p=3]([a=2]-[d=p])\pmod{p^2},\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}D':=&\sum_{0\leqslant k\leqslant d-2}(d-2-k)\binom{2k}{k+p-d}\\ &+\frac{2}{3}\left(d-1-\left(\frac{p+d-1}{3}\right)\right)(3[3\mid p+d-1]-1).\end{aligned}$$

Proof. By [PS, (1.2) and (1.3)], for any $l,m\in\mathbb{N}$ we have

$$\begin{aligned}&\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{l+1}\binom{2k}{k+m-2l}\\ &=[3\nmid m-1]\binom{l}{\lceil m/3\rceil}\binom{2\lceil m/3\rceil}{l+1}\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{l+2}\binom{2k}{k+m-2l}\\ &=(1+[3\mid m+1])\binom{l}{\lceil m/3\rceil}\binom{2\lceil m/3\rceil}{l+2}.\end{aligned}$$

In the case $l=p^a-1$ and $m=2l+d$, this yields

$$\begin{aligned}&\sum_{k=d}^{p^a-1}(-1)^{k+d}\binom{p^a-1}{k}\binom{2p^a-2+d-k}{p^a}\binom{2k}{k+d}\\ &=[3\nmid d-p^a]\binom{p^a-1}{k_0}\binom{2k_0}{p^a}\end{aligned}\tag{3.6}$$

and

$$\sum_{k=d}^{p^a-1}(-1)^{k+d}\binom{p^a-1}{k}\binom{2p^a-2+d-k}{p^a+1}\binom{2k}{k+d} = (1+[3\mid d-p^a-1])\binom{p^a-1}{k_0}\binom{2k_0}{p^a+1}, \tag{3.7}$$

where $k_0=\lceil(2(p^a-1)+d)/3\rceil\geqslant d$.

Fix $d\leqslant k<p^a$. Clearly

$$(-1)^k\binom{p^a-1}{k}=\prod_{0<j\leqslant k}\frac{j-p^a}{j}$$
$$\equiv\prod_{0<i\leqslant\lfloor k/p\rfloor}\frac{pi-p^a}{pi}=\prod_{0<i\leqslant\lfloor k/p\rfloor}\frac{i-p^{a-1}}{i}\pmod{p^a}.$$

Assume that $d\neq0$ or $k\neq p^a-1$. Then $k\leqslant p^a-2+d$ and hence

$$\binom{2p^a-2+d-k}{p^a}=\binom{2p^a-2+d-k}{p^a-2+d-k}=\prod_{0<j\leqslant p^a-2+d-k}\frac{p^a+j}{j}$$
$$\equiv\prod_{0<i\leqslant\lfloor(p^a-2+d-k)/p\rfloor}\frac{p^{a-1}+i}{i}\pmod{p^a}.$$

Note that

$$\left\lfloor\frac{p^a-2+d-k}{p}\right\rfloor+\left\lfloor\frac{k}{p}\right\rfloor=p^{a-1}+\left\lfloor-\frac{\{k\}_p+2-d}{p}\right\rfloor=p^{a-1}-1-\varepsilon_{d,k},$$

where

$$\varepsilon_{d,k}=\begin{cases}1 & \text{if } d=0\ \&\ p\mid k+1,\\ -1 & \text{if } d\geqslant 2+\{k\}_p,\\ 0 & \text{otherwise.}\end{cases}$$

Since

$$0\leqslant\varepsilon_{d,k}+\left\lfloor\frac{k}{p}\right\rfloor=p^{a-1}-1-\left\lfloor\frac{p^a-(k-d+2)}{p}\right\rfloor<p^{a-1}$$

and

$$\prod_{i=1}^{p^{a-1}-1}\frac{p^{a-1}+i}{i}=\frac{1}{2}\binom{2p^{a-1}}{p^{a-1}}=\prod_{j=0}^{a-2}\frac{\binom{2p^{j+1}}{p^{j+1}}}{\binom{2p^j}{p^j}}$$
$$\equiv\prod_{j=0}^{a-2}(1+[p=2]pp^j(2p^j-p^j))\equiv1+[p=2]p\equiv(-1)^{p-1}\pmod{p^2}$$

(with the help of Lemma 2.1), we have

$$\begin{aligned}\binom{2p^a-2+d-k}{p^a}\equiv&\prod_{i=1}^{p^{a-1}-1}\frac{p^{a-1}+i}{i}\times\prod_{i=p^{a-1}-\varepsilon_{d,k}-\lfloor k/p\rfloor}^{p^{a-1}-1}\frac{i}{p^{a-1}+i}\\ \equiv&(-1)^{p-1}\prod_{i=1}^{\varepsilon_{d,k}+\lfloor k/p\rfloor}\frac{p^{a-1}-i}{2p^{a-1}-i}\pmod{p^{\min\{a,2\}}}.\end{aligned}$$

Therefore

$$\begin{aligned}&(-1)^k\binom{p^a-1}{k}\binom{2p^a-2+d-k}{p^a}\\ \equiv&(-1)^{p-1}e_{d,k}\prod_{0<i\leqslant\lfloor k/p\rfloor}\frac{(i-p^{a-1})(p^{a-1}-i)}{i(2p^{a-1}-i)}\equiv(-1)^{p-1}e_{d,k}\pmod{p^2},\end{aligned}$$

where

$$e_{d,k}=\begin{cases}\frac{p^{a-1}-(k+1)/p}{2p^{a-1}-(k+1)/p}=1-\frac{p^{a-1}}{2p^{a-1}-(k+1)/p} & \text{if } d=0\ \&\ p\mid k+1,\\ \frac{2p^{a-1}-\lfloor k/p\rfloor}{p^{a-1}-\lfloor k/p\rfloor}=1+\frac{p^{a-1}}{p^{a-1}-\lfloor k/p\rfloor} & \text{if } d\geqslant 2+\{k\}_p,\\ 1 & \text{otherwise.}\end{cases}$$

If $d\geqslant 2$ or $k\neq p^a-1$, then $k\leqslant p^a-2+(d-1)$ and hence

$$\begin{aligned}&(-1)^k\binom{p^a-1}{k}\binom{2p^a-2+d-k}{p^a+1}\\ =&\frac{2p^a-2+d-k}{p^a+1}(-1)^k\binom{p^a-1}{k}\binom{2p^a-2+(d-1)-k}{p^a}\\ \equiv&(d-2-k)(-1)^{p-1}e_{d-1,k}\pmod{p^2}.\end{aligned}$$

In the case $d=0$ and $k=p^a-1$, clearly

$$\binom{2k}{k+d}=p^aC_{p^a-1}\equiv 0\pmod{p^2}.$$

When $d=1$ and $k=p^a-1$, we have $d-2-k=-p^a$ and

$$(d-2-k)\binom{2k}{k+d}=-p^a\binom{2p^a-2}{p^a}\equiv 0\pmod{p^2}.$$

If $d=0$ and $p\mid k+1$, then

$$\binom{2k}{k+d}=(k+1)C_k\equiv 0\pmod{p};$$

if $d=1$ and $p\mid k+1$, then $d-2-k\equiv 0\pmod p$.

In view of the above,

$$\begin{aligned}(-1)^{p-1}&\sum_{k=0}^{p^a-1}(-1)^k\binom{p^a-1}{k}\binom{2p^a-2+d-k}{p^a}\binom{2k}{k+d}\\&\equiv\sum_{k=0}^{p^a-1}\binom{2k}{k+d}+L_0\pmod{p^2}\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}(-1)^{p-1}&\sum_{k=0}^{p^a-1}(-1)^k\binom{p^a-1}{k}\binom{2p^a-2+d-k}{p^a+1}\binom{2k}{k+d}\\&\equiv\sum_{k=0}^{p^a-1}(d-2-k)\binom{2k}{k+d}+L_1\pmod{p^2},\end{aligned}\tag{3.9}$$

where

$$\begin{aligned}L_0:=&\sum_{\substack{0\leqslant k<p^a\\ \{k\}_p\leqslant d-2}}\frac{p^{a-1}}{p^{a-1}-\lfloor k/p\rfloor}\binom{2k}{k+d}\\ \equiv&[a=2\ \&\ 3\mid p+1]3p\sum_{0\leqslant k\leqslant d-2}\binom{2k}{k+p-d}\pmod{p^2}\\ &\qquad\text{(by Lemma 3.2)}\end{aligned}$$

and

$$\begin{aligned}L_1:=&\sum_{\substack{0\leqslant k<p^a\\ \{k\}_p\leqslant (d-1)-2}}(d-2-k)\frac{p^{a-1}}{p^{a-1}-\lfloor k/p\rfloor}\binom{2k}{k+d}\\ \equiv&\sum_{\substack{0\leqslant k<p^a\\ \{k\}_p\leqslant d-2}}(d-2-k)\frac{p^{a-1}}{p^{a-1}-\lfloor k/p\rfloor}\binom{2k}{k+d}\\ \equiv&[a=2\ \&\ 3\mid p+1]3p\sum_{0\leqslant k\leqslant d-2}(d-2-k)\binom{2k}{k+p-d}\pmod{p^2}\\ &\qquad\text{(by Lemma 3.2).}\end{aligned}$$

Note that $k_0:=(2p^a-2+d')/3\geqslant d'$, where

$$d'=d+1+\left(\frac{p^a-d+1}{3}\right)=\begin{cases}d&\text{if }d\equiv p^a-1\pmod 3,\\ d+1&\text{if }d\equiv p^a+1\pmod 3,\\ d+2&\text{if }d\equiv p^a\pmod 3.\end{cases}$$

If $d \in \{0,1\}$, then $d' \in \{0,1,2,3\}$ and hence

$$k_0 = \frac{2(p^a-1)+d'}{3} \not\equiv -1 \pmod p.$$

In particular, $k_0 \neq p^a - 1$ or $d' \geqslant d \geqslant 2$.

By the above,

$$\begin{aligned}
&(-1)^{p-1+d'}\binom{p^a-1}{k_0}\binom{2k_0}{p^a}\\
=&(-1)^{p-1+k_0}\binom{p^a-1}{k_0}\binom{2p^a-2+d'-k_0}{p^a}\\
\equiv&e_{d',k_0} = 1 + [\{k_0\}_p \leqslant d'-2]\frac{p^{a-1}}{p^{a-1}-\lfloor k_0/p\rfloor} \pmod{p^2},
\end{aligned}$$

and

$$\begin{aligned}
&(-1)^{p-1+d'}\binom{p^a-1}{k_0}\binom{2k_0}{p^a+1}\\
=&(-1)^{p-1+k_0}\binom{p^a-1}{k_0}\binom{2p^a-2+d'-k_0}{p^a+1}\\
\equiv&(d'-2-k_0)e_{d'-1,k_0} = (d'-2-k_0)\left(1+\frac{[\{k_0\}_p \leqslant d'-1-2]p^{a-1}}{p^{a-1}-\lfloor k_0/p\rfloor}\right)\\
\equiv&(d'-2-k_0)+(d'-2-k_0)[\{k_0\}_p \leqslant d'-2]\frac{p^{a-1}}{p^{a-1}-\lfloor k_0/p\rfloor} \pmod{p^2}
\end{aligned}$$

provided $d' > 0$. If $d' = 0$, then $d = 0$ and $k_0 = 2(p^a-1)/3$, hence we have

$$\begin{aligned}
&(-1)^{d'}\binom{p^a-1}{k_0}\binom{2k_0}{p^a+1}\\
=&\binom{p^a-1}{(p^a-1)/3}\frac{4(p^a-1)/3}{p^a+1}\binom{(4p^a-7)/3}{p^a}\\
\equiv&-\frac{4}{3}\prod_{j=1}^{(p^a-1)/3}\frac{p^a-j}{j}\times\prod_{j=1}^{(p^a-7)/3}\frac{p^a+j}{j}\\
\equiv&-\frac{4}{3}\cdot\frac{p^a-(p^a-1)/3}{(p^a-1)/3}\cdot\frac{p^a-(p^a-4)/3}{(p^a-4)/3}\prod_{j=1}^{(p^a-7)/3}\frac{p^{2a}-j^2}{j^2}\\
\equiv&-\frac{4}{3}(-1)^{p-1} \equiv (-1)^{p-1}(d'-2-k_0) \pmod{p^2}.
\end{aligned}$$

Now we distinguish three cases to discuss $\lfloor k_0/p\rfloor$ and $\{k_0\}_p$.

Case 1. $p=3$.

In this case, $d'=2$ for $d=0,1$. Thus we always have $d'\geqslant 2$. Observe that

$$\left\lfloor\frac{k_0}{p}\right\rfloor=\frac{2}{3}p^{a-1}+\left\lfloor\frac{d'-2}{3p}\right\rfloor=\frac{2}{3}p^{a-1}=2p^{a-2}$$

and

$$\{k_0\}_p=\frac{d'-2}{3}\leqslant d'-2.$$

Case 2. $3\mid p^{a-1}+1$.

In this case,

$$\left\lfloor\frac{k_0}{p}\right\rfloor=\frac{2p^{a-1}-1}{3}+\left\lfloor\frac{p+d'-2}{3p}\right\rfloor=\frac{2p^{a-1}-1}{3},$$

and

$$\begin{aligned}\{k_0\}_p=\frac{p-2}{3}+\frac{d'}{3}&\leqslant d'-2\\ \iff p-2+d'&\leqslant 3d'-6\\ \iff d'&>\frac{p+3}{2}.\end{aligned}$$

If $d'>(p+3)/2$, then $d\geqslant d'-2\geqslant (p-1)/2$. Clearly $d'=d+1=(p+1)/2$ for $d=(p-1)/2$, $d'=(p+1)/2$ for $d=(p+1)/2$, and $d'>d$ for $d=(p+3)/2$. Also, $d'=3$ for $d=1,2$. Thus, no matter $p=2$ or not, $d'>(p+3)/2$ if and only if $d>(p+(-1)^{p-1})/2$.

Case 3. $3\mid p^{a-1}-1$.

In this case, if $d\neq p$, then

$$\left\lfloor\frac{k_0}{p}\right\rfloor=\frac{2p^{a-1}-2}{3}+\left\lfloor\frac{2p+d'-2}{3p}\right\rfloor=\frac{2p^{a-1}-2}{3}$$

and

$$\{k_0\}_p=\frac{2(p-1)+d'}{3}>d'-2.$$

If $d=p$, then $d\equiv p^a \pmod 3$ and $d'=d+2=p+2$, hence

$$\left\lfloor\frac{k_0}{p}\right\rfloor=\frac{2p^{a-1}+1}{3}$$

and

$$\{k_0\}_p=\frac{2(p-1)+d'}{3}=p=d'-2.$$

Since

$$\begin{aligned}&(-1)^{p-1}\sum_{k=0}^{p^a-1}(-1)^k\binom{p^a-1}{k}\binom{2p^a-2+d-k}{p^a}\binom{2k}{k+d}\\=&[3\nmid d-p^a](-1)^{p-1+d}\binom{p^a-1}{k_0}\binom{2k_0}{p^a}\\=&\left(\frac{p^a-d}{3}\right)(-1)^{p-1+d'}\binom{p^a-1}{k_0}\binom{2k}{p^a},\end{aligned}$$

in view of the above we have

$$\begin{aligned}&(-1)^{p-1}\sum_{k=0}^{p^a-1}(-1)^k\binom{p^a-1}{k}\binom{2p^a-2+d-k}{p^a}\binom{2k}{k+d}\\\equiv&\left(\frac{p^a-d}{3}\right)\left(1+[p=3]\frac{p^{a-1}}{p^{a-1}-2p^{a-1}/3}\right)\\&+\left(\frac{p^a-d}{3}\right)[3\mid p^{a-1}+1\ \&\ 2d>p+(-1)^{p-1}]\frac{p^{a-1}}{p^{a-1}-(2p^{a-1}-1)/3}\\\equiv&\left(\frac{p^a-d}{3}\right)-\left(\frac{d}{3}\right)[p=3]p\\&+[3\mid p^{a-1}+1\ \&\ 2d>p+(-1)^{p-1}]\left(\frac{-p-d}{3}\right)\frac{3p^{a-1}}{p^{a-1}+1}\\\equiv&\left(\frac{p^a-d}{3}\right)-\left(\frac{d}{3}\right)[p=3]p-[\![a,d,p]\!]\left(\frac{p+d}{3}\right)3p\pmod{p^2}\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{k=0}^{p^a-1}\binom{2k}{k+d}-\left(\frac{p^a-d}{3}\right)\\\equiv&-\left(\frac{d}{3}\right)[p=3]p-[\![a,d,p]\!]\left(\frac{p+d}{3}\right)3p-L_0\\\equiv&-\left(\frac{d}{3}\right)[p=3]p-[\![a,d,p]\!]3pD\pmod{p^2}.\end{aligned}$$

This proves (3.4).

Let $\varepsilon=(\frac{p^a-d+1}{3})$. Then $d'-2=d-1+\varepsilon$, which is a multiple of 3 if $p=3$. Observe that

$$\begin{aligned}d'-2-k_0=&\frac{2}{3}(d'-2)-\frac{2}{3}p^a\\\equiv&2\times\frac{d-1+\varepsilon}{3}+p[p=3\ \&\ a=2]\pmod{p^2}.\end{aligned}$$

Similar to the proof of (3.4), we have

$$\begin{aligned}&(-1)^{p-1}(1+[3\mid d-p^a-1])(-1)^d\binom{p^a-1}{k_0}\binom{2k_0}{p^a+1}\\ =&(1-3[3\mid p^a-d+1])(-1)^{p-1+d'}\binom{p^a-1}{k_0}\binom{2k_0}{p^a+1}\\ \equiv&(1-3[3\mid p^a-d+1])(d'-2-k_0)\\ &\times\left(1+\frac{[p=3]p^{a-1}}{p^{a-1}-2p^{a-1}/3}+\frac{[3\mid p^{a-1}+1\ \&\ 2d>p+(-1)^{p-1}]p^{a-1}}{p^{a-1}-(2p^{a-1}-1)/3}\right)\\ \equiv&2(1-3[3\mid p^a-d+1])\frac{d-1+\varepsilon}{3}(1+p[p=3]+[\![a,d,p]\!]3p)\\ &+(1-3[3\mid p^a-d+1])p[p=3\ \&\ a=2](1+p[p=3])\\ \equiv&2(1-3[3\mid p^a-d+1])\frac{d-1+\varepsilon}{3}(1+[\![a,d,p]\!]3p)+\Delta\pmod{p^2},\end{aligned}$$

where

$$\begin{aligned}\Delta=&p[p=3](1-3[3\mid d-1])\left(2\times\frac{d-1-(\frac{d-1}{3})}{3}+[a=2]\right)\\ \equiv&p[p=3]([a=2]-[d=p])\pmod{p^2}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{k=0}^{p^a-1}(d-2-k)\binom{2k}{k+d}-2(1-3[3\mid p^a-d+1])\frac{d-1+\varepsilon}{3}\\ \equiv&[\![a,d,p]\!]3p(1-3[3\mid p+d-1])\frac{2}{3}\left(d-1+\left(\frac{-p-d+1}{3}\right)\right)+\Delta-L_1\\ \equiv&p[p=3]([a=2]-[d=p])-[\![a,d,p]\!]3pD'\pmod{p^2}.\end{aligned}$$

So (3.5) holds.

The proof of Theorem 3.1 is now complete. □

4. Proofs of Theorems 1.1-1.3

Lemma 4.1. *Let p be a prime and let $d\in\{0,\dots,p\}$. Then* (1.10) *holds for all $a=1,2,3,\dots$.*

Proof. By Theorem 3.1, (1.10) is valid for $a=3,4,\dots$. Below we show (1.10) for $a=1,2$.

In light of Theorem 2.1,

$$\begin{aligned}
&\sum_{k=0}^{p^3-1}\binom{2k}{k+d}=\sum_{n=0}^{p^2-1}\sum_{k=0}^{p-1}\binom{2(pn+k)}{pn+k+d}\\
\equiv&\sum_{n=0}^{p^2-1}\binom{2n}{n}\sum_{k=0}^{p-1}\binom{2k}{k+d}\\
&+\sum_{n=0}^{p^2-1}nC_n(1+[p=2]p(n+1)(n-1))\sum_{k=0}^{p-1}\binom{2k}{k+p-d}\\
&+\sum_{n=0}^{p^2-1}(4n^2-(2n-1)n+n^2)C_n\sum_{0<k<d}\binom{p}{k}\left(\frac{d-k}{3}\right)\\
&+p[p\leqslant 3]\sum_{n=0}^{p^2-1}(n+1)nC_n\left(\left(\frac{p+d}{3}\right)+\left(\frac{p-d}{3}\right)\right)\pmod{p^2}.
\end{aligned}$$

Note that

$$nC_n=\binom{2n}{n+1}\ \text{and}\ (4n^2-(2n-1)n+n^2)C_n=3n\binom{2n}{n}-2\binom{2n}{n+1}.$$

Also,

$$\binom{p}{k}=\frac{p}{k}\prod_{0<j<k}\left(\frac{p}{j}-1\right)\equiv p\frac{(-1)^{k-1}}{k}\pmod{p^2}\quad\text{for } k=1,\ldots,p-1.$$

So we have

$$\begin{aligned}
&\sum_{k=0}^{p^3-1}\binom{2k}{k+d}-\sum_{n=0}^{p^2-1}\binom{2n}{n}\sum_{k=0}^{p-1}\binom{2k}{k+d}-\sum_{n=0}^{p^2-1}\binom{2n}{n+1}\sum_{k=0}^{p-1}\binom{2k}{k+p-d}\\
&\quad\equiv\left(3\sum_{n=0}^{p^2-1}n\binom{2n}{n}-2\sum_{n=0}^{p^2-1}\binom{2n}{n+1}\right)pS_d\pmod{p^2}.
\end{aligned}\tag{4.1}$$

Similarly,

$$\begin{aligned}
&\sum_{k=0}^{p^2-1}\binom{2k}{k+d}-\sum_{n=0}^{p-1}\binom{2n}{n}\sum_{k=0}^{p-1}\binom{2k}{k+d}-\sum_{n=0}^{p-1}\binom{2n}{n+1}\sum_{k=0}^{p-1}\binom{2k}{k+p-d}\\
&\quad\equiv\left(3\sum_{n=0}^{p-1}n\binom{2n}{n}-2\sum_{n=0}^{p-1}\binom{2n}{n+1}\right)pS_d\pmod{p^2}.
\end{aligned}\tag{4.2}$$

By Theorem 3.1,

$$\sum_{n=0}^{p^2-1}\binom{2n}{n+\delta}\equiv\left(\frac{p^2-\delta}{3}\right)-\left(\frac{\delta}{3}\right)[p=3]p \pmod{p^2}\quad\text{ for }\delta=0,1.$$

Also,

$$\begin{aligned}\sum_{n=0}^{p^2-1}n\binom{2n}{n}&=-2\sum_{n=0}^{p^2-1}\binom{2n}{n}-\sum_{n=0}^{p^2-1}(0-2-n)\binom{2n}{n}\\&\equiv-2\left(\frac{p^2}{3}\right)-\left(2(1-3[3\mid p^2+1])\frac{-1+(\frac{p^2+1}{3})}{3}+p[p=3]\right)\\&\equiv-\frac{2}{3}[p\neq 3]-[p=3]p \pmod{p^2}\end{aligned}$$

and

$$\sum_{k=0}^{p^3-1}\binom{2k}{k+d}\equiv\left(\frac{p^3-d}{3}\right)-\left(\frac{d}{3}\right)[p=3]p \pmod{p^2}.$$

Thus, when $p\neq 3$, (4.1) yields

$$\left(\frac{p^3-d}{3}\right)-\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv-2pS_d \pmod{p^2},$$

that is,

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}-\left(\frac{p-d}{3}\right)\equiv 2pS_d \pmod{p^2}.$$

In the case $p=3$, as

$$\sum_{n=0}^{p^2-1}\binom{2n}{n}\equiv 0 \pmod{p^2}\text{ and }\sum_{n=0}^{p^2-1}\binom{2n}{n+1}\equiv-1-p \pmod{p^2},$$

by (4.1) we have

$$\left(\frac{-d}{3}\right)-\left(\frac{d}{3}\right)p+(p+1)\sum_{k=0}^{p-1}\binom{2k}{k+p-d}\equiv-2(-1-p)pS_d \pmod{p^2}$$

and hence

$$\sum_{k=0}^{p-1}\binom{2k}{k+p-d}\equiv\left(\frac{d}{3}\right)+2pS_d \pmod{p^2}.$$

Therefore, if $p=3$ then

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}-\left(\frac{p-d}{3}\right)\equiv 2pS_{p-d}\equiv 2p\left(S_d+\left(\frac{d}{3}\right)\right)\ (\mathrm{mod}\ p^2)$$

since $S_3=-3/2\equiv 0=S_0 \pmod 3$ and $S_2-S_1=S_2=1$. In view of the above, (1.10) does hold for $a=1$.

By [PS, (1.5)],

$$\sum_{n=0}^{p-1}n\binom{2n}{n}\equiv -\frac{2}{3}\left(\frac{p}{3}\right)-[p=3]\ (\mathrm{mod}\ p).$$

With the helps of this and (1.10) in the case $a=1$, we deduce from (4.2) that

$$\begin{aligned}&\sum_{k=0}^{p^2-1}\binom{2k}{k+d}\\ &\equiv\left(\frac{p}{3}\right)\left(\left(\frac{p-d}{3}\right)-p[p=3]\left(\frac{d}{3}\right)+2pS_d\right)\\ &\quad+\left(\left(\frac{p-1}{3}\right)-p[p=3]\right)\left(\left(\frac{d}{3}\right)-p[p=3]\left(\frac{p-d}{3}\right)+2pS_{p-d}\right)\\ &\quad+\left(-2\left(\frac{p}{3}\right)-3[p=3]-2\left(\frac{p-1}{3}\right)\right)pS_d\ (\mathrm{mod}\ p^2).\end{aligned}\tag{4.3}$$

Recall that

$$\left(\frac{p}{3}\right)\left(\frac{p-d}{3}\right)+\left(\frac{p-1}{3}\right)\left(\frac{d}{3}\right)=\left(\frac{p^2-d}{3}\right)$$

by (3.2). For $p>3$, by Lemma 2.2(i) we have

$$\begin{aligned}S_{p-d}-S_d&=\sum_{0<j<p-d}\frac{(-1)^{j-1}}{j}\left(\frac{p-d-j}{3}\right)-\sum_{0<k<p}\frac{(-1)^{k-1}}{k}\left(\frac{d-k}{3}\right)\\ &\equiv\sum_{d<k<p}\frac{(-1)^{p-1-k}}{p-k}\left(\frac{k-d}{3}\right)+\sum_{0<k<p}\frac{(-1)^{k-1}}{k}\left(\frac{k-d}{3}\right)\\ &\equiv\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\left(\frac{k-d}{3}\right)\equiv\frac{1}{p}\sum_{k=1}^{p-1}\binom{p}{k}\left(\frac{k-d}{3}\right)=0\ (\mathrm{mod}\ p).\end{aligned}$$

Therefore, (1.10) in the case $a = 2$ and $p \neq 3$, follows from (4.3). For $p = 3$, as $S_{p-d} \equiv S_d + (\frac{d}{3}) \pmod p$, by (4.3) we have

$$\begin{aligned}&\sum_{k=0}^{p^2-1}\binom{2k}{k+d}\\ \equiv&(-1-p)\left(\left(\frac{d}{3}\right)-p\left(\frac{-d}{3}\right)+2p\left(\frac{d}{3}\right)+2pS_d\right)+(-p+2)pS_d\\ \equiv&\left(\frac{p^2-d}{3}\right)-\left(\frac{d}{3}\right)p \pmod{p^2}.\end{aligned}$$

So (1.10) also holds when $a = 2$ and $p = 3$. We are done. □

Lemma 4.2. *Let d and k be nonnegative integers. Then we have*

$$(k+1)\left(\binom{2k}{k+d}-\binom{2k}{k+d+2}\right)=(d+1)\binom{2k+2}{k+d+2}. \tag{4.4}$$

Proof. If $k < d$, then both sides of (4.4) vanish.

Now assume that $k \geqslant d$. It is clear that

$$\begin{aligned}&\binom{2k}{k+d}-\binom{2k}{k+d+2}\\ =&\frac{(2k)!}{(k-d)!(k+d+2)!}((k+d+1)(k+d+2)-(k-d)(k-d-1))\\ =&\frac{2(d+1)(2k+1)(2k)!}{(k-d)!(k+d+2)!}=\frac{d+1}{k+1}\binom{2k+2}{k+d+2}.\end{aligned}$$

So (4.4) follows. □

Proof of Theorem 1.3. We first consider the case $0 \leqslant d \leqslant p = 3$. By Theorem 3.1,

$$\sum_{k=0}^{3^a-1}\binom{2k}{k+d}\equiv -4\left(\frac{d}{3}\right) \pmod{3^2}$$

and

$$\begin{aligned}&\sum_{k=0}^{3^a-1}(d-2-k)\binom{2k}{k+d}\\ \equiv&2(1-3[3\mid d-1])\frac{d-1-(\frac{d-1}{3})}{3}+3([a=2]-[d=3])\\ \equiv&2[d=3]+3([a=2]-[d=3])=3[a=2]-[d=3] \pmod{3^2}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=0}^{3^a-1} k\binom{2k}{k+d} \equiv&(d-2)\left(-4\left(\frac{d}{3}\right)\right)-(3[a=2]-[d=3])\\ \equiv&-(4d+1)\left(\frac{d}{3}\right)-3[a=2]+[d=3] \pmod{3^2}.\end{aligned}$$

This proves (1.12).

Below we assume $p \neq 3$. For $d \in \{0, \ldots, \lfloor (p+1)/2 \rfloor\}$, by (3.5) we have

$$\begin{aligned}&\sum_{k=0}^{p^a-1}(d-2-k)\binom{2k}{k+d}\\ \equiv&\frac{2}{3}(1-3[3\mid p^a-d+1])\left(d-1+\left(\frac{p^a-d+1}{3}\right)\right) \pmod{p^2}.\end{aligned}$$

(Note that if $p = 2$ and $d = 1$ then $D' \equiv 0 \pmod p$.) Combining this with (1.10) we obtain (1.11) for $d = 0, 1, \ldots, \lfloor (p+1)/2 \rfloor$.

Now we use induction argument. Assume that $0 \leqslant d \leqslant p-2$ and (1.11) holds. We want to show that (1.11) with d replaced by $d+2$ remains valid. By Lemmas 4.1-4.2,

$$\begin{aligned}&\sum_{k=0}^{p^a-1} k\binom{2k}{k+d+2}-\sum_{k=0}^{p^a-1} k\binom{2k}{k+d}\\ &+\left(\frac{p^a-d-2}{3}\right)-p[p=3]\left(\frac{d+2}{3}\right)-\left(\left(\frac{p^a-d}{3}\right)-p[p=3]\left(\frac{d}{3}\right)\right)\\ \equiv&-(d+1)\sum_{k=0}^{p^a-1}\binom{2(k+1)}{k+1+d+1}=-(d+1)\sum_{k=1}^{p^a}\binom{2k}{k+d+1}\\ \equiv&-(d+1)\left(\left(\frac{p^a-d-1}{3}\right)-p[p=3]\left(\frac{d+1}{3}\right)\right)\\ &-(d+1)\frac{2p^a}{p^a+d+1}\binom{2p^a-1}{p^a+d} \pmod{p^2}\end{aligned}$$

and thus

$$\begin{aligned}&\sum_{k=0}^{p^a-1} k\binom{2k}{k+d+2}-\sum_{k=0}^{p^a-1} k\binom{2k}{k+d}\\ \equiv&-\left(\frac{p^a-d-2}{3}\right)+\left(\frac{p^a-d}{3}\right)-(d+1)\left(\frac{p^a-d-1}{3}\right) \pmod{p^2}.\end{aligned}$$

This, together with (1.11), shows that $\sum_{k=0}^{p^a-1}k\binom{2k}{k+d+2}$ is congruent to

$$\begin{aligned}&-\frac{2}{3}\left(\frac{p^a-d}{3}\right)+\frac{d}{3}\left(\frac{p^a-d-1}{3}\right)-\frac{d}{3}\left(\frac{p^a-d+1}{3}\right)\\&-\left(\frac{p^a-d-2}{3}\right)+\left(\frac{p^a-d}{3}\right)-(d+1)\left(\frac{p^a-d-1}{3}\right)\\=&\frac{1}{3}\left(\frac{p^a-d}{3}\right)-\frac{d+3}{3}\left(\frac{p^a-d+1}{3}\right)-\frac{2d+3}{3}\left(\frac{p^a-d-1}{3}\right)\end{aligned}$$

modulo p^2. Since

$$\frac{d+1}{3}\left(\left(\frac{p^a-d-1}{3}\right)+\left(\frac{p^a-d}{3}\right)+\left(\frac{p^a-d+1}{3}\right)\right)=0,$$

we have

$$\begin{aligned}&\sum_{k=0}^{p^a-1}k\binom{2k}{k+d+2}+\frac{2}{3}\left(\frac{p^a-(d+2)}{3}\right)\\\equiv&\frac{d+2}{3}\left(\left(\frac{p^a-(d+2)-1}{3}\right)-\left(\frac{p^a-(d+2)+1}{3}\right)\right)\pmod{p^2}\end{aligned}$$

as desired. This concludes the induction step, and we are done. □

Remark 4.1. Let p be a prime. Similar to the proof of (1.11), for $d\in\{0,\ldots,p-2\}$ we have

$$\begin{aligned}&\sum_{k=0}^{p-1}(k+1)\binom{2k}{k+d+2}-\sum_{k=0}^{p-1}(k+1)\binom{2k}{k+d}\\=&-(d+1)\sum_{k=1}^{p}\binom{2k}{k+d+1}\\\equiv&(-1)^{d+1}2p-(d+1)\sum_{k=0}^{p-1}\binom{2k}{k+d+1}\pmod{p^2}\end{aligned}$$

because

$$(d+1)\binom{2p}{p+d+1}=\frac{p(d+1)}{p+d+1}\binom{2p}{p}\prod_{0<j\leqslant d}\frac{p-j}{p+j}\equiv(-1)^d2p\pmod{p^2}.$$

So we can compute $\sum_{k=0}^{p-1}k\binom{2k}{k+d}$ mod p^2 for any $d=0,1,\ldots,p$, by means of (1.10) and the forthcoming (4.6) with $d\in\{0,1\}$, though the result is somewhat complicated.

Proof of Theorem 1.2. We divide the proof into four steps.

`Step I.` Given $d \in \{0,1\}$ we prove (1.4) and (1.5) with $a=1$ for $p \neq 2$ and $p>3$ respectively.

By (2.3) and (1.10),

$$\begin{aligned}&\binom{m}{n}^{-1}\sum_{k=0}^{p-1}\binom{pm+2k}{pn+k+d}-d\frac{m-n}{n+1}\\ \equiv&(1+[p=2]pn(m-n))\left(\left(\frac{p-d}{3}\right)-p[p=3]\left(\frac{d}{3}\right)\right)\\ &+p[p\leqslant 3]\left(n\left(\frac{p+d}{3}\right)+(m-n)\left(\frac{p-d}{3}\right)\right) \pmod{p^2}\end{aligned}$$

and thus

$$\begin{aligned}&\binom{m}{n}^{-1}\sum_{k=0}^{p-1}\binom{pm+2k}{pn+k+d}-d\frac{m-n}{n+1}\\ \equiv&\left(\frac{p-d}{3}\right)-[p=3]pd(m+n+1)\\ &+[p=2]p[d[2\nmid m\ \&\ 2\mid n]+(1-d)[2\nmid m \text{ or } 2\nmid n]) \pmod{p^2}.\end{aligned}\tag{4.5}$$

This proves (1.4) with $a=1$ for $p\neq 2$. (4.5) for $p=2$ will be used later.

Clearly $(\frac{3+d}{3})+(\frac{3-d}{3})=0$ and $(-1)^d2+d\equiv 2 \pmod 3$. Also, $\binom{2q}{q}q=q(q+1)C_q\equiv 0 \pmod 2$ for any $q\in\mathbb{N}$. Thus, in light of Theorem 2.1,

$$\begin{aligned}\sum_{k=0}^{p^3-1}k\binom{2k}{k+d}=&\sum_{q=0}^{p^2-1}\sum_{k=0}^{p-1}(pq+k)\binom{2(pq+k)}{pq+k+d}\\ \equiv&\sum_{q=0}^{p^2-1}pq\left(\binom{2q}{q}\sum_{k=0}^{p-1}\binom{2k}{k+d}+d\binom{2q}{q+1}\right)\\ &+\sum_{q=0}^{p^2-1}\binom{2q}{q}\left(\sum_{k=0}^{p-1}k\binom{2k}{k+d}+\left(\frac{p-1}{3}\right)p\left((-1)^d2q+dq\right)\right)\\ &+d(p-1)\sum_{q=0}^{p^2-1}\binom{2q}{q+1} \pmod{p^2}.\end{aligned}$$

We claim that

$$\sum_{k=0}^{p-1}k\binom{2k}{k+d}\equiv\frac{2}{3}\left(p-\left(\frac{p}{3}\right)\right)-d\left(\frac{p-1}{3}\right)+(-1)^dp[p=3] \pmod{p^2}.\tag{4.6}$$

For $p=2,3$ this can be verified directly. Now assume $p>3$. By (1.10) and Theorem 1.3,

$$\sum_{q=0}^{p^2-1}\binom{2q}{q}\equiv\left(\frac{p^2}{3}\right)=1 \pmod{p^2},$$
$$\sum_{q=0}^{p^2-1}\binom{2q}{q+1}\equiv\left(\frac{p^2-1}{3}\right)=0 \pmod{p^2},$$
$$\sum_{q=0}^{p^2-1}q\binom{2q}{q}\equiv\sum_{q=0}^{p^2-1}q\binom{2q}{q+1}\equiv-\frac{2}{3} \pmod{p^2}.$$

Therefore, the last congruence before (4.6) yields

$$\begin{aligned}&\sum_{k=0}^{p^3-1}k\binom{2k}{k+d}-\sum_{k=0}^{p-1}k\binom{2k}{k+d}\\ \equiv&-\frac{2}{3}p\left(\left(\frac{p-d}{3}\right)+d+\left(\frac{p-1}{3}\right)((-1)^d2+d)\right)=-\frac{2}{3}p \pmod{p^2}.\end{aligned}$$

In light of Theorem 1.3,

$$\sum_{k=0}^{p^3-1}k\binom{2k}{k}\equiv-\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p^2}$$

and hence

$$\begin{aligned}\sum_{k=0}^{p^3-1}k\binom{2k}{k+1}&=\sum_{k=0}^{p^3-1}k\binom{2k}{k}-\sum_{k=0}^{p^3-1}\binom{2k}{k+1}\\ &\equiv-\frac{2}{3}\left(\frac{p}{3}\right)-\left(\frac{p^3-1}{3}\right) \pmod{p^2}.\end{aligned}$$

Thus

$$\sum_{k=0}^{p-1}k\binom{2k}{k}\equiv\frac{2}{3}\left(p-\left(\frac{p}{3}\right)\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}k\binom{2k}{k+1}\equiv\frac{2}{3}\left(p-\left(\frac{p}{3}\right)\right)-\left(\frac{p-1}{3}\right) \pmod{p^2}.$$

This proves (4.6) for $p>3$.

Combining (4.6) with (2.4) we get

$$\begin{aligned}&\binom{m}{n}^{-1}\sum_{k=0}^{p-1}k\binom{pm+2k}{pn+k+d}-d(p-1)\frac{m-n}{n+1}\\ \equiv&\frac{2}{3}\left(p-\left(\frac{p}{3}\right)\right)-d\left(\frac{p-1}{3}\right)\\ &+\begin{cases}(\frac{p-1}{3})p(m(-1)^d+dn)n^{[p=2\ \&\ d=1]}\pmod{p^2} & \text{if } p\neq 3,\\ p((-1)^d-m)\pmod{p^2} & \text{if } p=3.\end{cases}\end{aligned}\tag{4.7}$$

This proves (1.5) with $a=1$ for $p>3$. (4.7) in the case $p\in\{2,3\}$ will be used later.

`Step II.` Assuming that (1.4) (with $p\neq 2$) and (1.5) (with $p>3$) hold for $d=0,1$ with a replaced by $a-1>0$, we show (1.4) and (1.5) for any $d\in\{0,1\}$.

For $q\in\{0,1,\ldots,p-1\}$ we set

$$a_q:=\binom{pm+2q}{pn+q}\Big/\binom{m}{n}.\tag{4.8}$$

With the help of Lemma 2.1,

$$a_q=\frac{\binom{pm}{pn}}{\binom{m}{n}}\cdot\frac{\prod_{0<i\leqslant 2q}(pm+i)}{\prod_{0<j\leqslant q}(pn+j)(p(m-n)+j)}\equiv\frac{(2q)!}{q!q!}=\binom{2q}{q}\pmod{p}.$$

By (4.5) in the case $d=0$,

$$\sum_{q=0}^{p-1}a_q=\binom{m}{n}^{-1}\sum_{q=0}^{p-1}\binom{pm+2q}{pn+q}\equiv\left(\frac{p}{3}\right)+p[p=2][2\nmid m\text{ or }2\nmid n]\pmod{p^2}.\tag{4.9}$$

(4.5) with $d=1$ gives

$$\begin{aligned}&\sum_{q=0}^{p-1}a_q\frac{p(m-n)+q}{pn+q+1}-\frac{m-n}{n+1}\\ =&\binom{m}{n}^{-1}\sum_{q=0}^{p-1}\binom{pm+2q}{pn+q+1}-\frac{m-n}{n+1}\\ \equiv&\left(\frac{p-1}{3}\right)-[p=3]p(m+n+1)+[p=2]p[2\nmid m\ \&\ 2\mid n]\pmod{p^2}.\end{aligned}\tag{4.10}$$

In view of (4.7) in the case $d = 0$, we have

$$\begin{aligned}\sum_{q=0}^{p-1} q a_q &= \binom{m}{n}^{-1} \sum_{q=0}^{p-1} q \binom{pm+2q}{pn+q} \\ &\equiv \frac{2}{3}\left(p - \left(\frac{p}{3}\right)\right) + \left(\frac{p-1}{3}\right) pm + p[p=3] \pmod{p^2}.\end{aligned} \tag{4.11}$$

Also, by (4.7) with $d = 1$, if $p > 3$ then

$$\begin{aligned}&\sum_{q=0}^{p-1} q a_q \frac{p(m-n)+q}{pn+q+1} - (p-1)\frac{m-n}{n+1} \\ =&\binom{m}{n}^{-1} \sum_{q=0}^{p-1} q \binom{pm+2q}{pn+q+1} - (p-1)\frac{m-n}{n+1} \\ \equiv&\frac{2}{3}\left(p - \left(\frac{p}{3}\right)\right) + \left(\frac{p-1}{3}\right)(p(n-m)-1) \pmod{p^2}.\end{aligned} \tag{4.12}$$

Let $q \in \{0, \ldots, p-1\}$. Define

$$\begin{aligned}b_q =& \binom{pm+2q}{pn+q}^{-1} \sum_{k=0}^{p^{a-1}-1} \binom{p^{a-1}(pm+2q)+2k}{p^{a-1}(pn+q)+k+d} \\ &- d\frac{pm+2q-(pn+q)}{pn+q+1}\end{aligned}$$

and

$$\begin{aligned}b'_q =& \binom{pm+2q}{pn+q}^{-1} \sum_{k=0}^{p^{a-1}-1} k \binom{p^{a-1}(pm+2q)+2k}{p^{a-1}(pn+q)+k+d} \\ &- d(p^{a-1}-1)\frac{pm+2q-(pn+q)}{pn+q+1}.\end{aligned}$$

By the induction hypothesis, if $p \neq 2$ then

$$b_q - \left(\frac{p^{a-1}-d}{3}\right) \equiv -[p=3]pd(pm+2q+pn+q+1) \equiv -[p=3]pd \pmod{p^2};$$

also, if $p > 3$ then

$$\begin{aligned}&b'_q - \frac{2}{3}\left(p^{a-1} - \left(\frac{p^{a-1}}{3}\right)\right) + d\left(\frac{p^{a-1}-1}{3}\right) \\ \equiv&\left(\frac{p^{a-1}-1}{3}\right) p^{a-1}((-1)^d(pm+2q) + d(pn+q)) \\ \equiv&\left(\frac{p^{a-1}-1}{3}\right) p^{a-1} q((-1)^d 2 + d) \pmod{p^2}.\end{aligned}$$

In view of (4.9)-(4.10) and the congruence for b_q mod p^2, if $p \neq 2$ then

$$\begin{aligned}
&\binom{m}{n}^{-1}\sum_{k=0}^{p^a-1}\binom{p^am+2k}{p^an+k+d}-d\frac{m-n}{n+1}\\
=&\binom{m}{n}^{-1}\sum_{q=0}^{p-1}\sum_{k=0}^{p^{a-1}-1}\binom{p^am+2(p^{a-1}q+k)}{p^an+p^{a-1}q+k+d}-d\frac{m-n}{n+1}\\
=&\sum_{q=0}^{p-1}a_qb_q+d\bigg(\sum_{q=0}^{p-1}a_q\frac{pm+2q-(pn+q)}{pn+q+1}-\frac{m-n}{n+1}\bigg)\\
\equiv&\left(\frac{p}{3}\right)\bigg(\left(\frac{p^{a-1}-d}{3}\right)-[p=3]pd\bigg)\\
&+d\bigg(\left(\frac{p-1}{3}\right)-[p=3]p(m+n+1)\bigg)\\
\equiv&\left(\frac{p^a-d}{3}\right)-[p=3]pd(m+n+1)\ (\mathrm{mod}\ p^2).
\end{aligned}$$

This proves (1.4).

Observe that

$$\begin{aligned}
&\binom{m}{n}^{-1}\sum_{k=0}^{p^a-1}k\binom{p^am+2k}{p^an+k+d}-d(p^a-1)\frac{m-n}{n+1}\\
=&\sum_{q=0}^{p-1}\sum_{k=0}^{p^{a-1}-1}\frac{p^{a-1}q+k}{\binom{m}{n}}\binom{p^am+2(p^{a-1}q+k)}{p^an+p^{a-1}q+k+d}-d(p^a-1)\frac{m-n}{n+1}\\
=&\sum_{q=0}^{p-1}p^{a-1}qa_q\left(b_q+d\frac{p(m-n)+q}{pn+q+1}\right)\\
&+\sum_{q=0}^{p-1}a_q\bigg(b'_q+d(p^{a-1}-1)\frac{p(m-n)+q}{pn+q+1}\bigg)-d(p^a-1)\frac{m-n}{n+1}\\
=&p^{a-1}\bigg(\sum_{q=0}^{p-1}qa_qb_q+d\bigg(\sum_{q=0}^{p-1}qa_q\frac{p(m-n)+q}{pn+q+1}-(p-1)\frac{m-n}{n+1}\bigg)\bigg)\\
&+\sum_{q=0}^{p-1}a_qb'_q+d(p^{a-1}-1)\bigg(\sum_{q=0}^{p-1}a_q\frac{p(m-n)+q}{pn+q+1}-\frac{m-n}{n+1}\bigg).
\end{aligned}$$

Thus, with the helps of (4.10), (4.12) and the congruences for b_q and b'_q

modulo p^2, when $p > 3$ we have

$$\begin{aligned}
&\binom{m}{n}^{-1}\sum_{k=0}^{p^a-1} k\binom{p^a m+2k}{p^a n+k+d} - d(p^a-1)\frac{m-n}{n+1}\\
\equiv& p^{a-1}\sum_{q=0}^{p-1} q a_q \left(\frac{p^{a-1}-d}{3}\right) + p^{a-1}d\left(-\frac{2}{3}\left(\frac{p}{3}\right)-\left(\frac{p-1}{3}\right)\right)\\
&+\sum_{q=0}^{p-1} a_q\left(\frac{2}{3}\left(p^{a-1}-\left(\frac{p^{a-1}}{3}\right)\right)-d\left(\frac{p^{a-1}-1}{3}\right)\right)\\
&+p^{a-1}\sum_{q=0}^{p-1} q a_q\left(\frac{p^{a-1}-1}{3}\right)((-1)^d 2+d)+d(p^{a-1}-1)\left(\frac{p-1}{3}\right)\\
\equiv& p^{a-1}(1-d)\sum_{q=0}^{p-1} q a_q + \frac{2}{3}p^{a-1}\left(\sum_{q=0}^{p-1} a_q - \left(\frac{p}{3}\right)d\right)\\
&-\frac{2}{3}\left(\frac{p^{a-1}}{3}\right)\sum_{q=0}^{p-1} a_q - d\left(\frac{p^{a-1}-1}{3}\right)\sum_{q=0}^{p-1} a_q - d\left(\frac{p-1}{3}\right) \pmod{p^2}.
\end{aligned}$$

By (4.9) and (4.11),

$$\sum_{q=0}^{p-1} a_q \equiv \left(\frac{p}{3}\right) \pmod{p^2} \quad\text{and}\quad \sum_{q=0}^{p-1} q a_q \equiv -\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p}.$$

Therefore (1.5) follows. (Note that $p^a \equiv 0 \pmod{p^2}$ since $a \geqslant 2$.)

`Step III`. Given $d \in \{0, \ldots, p\}$, we prove (1.6) in the case $a = 1$.
In view of (2.5) and Lemma 4.1,

$$\begin{aligned}
&\frac{n+1}{\binom{m}{n}}\sum_{k=0}^{p-1}\binom{pm+2k}{pn+k+d} - (n+1)\left(\left(\frac{p-d}{3}\right) - p[p=3]\left(\frac{d}{3}\right) + 2pS_d\right)\\
\equiv&(m-n)\left(\left(\frac{d}{3}\right) - p[p=3]\left(\frac{p-d}{3}\right) + 2pS_{p-d}\right)\\
&+(m^2-(m-1)n+n^2)pS_d\\
&+p[p\leqslant 3](n+1)\left(n\left(\frac{p+d}{3}\right)+(m-n)\left(\frac{p-d}{3}\right)\right) \pmod{p^2}.
\end{aligned}$$

As in the proof of Lemma 4.1, $S_{p-d} \equiv S_d + [p=3](\frac{d}{3}) \pmod{p}$ if $p \neq 2$.

So we get

$$\begin{aligned}
&\frac{n+1}{\binom{m}{n}}\sum_{k=0}^{p-1}\binom{pm+2k}{pn+k+d}-(n+1)\left(\frac{p-d}{3}\right)-(m-n)\left(\frac{d}{3}\right)\\
\equiv&(2(n+1)+2(m-n)+m^2-(m-1)n+n^2)pS_d\\
&+p[p=3]\left(-(n+1)\left(\frac{d}{3}\right)-(m-n)\left(\frac{-d}{3}\right)+2(m-n)\left(\frac{d}{3}\right)\right)\\
&+p[p\leqslant 3](n+1)\left(n\left(\frac{p+d}{3}\right)+(m-n)\left(\frac{p-d}{3}\right)\right)\\
\equiv&((m-n)^2+(m+1)(n+2))pS_d-p[p=3]\left(\frac{d}{3}\right)(n+1)(m+n+1)\\
&+p[p=2\ \&\ 3\nmid d+1]m(n+1)\pmod{p^2}.
\end{aligned}$$

This proves (1.6) with $a=1$.

`Step IV.` Assuming that (1.6) holds for $d=0,1,\ldots,p$ with a replaced by $a-1>0$, we show (1.6) for any $d\in\{0,1,\ldots,p\}$.

Define

$$a'_q=\frac{(n+1)a_q}{pn+q+1}\qquad\text{for } q=0,1,\ldots,p-1.$$

As $a_q\equiv\binom{2q}{q}\pmod p$, a'_q is a p-adic integer if $0\leqslant q<p-1$. Note that

$$a'_{p-1}=\frac{a_{p-1}}{p}=\frac{a_{p-1}-\binom{2(p-1)}{p-1}}{p}+C_{p-1}\in\mathbb{Z}.$$

For

$$\sigma:=\frac{n+1}{\binom{m}{n}}\sum_{k=0}^{p^a-1}\binom{p^am+2k}{p^an+k+d},$$

clearly

$$\begin{aligned}
\sigma=&\frac{n+1}{\binom{m}{n}}\sum_{q=0}^{p-1}\sum_{k=0}^{p^{a-1}-1}\binom{p^am+2(p^{a-1}q+k}{p^an+p^{a-1}q+k+d}\\
=&\sum_{q=0}^{p-1}a'_q\frac{pn+q+1}{\binom{pm+2q}{pn+q}}\sum_{k=0}^{p^{a-1}-1}\binom{p^{a-1}(pm+2q)+k}{p^{a-1}(pn+q)+k+d}.
\end{aligned}$$

Thus, by the induction hypothesis,

$$\begin{aligned}\sigma \equiv &\sum_{q=0}^{p-1} a'_q\bigg((pn+q+1)\left(\frac{p^{a-1}-d}{3}\right)+(pm+2q-pn-q)\left(\frac{d}{3}\right)\bigg)\\&+p^{a-1}S_d\sum_{q=0}^{p-1}a'_q((pm+2q-pn-q)^2+(pm+2q+1)(pn+q+2))\\&-p[p=3]\left(\frac{d}{3}\right)\sum_{q=0}^{p-1}a'_q(pn+q+1)(pm+2q+pn+2q+1)\\&+p[p=2\ \&\ 3\nmid d-(-1)^{a-1}]\sum_{q=0}^{p-1}a'_q(pm+2q)(pn+q+1)\ (\mathrm{mod}\ p^2).\end{aligned}$$

and hence

$$\begin{aligned}\sigma \equiv &(n+1)\sum_{q=0}^{p-1}a_q\bigg(\left(\frac{p^{a-1}-d}{3}\right)+\frac{p(m-n)+q}{pn+q+1}\left(\frac{d}{3}\right)\bigg)\\&+p^{a-1}S_d\sum_{q=0}^{p-1}a'_q(q^2+(2q+1)(q+2))-p[p=3]\left(\frac{d}{3}\right)\sum_{q=0}^{p-1}(n+1)a_q\\\equiv&\bigg(\left(\frac{p^{a-1}-d}{3}\right)-p[p=3]\left(\frac{d}{3}\right)\bigg)(n+1)\sum_{q=0}^{p-1}a_q\\&+\left(\frac{d}{3}\right)(n+1)\sum_{q=0}^{p-1}a_q\frac{p(m-n)+q}{pn+q+1}\\&+p^{a-1}S_d\sum_{q=0}^{p-1}a'_q(q+1)(3q+2)\ (\mathrm{mod}\ p^2).\end{aligned}$$

If $q\in\{0,\ldots,p-2\}$ then

$$(q+1)a'_q=\frac{(q+1)(n+1)}{pn+q+1}a_q\equiv(n+1)a_q\ (\mathrm{mod}\ p).$$

For $q=p-1$, as $a_q\equiv\binom{2q}{q}=pC_{p-1}\equiv0\ (\mathrm{mod}\ p)$ we have

$$(q+1)a'_q=a_q\equiv(n+1)a_q\ (\mathrm{mod}\ p).$$

With the helps of (4.9) and (4.11),

$$\begin{aligned}\sum_{q=0}^{p-1}a'_q(q+1)(3q+2)\equiv&(n+1)\sum_{q=0}^{p-1}(3qa_q+2a_q)\\\equiv&(n+1)\left(-2\left(\frac{p}{3}\right)+2\left(\frac{p}{3}\right)\right)=0\ (\mathrm{mod}\ p).\end{aligned}$$

In light of (4.9) and (4.10), by the above we have

$$\begin{aligned}\sigma \equiv&(n+1)\bigg(\left(\frac{p^{a-1}-d}{3}\right)-p[p=3]\left(\frac{d}{3}\right)\bigg)\\ &\times\bigg(\left(\frac{p}{3}\right)+p[p=2][2\nmid m \text{ or } 2\nmid n]\bigg)\\ &+\left(\frac{d}{3}\right)\bigg(m-n+(n+1)\left(\frac{p-1}{3}\right)-[p=3]p(n+1)(m+n+1)\bigg)\\ &+\left(\frac{d}{3}\right)[p=2]p(n+1)[2\nmid m \ \&\ 2\mid n]\\ \equiv&(n+1)\left(\frac{p^a-d}{3}\right)+(m-n)\left(\frac{d}{3}\right)\\ &-p[p=3]\left(\frac{d}{3}\right)(n+1)(m+n+1)\\ &+p[p=2][2\nmid m\ \&\ 2\mid n]\bigg(\left(\frac{2^{a-1}-d}{3}\right)+\left(\frac{d}{3}\right)\bigg)\pmod{p^2}.\end{aligned}$$

Note that

$$\begin{aligned}&\left(\frac{2^{a-1}-d}{3}\right)+\left(\frac{d}{3}\right)\equiv 1 \pmod 2\\ \iff& d\equiv 0 \pmod 3 \text{ or } d\equiv 2^{a-1}\equiv(-1)^{a-1}\pmod 3\\ \iff& d\not\equiv(-1)^a \pmod 3.\end{aligned}$$

So (1.6) holds.

In view of the above four steps, we have completed the induction proof of Theorem 1.2. □

Proof of Theorem 1.1. By (1.8) in the case $d\in\{0,1\}$, we have

$$\begin{aligned}\frac{1}{C_n}\sum_{k=0}^{p^a-1}C_{p^an+k}=&\frac{1}{C_n}\sum_{k=0}^{p^a-1}\left(\binom{2(p^an+k)}{p^an+k}-\binom{2(p^an+k)}{p^an+k+1}\right)\\ \equiv&(n+1)\left(\frac{p^a}{3}\right)-\left(n+(n+1)\left(\frac{p^a-1}{3}\right)-[p=3]p(n+1)\right)\\ \equiv&1-3(n+1)\left(\frac{p^a-1}{3}\right)\pmod{p^2}.\end{aligned}$$

This proves (1.1). On the other hand, (1.8) in the case $d=0$ yields

$$\frac{1}{C_n}\sum_{k=0}^{p^a-1}(p^an+k+1)C_{p^an+k}\equiv(n+1)\left(\frac{p^a}{3}\right)\pmod{p^2}.$$

So we have

$$\begin{aligned}\frac{1}{C_n}\sum_{k=0}^{p^a-1} kC_{p^a n+k} \equiv&(n+1)\binom{p^a}{3}-\frac{p^a n+1}{C_n}\sum_{k=0}^{p^a-1}C_{p^a n+k}\\ \equiv&(n+1)\binom{p^a}{3}-(p^a n+1)\left(1-3(n+1)\binom{p^a-1}{3}\right)\\ \equiv&(3p^a n+1)(n+1)\binom{p^a-1}{3}-p^a n-1\\ &+(n+1)\left(2\binom{p^a-1}{3}+\binom{p^a}{3}\right) \pmod{p^2}.\end{aligned}$$

Since

$$2\binom{p^a-1}{3}+\binom{p^a}{3}=1-p[p=3],$$

(1.2) follows at once.

Now we assume $p>3$. Applying (1.5) with $d=0$ and $m=2n$, we get

$$\frac{1}{\binom{2n}{n}}\sum_{k=0}^{p^a-1}k\binom{2p^a n+2k}{p^a n+k}\equiv\frac{2}{3}\left(p^a-\binom{p^a}{3}\right)+2p^a n\binom{p^a-1}{3} \pmod{p^2}.$$

It follows that

$$\begin{aligned}&\frac{1}{C_n}\sum_{k=0}^{p^a-1}k(p^a n+k+1)C_{p^a n+k}\\ \equiv&\frac{2}{3}(n+1)\left(p^a-\binom{p^a}{3}\right)+2p^a n(n+1)\binom{p^a-1}{3} \pmod{p^2}.\end{aligned}$$

Combining this with (1.2) we immediately obtain (1.3).

The proof of Theorem 1.1 is now complete. □

5. Related conjectures and results

In this section we pose three conjectures and announce three theorems. To make this paper not too long, we'll include the detailed proofs of the announced theorems in a subsequent article on the basis of this paper, since they involve more complicated computations and some new techniques.

As a supplement to Problem 1.1, we have the following conjecture.

Conjecture 5.1. *If $n\in\mathbb{Z}^+$ is a power of 3, then*

$$\frac{1}{n^2}\sum_{k=0}^{n-1}\binom{2k}{k}\equiv 1 \pmod 3. \tag{5.1}$$

It is interesting to investigate the asymptotic behaviour of $\sum_{k=0}^{n}\binom{2k}{k}$ and $\sum_{k=0}^{n}C_k$ as $n\to+\infty$. Based on Stirling's approximation formula for $n!$ and some numerical computations via Maple, we raise the following conjecture.

Conjecture 5.2. *We have*

$$\lim_{n\to+\infty}\frac{\sum_{k=0}^n\binom{2k}{k}}{4^n/\sqrt{n}}=\lim_{n\to+\infty}\frac{\sum_{k=0}^n C_k}{4^n/(n\sqrt{n})}=\frac{4}{3\sqrt{\pi}}=0.7522527\ldots. \tag{5.2}$$

In view of (1.10), (1.11) and (3.5), for D' in Theorem 3.1 we must have $D'\equiv 0 \pmod p$ if $3\mid p+1$ and $(p+1)/2<d\leqslant p$. Replacing d by $p-d$ we see that the following conjecture holds when $a=1$ and $p\equiv 2\pmod 3$.

Conjecture 5.3. *Let $p>3$ be a prime, and let $a\in\mathbb{Z}^+$. Then, for every $d=0,\ldots,\min\{p,(p^a-3)/2\}$ we have*

$$\begin{aligned}&\sum_{k=0}^{p^a-d-2}(k+d+2)\binom{2k}{k+d}\\ \equiv&\frac{2}{3}(1-3[3\mid p^a+d+1])\left(d+1-\left(\frac{p^a+d+1}{3}\right)\right)\\ &+\left[a=2\ \&\ d\geqslant\frac{p-1}{2}\right]\frac{2}{3}p(1-3[3\mid p+d+1])\\ &\times\left(d+3-\left(\frac{p+d}{3}\right)\right)\pmod{p^{\min\{a,2\}}}.\end{aligned}\tag{5.3}$$

Now we announce some results close to this paper.

Theorem 5.1. *Let $p>3$ be a prime and let $a\in\mathbb{Z}^+$. Then, for every $d=0,\ldots,\min\{p,(p^a-3)/2\}$ we have*

$$\begin{aligned}&\sum_{k=0}^{p^a-d-2}\binom{2k}{k+d}-\left(\frac{p^a+d}{3}\right)\\ \equiv&\left[a=2\ \&\ d\geqslant\frac{p-1}{2}\right]\left(\frac{p+d}{3}\right)p\pmod{p^{\min\{a,2\}}}.\end{aligned}\tag{5.4}$$

Remark 5.1. Comparing (1.10) with (3.4) in the case $a=2$, we obtain Theorem 5.1 in the case $a=1$ and $p\equiv 2\pmod 3$.

Theorem 5.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod{p^2}\ \text{and}\ \sum_{k=1}^{p-1}\frac{\binom{2k}{k+1}}{k}\equiv -3\left(\frac{p+1}{3}\right)-3\pmod{p^2}. \tag{5.5}$$

Remark 5.2. The two congruences in (5.5) were shown to be true mod p by Pan and Sun [PS, (1.6)].

Theorem 5.3. *Let $p > 3$ be a prime, and let $a \in \mathbb{Z}^+$. Let $d \in \{0, \dots, p\}$ and $r \in \{0, \dots, p-1\}$. Then*

$$\begin{aligned}
&(-1)^{r+\varepsilon+1} \sum_{k=0}^{p^a-1} \binom{k-d+r+1}{r}\binom{2k}{k+d} \\
&\equiv \sum_{k\geqslant 0} (-1)^k \binom{r+1}{3k+1+\varepsilon}\binom{2k+2(d-1+\varepsilon)/3}{r} \pmod{p^{\min\{a,2\}}}
\end{aligned} \tag{5.6}$$

where $\varepsilon = (\frac{p^a-d+1}{3})$. Consequently, there is an explicit $\psi_r(p^a \bmod 3)$ only depending on the parameter r, and $p^a \bmod 3$, such that

$$\sum_{k=0}^{p^a-1} k^r C_k \equiv \psi_r(p^a \bmod 3) \pmod{p^{\min\{a,2\}}}.$$

Remark 5.3. For a prime p and $r \in \{0, 1, \dots, p-1\}$, the sum $\sum_{k=0}^{p-1}\binom{k+r}{r}C_k$ modulo p was determined in [PS, Theorem 1.3].

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