A ONE-PARAMETER FAMILY OF DENDRIFORM IDENTITIES

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ABSTRACT. We prove a q-identity in the dendriform dialgebra of colored free quasi-symmetric functions. For $q = 1$, we recover identities due to Ebrahimi-Fard, Manchon, and Patras, in particular the noncommutative Bohnenblust-Spitzer identity.

1. INTRODUCTION

The classical Spitzer and Bohnenblust–Spitzer identities [17, 1, 15] from probability theory can be formulated in terms of certain algebraic structures known as commutative Rota-Baxter algebras. Recently, Ebrahimi-Fard et al. [3] have extended these identities to noncommutative Rota-Baxter algebras. Their results can in fact be formulated in terms of dendriform dialgebras [4], a class of associative algebras whose multiplication split into two operations satisfying certain compatibility relations [10]. Here, we exploit a natural embedding of free dendriform dialgebras into free colored quasisymmetric functions in order to simplify the calculations, and to obtain a q-analog of the main formulas of [3, 4].

Notations – We assume that the reader is familiar with the standard notations of the theory of noncommutative symmetric functions [5, 2].

2. DENDRIFORM ALGEBRAS AND FREE QUASI-SYMMETRIC FUNCTIONS

2.1. **Dendriform algebras.** A dendriform dialgebra [9] is an associative algebra A whose multiplication $\cdot$ can be split into two operations

$$(1) \qquad a \cdot b = a \prec b + a \succ b$$

satisfying

$$(2) \qquad (x \prec y) \prec z = x \prec (y \cdot z) \,, \, (x \succ y) \prec z = x \succ (y \prec z) \,, \, (x \cdot y) \succ z = x \succ (y \succ z) \,.$$

2.2. **Free quasi-symmetric functions.** For example, the algebra of free quasi-symmetric functions **FQSym** [2] (or the Malvenuto-Reutenauer Hopf algebra of permutations [11]) is dendriform. Recall that for a totally ordered alphabet A, **FQSym**(A) is the algebra spanned by the noncommutative polynomials

$$(3) \qquad \mathbf{G}_\sigma(A) := \sum_{\substack{w \in A^n \\ \mathrm{std}(w) = \sigma}} w$$

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where σ is a permutation in the symmetric group $\mathfrak{S}_n$ and $\mathrm{std}(w)$ denotes the standardization of the word w. The multiplication rule is

$$(4) \qquad \mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\gamma \in \alpha * \beta} \mathbf{G}_\gamma,$$

where the convolution $\alpha * \beta$ is [11]

$$(5) \qquad \alpha * \beta = \sum_{\substack{\gamma = uv \\ \mathrm{std}(u) = \alpha;\, \mathrm{std}(v) = \beta}} \gamma \,.$$

The dendriform structure of $\mathbf{FQSym}$ is inherited from that of the free associative algebra over A, which is [12, 13]

$$(6) \qquad u \prec v = \begin{cases} uv & \text{if } \max(v) \leq \max(u) \\ 0 & \text{otherwise,} \end{cases}$$

$$(7) \qquad u \succ v = \begin{cases} uv & \text{if } \max(v) \geq \max(u) \\ 0 & \text{otherwise,} \end{cases}$$

This yields

$$(8) \qquad \mathbf{G}_\alpha \mathbf{G}_\beta = \mathbf{G}_\alpha \prec \mathbf{G}_\beta + \mathbf{G}_\alpha \succ \mathbf{G}_\beta \,,$$

where

$$(9) \qquad \mathbf{G}_\alpha \prec \mathbf{G}_\beta = \sum_{\substack{\gamma = uv \in \alpha * \beta \\ |u| = |\alpha|;\, \max(v) < \max(u)}} \mathbf{G}_\gamma \,,$$

$$(10) \qquad \mathbf{G}_\alpha \succ \mathbf{G}_\beta = \sum_{\substack{\gamma = u.v \in \alpha * \beta \\ |u| = |\alpha|;\, \max(v) \geq \max(u)}} \mathbf{G}_\gamma \,.$$

Then $x = \mathbf{G}_1$ generates a free dendriform dialgebra in $\mathbf{FQSym}$, isomorphic to $\mathbf{PBT}$, the Loday-Ronco algebra of planar binary trees [10].

There is a Hopf embedding $\iota : \mathbf{Sym} \to \mathbf{PBT}$ of noncommutative symmetric functions into $\mathbf{PBT}$ [5, 2, 6], which is given by

$$(11) \qquad \iota(S_n) = (\ldots((x \succ x) \succ x) \ldots) \succ x \quad (n \text{ times}).$$

One of the identites of [3] amounts to an expression of $\iota(\Psi_n)$ in terms of the dendriform operations. It reads

$$(12) \qquad \iota(\Psi_n) = (\ldots((x \triangleright x) \triangleright x) \cdots \triangleright x \quad (n \text{ times}),$$

where $a \triangleright b = a \succ b - b \prec a$. Interestingly enough, applying this identity to the dendriform products of a Rota-Baxter algebra yields a closed form solution to the Bogoliubov recursion in quantum field theory [3]. However, using the embedding in $\mathbf{FQSym}$, the proof of this identity is remarkably simple. Indeed,

$$(13) \qquad \mathbf{G}_\sigma \succ x = \mathbf{G}_{\sigma \cdot (n+1)} \text{ and } x \prec \mathbf{G}_\sigma = \mathbf{G}_{(n+1) \cdot \sigma} \,,$$

so that $\iota(S_n) = \mathbf{G}_{12...n}$. In terms of permutations, this is therefore the standard embedding of **Sym** in **FQSym** as the descent algebra, for which, identifying $\mathbf{G}_\sigma$ with σ,

(14) $$\Psi_n = [[\ldots, [1,2], \ldots, n-1], n].$$

We then clearly have

(15) $$x \triangleright x = \mathbf{G}_{12} - \mathbf{G}_{21} = R_2 - R_{11} = \Psi_2$$

(16) $$\Psi_2 \triangleright x = \mathbf{G}_{123} - \mathbf{G}_{213} - \mathbf{G}_{312} + \mathbf{G}_{321} = R_3 - R_{21} + R_{111} = \Psi_3$$

(17) $$\Psi_{n-1} \triangleright x = \mathbf{G}_{12...n} - \cdots \pm \mathbf{G}_{n...21} = \sum_{k=0}^{n-1} (-1)^k R_{1^k, n-k} = \Psi_n.$$

2.3. Colored free quasi-symmetric functions.

Similarly, the free dendriform dialgebra of r generators $x_1, \ldots, x_r$ can be realized inside the algebra $\mathbf{FQSym}^{(r)}$ of free quasi-symmetric functions of level r [14]. It is a straightforward generalization of **FQSym**, built from an r-colored alphabet

(18) $$\mathbf{A} := A^{(1)} \sqcup \cdots \sqcup A^{(r)}$$

where

(19) $$A^{(i)} := \{a_1^{(i)} < a_2^{(i)} < \ldots\}$$

are copies of A. Writing a colored word

(20) $$\mathbf{w} = a_{i_1}^{(u_1)} \ldots a_{i_n}^{(u_n)} = (w, u)$$

where $w = a_{i_1} \ldots a_{i_n}$ is the *underlying word* and $u = u_1 \ldots u_n$ the *color word*, we define

(21) $$\mathbf{G}_{\sigma, u} := \sum_{\mathrm{std}(w) = \sigma} (w, u) = \mathbf{G}_\sigma \otimes u.$$

Then,

(22) $$\mathbf{G}_{\alpha, u} \mathbf{G}_{\beta, v} = \sum_{\gamma \in \alpha * \beta} \mathbf{G}_{\gamma, u \cdot v} = (\mathbf{G}_\alpha \mathbf{G}_\beta) \otimes uv,$$

and we have again a natural dendriform structure, in which

(23) $$x_1 = \mathbf{G}_{1,1}, \ \ldots, \ x_r = \mathbf{G}_{1,r}$$

generate a free dendriform dialgebra.

3. THE IDENTITIES

3.1. A q-analog of the Ψ_n with colors.

For a color word u and a permutation σ of the same size, we introduce the *biword notation*

(24) $$\binom{u}{\sigma} := \mathbf{G}_{\sigma, u}.$$

With any color word $u = u_1 \ldots u_r$, we associate a nested q-bracketing

$$(25) \qquad \Psi^u := \left[\cdots \left[\begin{matrix} u_1 \, u_2 \\ 1 \, , \, 2 \end{matrix} \right]_q \cdots , \begin{matrix} u_p \\ p \end{matrix} \right]_q$$

where $[x, y]_q = xy - qyx$, the multiplication of nonparenthesized biletters being ordinary concatenation, the result being interpreted as a linear combination of parenthesized biwords. For example,

$$(26) \qquad \Psi^{312} = \left[\left[\begin{matrix} 3 \, 1 \\ 1 \, , \, 2 \end{matrix} \right]_q , \begin{matrix} 2 \\ 3 \end{matrix} \right]_q = \binom{312}{123} - q \binom{132}{213} - q \binom{231}{312} + q^2 \binom{213}{321}.$$

This is an element of the free dendriform dialgebra generated by $x_1, \ldots, x_r$:

$$(27) \qquad \Psi^u = (\ldots (x_{u_1} \triangleright_q x_{u_2}) \triangleright_q x_{u_3} \ldots) \triangleright_q x_{u_p}$$

where $x \triangleright_q y = x \succ y - q y \prec x$. For $q = 1$, $\triangleright_q$ is one of the two pre-Lie products always defined on a dendriform dialgebra.

A word $u = u_1 \ldots u_p$ is called *initially dominated* if $u_1 > u_i$ for all $i > 1$. Each word has a unique increasing factorization into initially dominated words $u^{(i)}$, *i.e.*,

$$(28) \qquad u = u^{(1)} \cdot u^{(2)} \cdots u^{(p)}$$

such that $u_1^{(1)} \le u_1^{(2)} \le \cdots \le u_1^{(p)}$.

With a permutation $\sigma \in \mathfrak{S}_n$ regarded as a word with increasing factorization $\sigma = u^{(1)} \cdots u^{(p)}$, we associate the following element of $\mathbf{FQSym}^{(n)}$:

$$(29) \qquad \mathbf{\Psi}^\sigma = \Psi^{u^{(1)}} \Psi^{u^{(2)}} \cdots \Psi^{u^{(p)}}.$$

For $q = 1$, this reduces to $T_\sigma(x_1, \ldots, x_n)$ in the notation of [3]. Our aim is to compute the equivalent of the sum of all T_σ in our context.

Example 3.1. With $n = 3$, one has

$$(30) \qquad \mathbf{\Psi}^{123} = \Psi^1 \Psi^2 \Psi^3 = \binom{123}{123} + \binom{123}{132} + \binom{123}{213} + \binom{123}{231} + \binom{123}{312} + \binom{123}{321}.$$

$$(31) \qquad \mathbf{\Psi}^{132} = \Psi^1 \Psi^{32} = \binom{132}{123} + \binom{132}{213} + \binom{132}{312} - q \binom{123}{132} - q \binom{123}{231} - q \binom{123}{321}.$$

$$(32) \qquad \mathbf{\Psi}^{213} = \Psi^{21} \Psi^3 = \binom{213}{123} + \binom{213}{132} + \binom{213}{231} - q \binom{123}{213} - q \binom{123}{312} - q \binom{123}{321}.$$

$$(33) \qquad \mathbf{\Psi}^{231} = \Psi^2 \Psi^{31} = \binom{231}{123} + \binom{231}{213} + \binom{231}{312} - q \binom{213}{132} - q \binom{213}{231} - q \binom{213}{321}.$$

$$(34) \qquad \mathbf{\Psi}^{312} = \Psi^{312} = \binom{312}{123} - q \binom{132}{213} - q \binom{231}{312} + q^2 \binom{213}{321}.$$

$$(35) \qquad \mathbf{\Psi}^{321} = \Psi^{321} = \binom{321}{123} - q \binom{231}{213} - q \binom{132}{312} + q^2 \binom{123}{321}.$$

Let

$$(36) \qquad \Sigma_n := \sum_{\sigma \in \mathfrak{S}_n} \mathbf{\Psi}^\sigma.$$

Summing Equations (30) to (35), one can observe that the coefficient of each individual biword is a power of $(-q)$ multiplied by a power of $(1-q)$. We shall see that this is true in general. By putting $q = 1$ into Σ_n, one then recovers the result of [3], namely that the sum of the T_σ is equal to the sum of all colorings of the identity permutation. To prove this fact, we first group permutations into classes having the same coefficient.

3.2. **Grouping the permutations.** If the sizes of the factors of a permutation σ into initially dominated words are $|u^{(1)}| = k_1, \ldots, |u^{(p)}| = k_p$, we set

$$(37) \qquad S(\sigma) := (k_1, \ldots, k_p) = K,$$

and call it the *saillance composition* of σ. The following tables represent the saillance compositions of all permutations of $\mathfrak{S}_3$ and $\mathfrak{S}_4$.

(38)

3	21	12	111
312	213	132	123
321		231	

(39)

4	31	22	211	13	121	112	1111
4123	3124	2143	2134	1423	1324	1243	1234
4132	3214	3142		1432	2314	1342	
4213		3241		2413		2341	
4231				2431			
4312				3412			
4321				3421			

The saillance composition is similar to the *descent composition* $D(\sigma) = (i_1, \ldots, i_s)$ whose parts are the sizes of the maximal increasing factors of σ. The *descent set* $\mathrm{Des}(I)$ of a composition $I = (i_1, \ldots, i_s)$ is

$$(40) \qquad \mathrm{Des}(I) = \{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_{s-1}\}.$$

If one writes $\mathrm{Des}(I) = \{d_1, \ldots\}$, we then define $\mathrm{Des}(I)^-$ as the set $\{d_1 - 1, d_2 - 1, \ldots\}$.

For a color word u of size n and compositions I and J of n, set

$$(41) \qquad R_I^u := \sum_{D(\tau) = I} \binom{u}{\tau},$$

and

$$(42) \qquad R_I^{(J)} := \sum_{S(\sigma) = J} R_I^\sigma = \sum_{\substack{D(\tau) = I \\ S(\sigma) = J}} \binom{\sigma}{\tau}.$$

Example 3.2. Regarding the biwords as bilinear operations, we have

$$(43) \qquad R_{211}^{(13)} := \begin{pmatrix} 1423 + 1432 + 2413 + 2431 + 3412 + 3421 \\ 2134 + 3124 + 4123 \end{pmatrix}.$$

$$(44) \qquad R_{31}^{(121)} := \begin{pmatrix} 1324 + 2314 \\ 2143 + 3142 + 3241 + 4132 + 4231 \end{pmatrix}.$$

We shall prove later that the sum Σ_n is a linear combination of $R_I^{(J)}$. Note that the linear span of the $R_I^{(J)}$ is not a subalgebra of the colored free quasi-symmetric functions. However, we shall refer to it as the space of colored noncommutative symmetric functions.

3.3. **Other bases of colored noncommutative symmetric functions.** The expression of Σ_n is simpler in a different basis. Let us define the colored elementary basis $\Lambda_I^{(J)}$ by

$$(45) \qquad \Lambda_I^{(J)} := \sum_{I' \geq \widetilde{I}} R_{I'}^{(J)},$$

where the sum runs over the compositions I' finer than the conjugate $\widetilde{\overline{I}}$ of the mirror image $\overline{I}$ of I. Note that this definition is independent of the color J.

For example,

$$(46) \qquad \Lambda_{32}^{(41)} = R_{1121}^{(41)} + R_{11111}^{(41)}.$$

3.4. **The main result.** We shall need a simple statistic on pairs of compositions. First recall the two basic operations on compositions $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$:

$$(47) \qquad I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s), \quad \text{and} \quad I \triangleright J = (i_1, \ldots, i_r + j_1, \ldots, j_s).$$

Now, let us define the I-decomposition of a composition J as the unique sequence of compositions

$$(48) \qquad J \downarrow I = (J^{(1)}, \ldots, J^{(r)})$$

such that $J^{(k)} \vDash i_k$ for all k and

$$(49) \qquad J = J^{(1)} \circ_1 J^{(2)} \circ_2 \cdots \circ_{r-1} J^{(r)},$$

where each $\circ_i$ is either $\cdot$ or $\triangleright$.

Let I and J be two compositions of n, and let $(J^{(1)}, \ldots, J^{(r)}) = J \downarrow I$. Write $J^{(k)} = (j_1^{(k)}, \ldots, j_{s_k}^{(k)})$. Then the statistic $D(I, J)$ is

$$(50) \qquad D(I, J) = n - \sum_{k=1}^{r} j_{s_k}^{(k)}.$$

For example, with $I = (6, 2, 2, 4, 1, 4)$ and $J = (3, 2, 4, 3, 2, 5)$, one has $J \downarrow I = ((3, 2, 1), (2), (1, 1), (2, 2), (1), (4))$. Hence $D(I, J) = 19 - 1 - 2 - 1 - 2 - 1 - 4 = 8$. The complete examples for sizes 3 and 4 are given in Section 5.1.

We then have the following simple expression:

Theorem 3.3.

$$(51) \qquad \Sigma_n = \sum_{I,J} (-1)^{l(I)-1} q^{D(I,J)} \Lambda_I^{(J)}.$$

The proof of this theorem relies on Theorem 4.5 and will be given in Section 4.

One easily derives from this result the expansion of Σ_n in terms of the ribbon basis. Note that one can work in **Sym** since the colors J do not interfere with the change of basis between Λ and R. We have

Corollary 3.4. *The sum of all Ψ^σ in $\mathbf{FQSym}^{(n)}$ is*

$$(52) \qquad \sum_{\sigma \in \mathfrak{S}_n} \Psi^\sigma = \sum_{I,J \vDash n} c_{IJ}(q) R_I^{(J)}$$

where
$$(53)$$
$$c_{IJ}(q) = \begin{cases} 0 & \text{if } \mathrm{Des}(I) \backslash \mathrm{Des}(I)^- \not\subset \mathrm{Des}(J), \\ (-q)^{|\mathrm{Des}(I) \backslash \mathrm{Des}(J)|} (1-q)^{|\mathrm{Des}(I) \cap \mathrm{Des}(J)|} & otherwise. \end{cases}$$

Corollary 3.5. *For $q = 1$, we recover the noncommutative Bohnenblust-Spitzer identity of [3]:*

$$(54) \qquad \sum_{\sigma \in \mathfrak{S}_n} T_\sigma(u_1, \ldots, u_n) = \sum_{\sigma \in \mathfrak{S}_n} \Psi^\sigma|_{q=1} = \sum_{J \vDash n} R_n^{(J)} = \sum_{\sigma \in \mathfrak{S}_n} \begin{pmatrix} 12 \ldots n \\ \sigma \end{pmatrix}.$$

Indeed, $c_{IJ}(1) \neq 0$ iff $\mathrm{Des}(I) \cap \mathrm{Des}(J) = \emptyset$ and $\mathrm{Des}(I) \backslash \mathrm{Des}(I)^- \subset \mathrm{Des}(J)$, that is, $\mathrm{Des}(I) \subset \mathrm{Des}(I)^-$, so that $I = (n)$ (no descents). So the sum simplifies to the sum of all colorings of the identity permutation.

4. A REFINEMENT OF THEOREM 3.3

The proof of Theorem 3.3 relies on an induction similiar to Newton's recursion for symmetric functions. We shall use this recursion to state a refinement of the theorem to prove. Let

$$(55) \qquad P_I := \sum_{S(\gamma)=I} \Psi^\gamma.$$

Lemma 4.1. *Let $I = (n)$ be a one-part composition. Then*

$$(56) \qquad P_I = P_{(n)} = \sum_{k=1}^{n} (-q)^{n-k} \sum_{J \vDash n-k} R_{1^{n-k},k}^{(J,k)}.$$

$$(57) \qquad P_{(n)} = \sum_{I,J} (-1)^{n-l(I)} q^{D(I,J)} \Lambda_I^{(J)},$$

where the sum is taken over all compositions $J = (j_1, \ldots, j_k)$ and all compositions I belonging to the interval of the composition lattice for the refinement order whose finest element is $(n - j_k + 1, 1^{j_k-1})$ and whose coarsest element is (n).

Proof – Formula (56) is immediate by definition of $P_{(n)}$. The second formula follows from the first one by a simple computation in the algebra of noncommutative symmetric functions. ∎

For an interval $[H, K]$ of the boolean lattice of compositions of n, let

$$\tag{58} \Lambda_I^{[H,K]} := \sum_{J \in [H,K]} \Lambda_I^{(J)}.$$

Using this notation, Equation (57) can be rewritten as

$$\tag{59} P_{(n)} = \sum_{I=(i_1,\ldots,i_p) \vDash n} (-1)^{n-l(I)} \sum_{k=n-i_1+1}^n q^{D(I,(n-k,k))} \Lambda_I^{[(n-k,k),(1^{n-k},k)]}.$$

Note 4.2. The characterization of the $\Lambda_I^{(J)}$ appearing in P_n is simple: it consists in the compositions I and J of n such that the sum of the first part of I and the last part of J is strictly greater than n.

We now need a very simple lemma on permutations.

Lemma 4.3. *Let $\sigma \in \mathfrak{S}_n$. Then $S(\sigma) = (l_1, \ldots, l_p)$ iff*

$$\tag{60} S(\sigma_1, \ldots, \sigma_{l_1+\cdots+l_{p-1}}) = (l_1, \ldots, l_{p-1}) \text{ and } \sigma_{l_1+\cdots+l_{p-1}+1} = n.$$

This lemma implies a recursion for P_L. For any totally ordered color alphabet C of size n, denote by $P_I[C]$ the result of replacing each color i by c_i in P_L. Then, by definition of P_L,

$$\tag{61} P_{(l_1,\ldots,l_p)} = \sum_{|C'|=n-l_p,\ |C''|=l_p,\ n \in C''} P_{(l_1,\ldots,l_{p-1})}[C']P_{(l_p)}[C''].$$

This can be rewritten in the more suggestive form

$$\tag{62} P_{(l_1,\ldots,l_p)} = P_{(l_1,\ldots,l_{p-1})} \gg P_{(l_p)}.$$

where the dendriform products $\ll$ and $\gg$ are defined in the biword notation of (24) by

$$\tag{63} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ll \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} := \begin{pmatrix} \alpha \prec \alpha' \\ \beta * \beta' \end{pmatrix}$$

$$\tag{64} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \gg \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} := \begin{pmatrix} \alpha \succ \alpha' \\ \beta * \beta' \end{pmatrix}$$

Thanks to (59) and (62), the evaluation of P_L reduces to the following:

Lemma 4.4. *Let I and $J = (j_1, \ldots, j_p)$ be two compositions of the same size and let I' be a composition of n. Then*

$$\tag{65} \Lambda_I^{(j_1,\ldots,j_p)} \gg \Lambda_{I'}^{[(n-k,k),(1^{n-k},k)]} = \Lambda_{I \cdot I'}^{[(j_1,\ldots,j_{p-1}+j_p,1^{n-k},k),(j_1,\ldots,j_{p-1},j_p,1^{n-k},k)]}.$$

Proof – From the characterization of the right dendriform product (64), we just have to evaluate in **FQSym**

$$(66) \qquad \left(\sum_{S(\sigma)=J; \sigma \in \mathfrak{S}_n} \mathbf{G}_\sigma \right) \succ \left(\sum_{\tau_{m-k+1}=m; \tau \in \mathfrak{S}_m} \mathbf{G}_\tau \right)$$

that is, thanks to Lemma 4.3,

$$(67) \qquad \sum_{\substack{\rho=u.v; S(u)=S(\sigma) \\ \rho_{n+m-k+1}=n+m}} \mathbf{G}_\rho = \sum_{\substack{S(\rho)=(j_1,\ldots,j_{p-1},K,k) \\ |K|=j_p+m-k; K_1 \geq j_p}} \mathbf{G}_\rho.$$

∎

We can now state our main result:

Theorem 4.5. *Let $L = (l_1, \ldots, l_p)$ be a composition of n. Then*

$$(68) \qquad P_L = \sum_{I,J} (-1)^{n-l(I)} q^{D(I,J)} \Lambda_I^{(J)},$$

where the sum is taken over all pairs of compositions (I, J) such that
- *I is finer than L,*
- *For $k = 1, \ldots, p-1$, $\mathrm{Des}(J) \cap [d_k, d_{k+1} - 1] \neq \emptyset$, where $d_k = l_1 + \cdots + l_k$,*
- *If $I \downarrow L = (I^{(1)}, \ldots, I^{(p)})$ and $J \downarrow L = (J^{(1)}, \ldots, J^{(p)})$, then, for all $k \in [1, p]$, the sum of the first part of $I^{(k)}$ and the last part of $J^{(k)}$ is strictly greater than l_k.*

Proof – First, Equation (59) and Lemma 4.4 imply that P_L is a linear combination of $\Lambda_I^{(J)}$. It is also clear that the theorem holds if $L = (n)$. The result now follows by induction, since it is obviously a multiplicity-free expansion thanks to Lemma 4.4 and since the characterization is the expected one thanks to Note 4.2.

The only point that remains to be proved is that the coefficient $(-1)^{n-l(I)} q^{D(I,J)}$ is what is expected but this follows directly from the fact that, following the notations of Lemma 4.4,

$$(69) \qquad D(I, J) + n - k = D(I \cdot I', K),$$

for all I' such that $I'_1 > n - k$ and for all K in the interval

$$(70) \qquad [(j_1, \ldots, j_{p-1}, j_p + 1^{n-k}, k), (j_1, \ldots, j_{p-1}, j_p, 1^{n-k}, k)].$$

∎

Proof – [of Theorem 3.3] Thanks to Theorem 4.5, there only remains to prove that each pair (I, J) appears in the expansion of exactly one P_L. Indeed, starting from I and J, one glues a part of I to the previous one if there is no descent of J in between those two parts. This gives a composition L such that $\Lambda_I^{(J)}$ appears in P_L since it satisfies all three conditions of Theorem 4.5: the third condition is the only one that remains to be checked. It is satisfied with $L = I$ and this property remains true after each gluing, by definition of the gluing. Any composition strictly finer than L and

coarser than I does not satisfy the second condition, any other composition coarser than I does not satisfy the third condition. All other compositions do not satisfy the first condition. ∎

5. EXAMPLES

5.1. Expressions of Σ_n in terms of Λ and R.

We have

$$(71) \qquad \Sigma_2 = -q\Lambda_2^{(11)} - \Lambda_2^{(2)} + \Lambda_{11}^{(11)} + \Lambda_{11}^{(2)}.$$

Arranging the coefficients into a matrix, whose row I and column J gives the value of $\Lambda_I^{(J)}$ in Σ_n, we have

$$(72) \qquad M_2 = \begin{array}{c} \\ 2 \\ 11 \end{array} \begin{array}{cc} 2 & 11 \\ \begin{pmatrix} -1 & -q \\ 1 & 1 \end{pmatrix} \end{array}$$

To save space and for better readability, we shall rather give the matrices of the exponent $D(I, J)$ itself, where 0 is represented by a dot.

$$(73) \qquad D_2 = \begin{array}{c} 2 \\ 11 \end{array} \begin{array}{cc} 2 & 11 \\ \begin{pmatrix} . & 1 \\ . & . \end{pmatrix} \end{array} \qquad D_3 = \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} \begin{array}{cccc} 3 & 21 & 12 & 111 \\ \begin{pmatrix} . & 2 & 1 & 2 \\ . & . & 1 & 1 \\ . & 1 & . & 1 \\ . & . & . & . \end{pmatrix} \end{array}$$

$$(74) \qquad D_4 = \begin{array}{c} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} \begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \begin{pmatrix} . & 3 & 2 & 3 & 1 & 3 & 2 & 3 \\ . & . & 2 & 2 & 1 & 1 & 2 & 2 \\ . & 1 & . & 1 & 1 & 2 & 1 & 2 \\ . & . & . & . & 1 & 1 & 1 & 1 \\ . & 2 & 1 & 2 & . & 2 & 1 & 2 \\ . & . & 1 & 1 & . & . & 1 & 1 \\ . & 1 & . & 1 & . & 1 & . & 1 \\ . & . & . & . & . & . & . & . \end{pmatrix} \end{array}$$

Note that all columns of M_n become equal when $q = 1$.

Here are now the matrices of Σ_2, Σ_3, and Σ_4 in the ribbon basis $R_I^{(J)}$.

$$(75) \qquad M_2' = \begin{array}{c} 2 \\ 11 \end{array} \begin{array}{cc} 2 & 11 \\ \begin{pmatrix} 1 & 1 \\ . & 1-q \end{pmatrix} \end{array}$$

$$(76) \qquad M'_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array} \begin{array}{cccc} 3 & 21 & 12 & 111 \\ \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ . & 1-q & . & 1-q \\ . & . & 1-q & 1-q \\ . & -q(1-q) & . & (1-q)^2 \end{array}\right) \end{array}$$

$$(77)$$

$$M'_4 = \begin{array}{c} 4 \quad 31 \quad\quad 22 \quad\quad 211 \quad\quad 13 \quad\quad 121 \quad\quad 112 \quad\quad 1111 \\ \left(\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1-q & . & 1-q & . & 1-q & . & 1-q \\ . & . & 1-q & 1-q & . & . & 1-q & 1-q \\ . & -q(1-q) & . & . & 1 & 1 & 1 & 1 \\ . & . & 1 & 2 & . & 2 & 1 & 2 \\ . & . & 1 & 1 & . & . & 1 & 1 \\ . & . & . & 1 & . & 1 & . & 1 \\ . & q^2(1-q) & . & . & . & . & . & . \end{array}\right) \end{array}$$

5.2. **Expressions of P_L in terms of Λ.** The entry (I, J) in the following matrices is the composition L such that $\Lambda_I^{(J)}$ appears in P_L.

$$(78) \qquad N_2 = \begin{array}{c} \\ 2 \\ 11 \end{array} \begin{array}{cc} 2 & 11 \\ \left(\begin{array}{cc} 2 & 2 \\ 2 & 11 \end{array}\right) \end{array} \qquad\qquad N_3 = \begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array} \begin{array}{cccc} 3 & 21 & 12 & 111 \\ \left(\begin{array}{cccc} 3 & 3 & 3 & 3 \\ 3 & 21 & 3 & 21 \\ 3 & 12 & 12 & 12 \\ 3 & 21 & 12 & 111 \end{array}\right) \end{array}$$

$$(79) \qquad N_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} \begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \\ \left(\begin{array}{cccccccc} 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 31 & 4 & 31 & 4 & 31 & 4 & 31 \\ 4 & 22 & 22 & 22 & 4 & 22 & 22 & 22 \\ 4 & 31 & 22 & 211 & 4 & 31 & 22 & 211 \\ 4 & 13 & 13 & 13 & 13 & 13 & 13 & 13 \\ 4 & 31 & 13 & 121 & 13 & 121 & 13 & 121 \\ 4 & 22 & 22 & 22 & 13 & 112 & 112 & 112 \\ 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array}\right) \end{array}$$

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