A characterization of all equilateral triangles in $\mathbb{Z}^3$

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Abstract

This paper is a continuation of the work started in [1] and [2]. We extend one of the theorems that gave a way to construct equilateral triangles whose vertices have integer coordinates to the general situation. An approximate extrapolation formula for the sequence $ET(n)$ of all equilateral triangles with vertices in $\{0, 1, 2, ..., n\}^3$ (A 102698) is given and the asymptotic behavior of this sequence is analyzed.

1 Introduction

It turns out that equilateral triangles in $\mathbb{Z}^3$ exist and there are unexpectedly many. Just to give an example, if we restrict our attention only to the cube $\{0, 1, 2, ..., 1105\}^3$ we have 2,474,524,936,846,512 of them. In [1] it was shown the first part of the following theorem and the second part about the converse was only proven under the hypothesis that $gcd(d, a) = 1$ or $gcd(d, b) = 1$ or $gcd(d, c) = 1$. The main result of this paper is to show that one can drop this condition.

Theorem 1.1. *Let a, b, c, d be odd positive integers such that $a^2 + b^2 + c^2 = 3d^2$ and $gcd(a, b, c) = 1$. Then the points $P(u, v, w)$ and $Q(x, y, z)$ whose coordinates given by*

$$\begin{cases} u = m_u m - n_u n, \\ v = m_v m - n_v n, \\ w = m_w m - n_w n, \end{cases} \qquad \text{and} \qquad \begin{cases} x = m_x m - n_x n, \\ y = m_y m - n_y n, \\ z = m_z m - n_z n, \end{cases} \tag{1}$$

with

$$
\begin{cases} m_x = -\frac{1}{2}[db(3r+s)+ac(r-s)]/q, & n_x = -(rac+dbs)/q \\ m_y = \frac{1}{2}[da(3r+s)-bc(r-s)]/q, & n_y = (das-bcr)/q \\ m_z = (r-s)/2, & n_z = r \end{cases}
$$
$$
\text{and} \qquad (2)
$$
$$
\begin{cases} m_u = -(rac+dbs)/q, & n_u = -\frac{1}{2}[db(s-3r)+ac(r+s)]/q \\ m_v = (das-rbc)/q, & n_v = \frac{1}{2}[da(s-3r)-bc(r+s)]/q \\ m_w = r, & n_w = (r+s)/2 \end{cases}
$$

where $q = a^2 + b^2$ *and* (r, s) *is a suitable solution of* $2q = s^2 + 3r^2$ *which makes all the numbers in (2) integers, together with the origin (O(0,0,0)) forms an equilateral triangle in* $\mathbb{Z}^3$ *contained in the plane*

$$\mathcal{P}_{a,b,c} := \{(\alpha, \beta, \gamma) | a\alpha + b\beta + c\gamma = 0\}$$

and having sides-lengths equal to $d\sqrt{2(m^2 - mn + n^2)}$.

Conversely, there exist a choice of the integers r *and* s *such that given an arbitrary equilateral triangle contained in the plane* $\mathcal{P}_{a,b,c}$ *with one of the vertices the origin, the other two vertices are of the form (2) for some integer values* m *and* n.

The condition

$$min(gcd(d, a), gcd(d, b), gcd(d, c)) > 1 \tag{3}$$

defines a, b, c, and d as a *degenerate solution* of the Diophantine equation

$$a^2 + b^2 + c^2 = 3d^2. \tag{4}$$

The first d that admits such degenerate decompositions in a, b, and c is $d = 1105$. There exactly seven of them:

$$(a, b, c) \in \{(731, 1183, 1315), (475, 1309, 1313), (299, 493, 1825),$$

$$(1027, 1139, 1145), (187, 415, 1859), (265, 533, 1819), (493, 1001, 1555)\}$$

Our study started with the intent of computing the sequence $ET(n)$ of all equilateral triangles with vertices in $\{0, 1, 2, ..., n\}$. These values were calculated by the first author for $n \leq 1105$ using a improved version of the code published in [2] and translated into Mathematica (see A102698 in The On-Line Encyclopedia of Integer Sequences).

One of the parametrizations like in (2) in the case $a = 731$, $b = 1183$ and $c = 1315$ is shown below:

$$
\begin{aligned}
&P = (901m - 1428n, -1157m + 221n, 540m + 595n), \\
&Q = (-527m - 901n, -936m + 1157n, 1135m - 540n), m, n \in \mathbb{Z}.
\end{aligned}
$$

For other values of r and s we get formally different parametrizations but they are all equivalent in the sense that they can be obtained from the above by changing the variables in the expected ways:

$$\begin{gathered}(m,n)\to(-m,-n),\ (m,n)\to(m-n,m),\ (m,n)\to(n-m,n),\\ (m,n)\to(m-n,-n)\ \textit{and}\ (m,n)\to(n-m,-m).\end{gathered} \tag{5}$$

These changes of variables leave invariant the quadratic form involved in the side-length of the triangle the various points given by (1) define the vertices of a tessellation of the plane $\mathcal{P}_{a,b,c}$ with equilateral triangles.

The above example is saying and it is supporting evidence that the parametrization (2) is essentially the only one and the Theorem 1.1 can be extended to cover the degenerate solutions case also.

2 The arguments of the Theorem 1.1.

Proof. The first part of the Theorem 1.1 follows from [1]. For the second part we are going to reconstitute some of the details started in the proof of the particular case: $gcd(d,c)=1$.

Let us start with a triangle in $\mathcal{P}_{a,b,c}$ say $\triangle OPQ$ with $P(u_0,v_0,w_0)$ and $Q(x_0,y_0,z_0)$.

By Theorem 4 in [1]we have

$$\begin{cases} x_0=\dfrac{u_0}{2}\pm\dfrac{cv_0-bw_0}{2d}\\ y_0=\dfrac{v_0}{2}\pm\dfrac{aw_0-cu_0}{2d}\\ z_0=\dfrac{w_0}{2}\pm\dfrac{bu_0-av_0}{2d}, \end{cases} \tag{6}$$

for some choice of the signs. This means that d must divide cv_0-bw_0, aw_0-cu_0 and bu_0-aw_0.

So, we need to look at the the following system of linear equations in m and n:

$$\begin{cases} u_0=m_um-n_un,\\ v_0=m_vm-n_vn,\\ w_0=m_wm-n_wn. \end{cases} \tag{7}$$

By the Kronecker-Capelli theorem this linear system of equations has a solution if and only the rank of the main matrix is the same as the rank of the extended matrix. Since $m_vn_w-m_wn_v=ad$, $m_wn_u-m_un_w=bd$ and $m_un_v-m_vn_u=cd$ (one can check these calculations based on the definitions in (2)) then the rank of the main matrix is two and the rank of the extended matrix is also two because its determinant is $u_0ad+v_0bd+w_0dc=0$. This implies that (7) has a unique real (in fact rational) solution in m and n.

We want to show that this solution is in fact an integer solution. Solving for n from each pair of equations in (7) we get

$$n=\frac{v_0m_w-w_0m_v}{ad}=\frac{w_0m_u-u_0m_w}{bd}=\frac{u_0m_v-v_0m_u}{cd}. \tag{8}$$

Because $gcd(a, b, c) = 1$, there exist integers a', b', c' such that $aa' + bb' + cc' = 1$. Then one can see that

$$n = \frac{a'(v_0 m_w - w_0 m_v) + b'(w_0 m_u - u_0 m_w) + c'(u_0 m_v - v_0 m_u)}{d} \tag{9}$$

Next, from (9) we observe that in order for n to be an integer it is enough to prove that d divides $v_0 m_w - w_0 m_v$, $w_0 m_u - u_0 m_w$ and $u_0 m_v - v_0 m_u$. Hence we calculate for example $v_0 m_w - w_0 m_v$ in more detail:

$$v_0 m_w - w_0 m_v = v_0 r - \frac{das - rbc}{q} w_0 = \frac{v_0 q r - (das - rbc) w_0}{q} =$$

$$\frac{-dasw_0 + v_0(3d^2 - c^2)r + rbcw_0}{q} = \frac{c(bw_0 - v_0 c)r + 3rv_0 d^2 - dasw_0}{q}.$$

From (6) we see that $bw_0 - v_0 c = \pm d(2x_0 - u_0)$. Hence,

$$v_0 m_w - w_0 m_v = \frac{d[\pm c(2x_0 - u_0)r + 3rv_0 d - asw_0]}{q}. \tag{10}$$

Assuming that $gcd(d, c) = \zeta$ we can write $d = \zeta d_1$ and $c = \zeta c_1$ with $gcd(d_1, c_1) = 1$. Also we see that ζ^2 must divide $q = 3d^2 - c^2$ so let us write $q = \zeta^2 q_1$. If p is a prime dividing ζ, it must be an odd prime and if it is of the form $4k + 3$ it must divide a and b which is contradicting the assumption that $gcd(a, b, c) = 1$. Therefore it must be a prime of the form $4k + 1$. Hence q_1 is still a sum of two squares.

In the proof of Theorem 13 in [1] one can choose r and s with the extra condition that r and s are divisible by ζ. Indeed, Lemma 14 in [1] is applied to $(ac)^2 + 3(db)^2 = \zeta^2[(ac_1)^2 + 3(d_1 b)^2]$ and to $q = \zeta^2 q_1$ but instead one can apply it to $(ac_1)^2 + 3(d_1 b)^2$ and to q_1 giving, let us, say r_1 and s_1. Then we put $r = \zeta r_1$ and $s = \zeta s_1$ and then all the arguments there go as stated. From (10) we see that

$$v_0 m_w - w_0 m_v =$$

$$\frac{d[\pm c(2x_0 - u_0)r + 3rv_0 d - asw_0]}{q} = \frac{\zeta d_1[\pm \zeta c_1(2x_0 - u_0)\zeta r_1 + 3\zeta r_1 v_0 \zeta d_1 - a\zeta s_1 w_0]}{\zeta^2 q_1} =$$

$$\frac{\zeta^2 d_1[\pm c_1(2x_0 - u_0) r_1 \zeta + 3 r_1 v_0 d_1 \zeta - a s_1 w_0]}{\zeta^2 q_1} = \frac{d_1 \xi}{q_1}, \tag{11}$$

where $\xi = \pm c_1(2x_0 - u_0) r_1 \zeta + 3 r_1 v_0 d_1 \zeta - a s_1 w_0$. This implies that d_1 must divide $v_0 m_w - w_0 m_v$. In a similar way we can show that d_1 divides $w_0 m_u - u_0 m_w$ and $u_0 m_v - v_0 m_u$ and so from (9) we see that n is a rational with denominator having ζ as a multiple. Similar arguments will give us that m is of the same form.

The triangle having the coordinates as in (1) with these m and n will give the length

$$l^2 = 2d^2(m^2 - mn + n^2).$$

But this whole construction can be repeated for a or b instead of c and we obtain that

$$l^2 = 2d^2(m_1^2 - m_1n_1 + n_1^2),$$

for some rational numbers m_1, n_1 with denominator having multiple $\eta = gcd(d, b)$ for example. Since $gcd(\zeta, \eta) = 1$ we see that $m^2 - mn + n^2 = m_1^2 - m_1n_1 + n_1^2$ must be an integer. Therefore

$$l^2 = 2d^2(\alpha^2 - \alpha\beta + \beta^2), \alpha, \beta \in \mathbb{Z}. \tag{12}$$

On the other hand if $u_0^2 + v_0^2 + w_0^2 = l^2$ and ζ divides d we can see that

$$u_0^2 + v_0^2 + w_0^2 \equiv 0 \ (mod \ \zeta^2). \tag{13}$$

We also know that $au_0 + bv_0 + cw_0 = 0$ and hence $a^2u_0^2 = b^2v_0^2 + 2bcv_0w_0 + c^2w_0^2$. This implies $a^2u_0^2 \equiv b^2v_0^2 + 2bcv_0w_0 \pmod{\zeta^2}$. But $a^2 + b^2 = 3d^2 - c^2 \equiv 0 \pmod{\zeta^2}$ too and then $a^2(u_0^2 + v_0^2) \equiv 2bcv_0w_0 \pmod{\zeta^2}$ which combined with (13) gives

$$a^2w_0^2 + 2bcv_0w_0 \equiv 0 \ (mod \ \zeta^2). \tag{14}$$

Because we must have $gcd(a, \zeta) = 1$, (14) implies that ζ divides w_0. Indeed, if p is a prime that has exponent one in the decomposition of ζ then (14) gives in particular $w_0^2 \equiv 0 \pmod{p}$ and so p must divide w_0. If the exponent of p in ζ is two, then (14) in particular implies that w_0 is divisible by p but then $aw_0^2 \equiv 0 \pmod{p^3}$ which implies p^2 divides w_0. Inductively if the exponent of p in ζ is k then this must be true for w_0 too. Hence we must have ζ a divisor of w_0 so $w_0 = \zeta w_0'$.

Now we can go back to (11) and observe that we can rewrite it as

$$v_0r - w_0m_v = \frac{\zeta d_1\xi'}{q_1}, \tag{15}$$

where $\xi' = \pm c_1(2x_0 - u_0)r_1 + 3r_1v_0d_1 - as_1w_0'$. Now we observe that the left hand side of (15) is a multiple of ζ since r and w_0 are. After simplification with ζ this will show that q_1 must divide in fact ξ' and so d divides $v_0m_w - w_0m_v$. Similar arguments can be used to deal with the other cases. Hence n must be an integer and so should be m. Changing the variables as in (5), one of the corresponding triangles given by (1) is going to match with the triangle OPQ. □

Remark: One can see that the condition on r and s to be divisible by ζ is implied by asking only that the numbers in (2) be integers. Indeed, given such a choice of r and s, they will define by (1), in which $m = 1$ and $n = 0$, an equilateral triangle with integer coordinates. According to the proof of Theorem 1.1, $w = r$ and $z = (r - s)/2$ must be divisible by ζ. This implies that r and s must be multiples of ζ. Therefore any parametrization as in the Theorem 1.1 is unique up to the transformations (5).

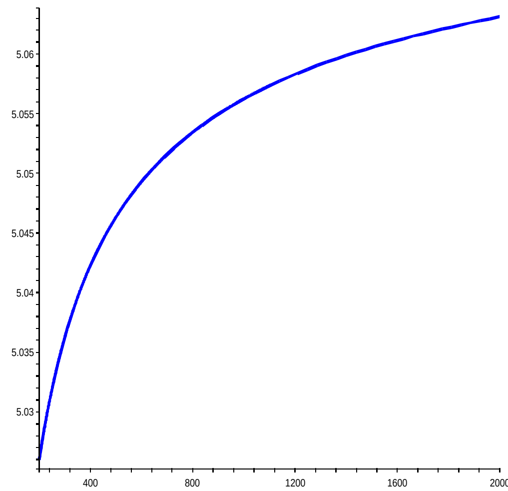

Figure 1: The graph of g extrapolating f over the interval $[1, 2000]$

3 Behavior of the sequence $ET(n)$

The calculations of the $ET(n)$ for all $n \leq 1105$ gave us enough data to be able to extrapolate the graph of $n \xrightarrow{f} \frac{\ln(ET(n))}{\ln(n+1)}$ as shown in Figure 1. The function we used to extrapolate is of the form $g(x) = a + \frac{b}{\sqrt{x+c}}$ having clearly a as limit at infinity. Then we made it agree with f on three points. That gave us $a := 5.079282921$, $b := -0.7091588389$, and $c := -0.8403164433$. Numerically then we discovered that the average of $|f(k) - g(k)|$ over all values of $k = 1, ..., 1105$ is approximately 0.002638971108.

One conjecture that we would like to make here is that $f(n)$ is a strictly increasing sequence and then as result it is convergent to a constant $C \approx 5.08$.

The graph of the "derivative" of $ET(n)$ (Figure 2) is almost like the graph of $h(x) = C(x+1)^k$ where $k := 4.151431798$ and $C := 2.660972140$. The third difference of $ET(n)$ as represented in Figure 3 seems to bring a chaotic flavor to this sequence and it is saying in a certain sense that no simple formula for $ET(n)$ can exist.

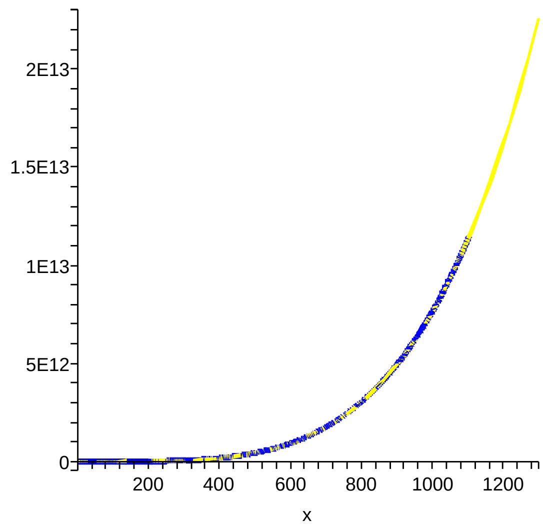

Figure 2: The graph of $n \to ET(n+1) - ET(n)$

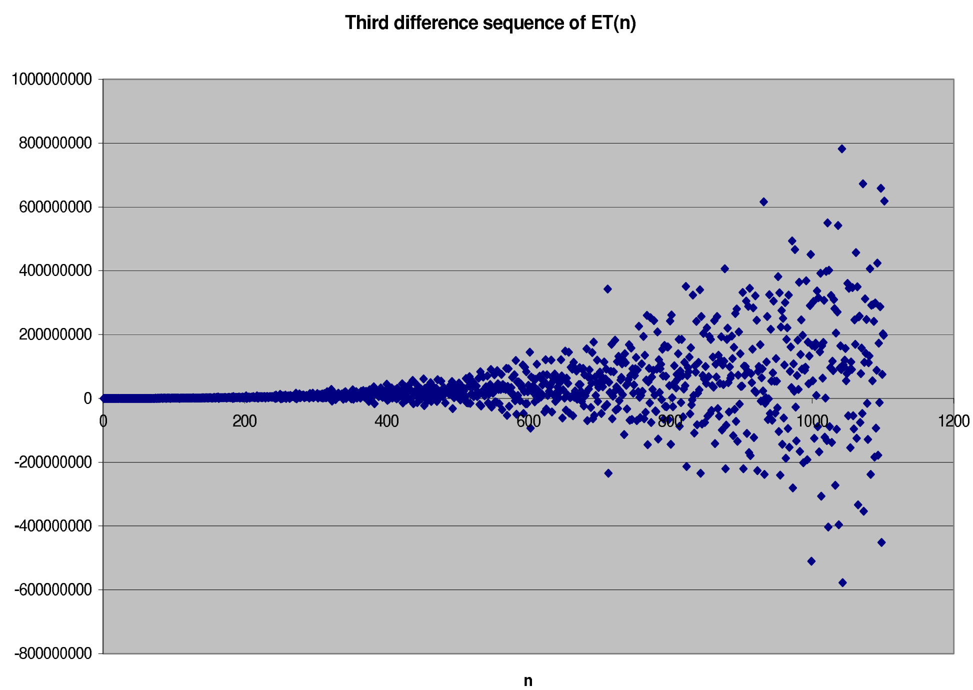

Figure 3: $\Delta^3 ET(n)$

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(Concerned with sequence A102698.)
