Fair Triangulations

Roland Bacher

October 5, 2007

Abstract: We describe the statistics of checkerboard triangulations obtained by colouring black every other triangle in triangulations of convex polygons.[1]

1 Introduction

Triangulations of strictly convex polygons are classical combinatorial objects studied almost ever since the sad moment when mankind made his biggest mistake and left his favourite tree. Out of remorse and longing for the lost paradise, mankind (in fact, only a small part of it, those successful with girls had more urgent and important matters to attend to) started the study of trees and discovered that plane rooted trees and triangulated convex polygons share many combinatorial properties. This caused some fellows to investigate triangles thus giving birth to geometry and eventually analysis and real numbers. Other chaps, tempted by enumeration, discovered the natural numbers which led later to arithmetics and algebra. They tried successfully to enumerate plane rooted trees (or, equivalently, triangulations of convex polygons) by introducing the now famous sequence of the ubiquituous Catalan numbers, namely

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, \dots ,$$

see sequence A108 in [2]. Exercice 6.19 in [5] lists 66 sequences of sets enumerated by them. [5] contains also some historical information which is obviously rather less reliable than the translation of the Holy Scriptures presented above.

The generating series of the Catalan sequence is the algebraic function

$$g(x) = \frac{1 - \sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1} \ .$$

It satisfies the equation $g(x) = 1 + xg(x)^2$.

[1]Math. Class: 05A15, Keywords: Catalan number, checkerboard triangulation, dissections of polygons

This paper is devoted to the study of triangulations which are coloured like a chessboard: every triangle is either black or white and edge-adjacent triangles have different colours. Such a colouring is unique, up to permutation of the colours. Call a triangulation τ *fair* if its two colourings involve the same number of black and white triangles.

Theorem 1.1. *Every strictly convex polygon with* $2n+2$ *vertices has exactly* $\binom{2n}{n}^2\frac{1}{n+1}$ *fair triangulations.*

The choice of a marked edge e_* on the boundary ∂P of a polygon P having at least 3 vertices selects in any triangulation of P a marked triangle Δ_*, which contains the marked edge e_*. The imposition of the colour black for the marked triangle Δ_* removes the colour-ambiguity. We get in this way a bijection between triangulations and "checkerboard triangulations" of convex polygons $P \supset e_*$ having a marked edge. Section 2 contains our main result, Theorem 2.5, which describes closed formulae for the the number of such checkerboard triangulations involving n_b black and n_w white triangles. Theorem 1.1 corresponds to the special case $n_b = n_w$. The proof of Theorem 2.5 uses probably well-known techniques usable for similar problems: algebraic generating series which can be expressed as finite sums of a kind of hypergeometric series in several variables.

We study also $d-$dissections, a generalisation of triangulations corresponding to the special case $d = 2$. (The choice of the notation can be motivated by higher dimensional analogues.) A $d-$dissection of a strictly convex polygon P is a decomposition of P into polygons with $(d+1)-$vertices, all contained in the vertex-set of P. Defining fair $d-$dissections in the obvious way, we have the following conjecture:

Conjecture 1.2. *A strictly convex polygon with* $2(d-1)n+2$ *vertices has exactly* $\binom{dn}{n}^2\frac{1}{(d-1)n+1}$ *fair* $d-$*dissections.*

Conjecture 1.2 holds of course for $d = 2$ by Theorem 1.1. For general d it is again a special case: Conjecture 3.5 of Section 3 gives the number of all $d-$dissections of a convex polygon P involving exactly n_b black and n_w white polygons such that a prescribed marked edge $e_* \subset P$ belongs to a black triangle. Obvious modifications of the proof of Theorem 2.5 establish Conjecture 3.5 for a few small values of d and show the truth of Conjecture 1.2 for $d \leq 6$.

Remark 1.3. *The sequence* $1, 2, 12, 100, 980, 10584, \ldots, \binom{2n}{n}^2\frac{1}{n+1}, \ldots$ *enumerating fair triangulations appears as A888 in [2], together with the following description, due to D. Callan:* $\binom{2n}{n}^2\frac{1}{n+1}$ *is the number of lattice paths consisting of* $2n$ *steps in* $\{(\pm1,0),(0,\pm1)\}$ *starting at* $(0,0)$ *and ending on the diagonal* $x = y$ *with the constraint of remaining in the tilted quarter plane* $-x \leq y \leq x$*. A short bijective proof by Callan ([1]) is as follows: The* *pair of orthogonal projections onto the lines $y = -x$ and $y = x$ induces a bijection between such paths and the product of Dyck paths of length $2n$ and of positive paths of length $2n$. Dyck paths of length $2n$, given by lattice walks of $2n$ steps ± 1 on $\mathbb{N}$, starting and ending at the origin, are enumerated by the Catalan number $\binom{2n}{n}\frac{1}{n+1}$. Positive paths of length $2n$, given by lattice walks of $2n$ steps ± 1 on $\mathbb{N}$, starting at the origin, are enumerated by the central binomial coefficient $\binom{2n}{n}$.*

Remark 1.4. *A separate paper will deal with vertex-colourings of triangulations. The vertices of a $\tau-$triangulated polygon P can be coloured with 3 colours such that every triangle of τ has vertices of all 3 colours. Such a $3-$colouring is unique up to permutations of the three colours. One can show that a strictly convex polygon with $3n$ vertices has exactly $\binom{2n-2}{n-1}^3 \frac{3n-2}{n^2}$ triangulations which involve each colour a common number n of times in every such $3-$colouring of its vertices. Methods and techniques are completely analogous to those used in the present paper (although the formulae are simpler and the necessary symbolic computations heavier).*

The next two sections contain definitions and our main results. The rest of the paper is devoted to proofs and a few complements.

2 Main results for checkerboard triangulations

Through the rest of this paper, P_n denotes always some fixed strictly convex polygon with $n + 2$ vertices and edges. We write $P_n \supset e_*$ if P_n is decorated with a marked edge $e_* \subset \partial P_n$. A *triangulation* of P_n is a decomposition of P_n into n non-overlapping triangles. A triangulation will always be denoted by the greek letter τ. A triangulation τ of $P_n \supset e_*$ selects a unique marked triangle $\Delta_* \in \tau$ of τ such that Δ_* contains the marked edge e_*. We will generally omit a separate discussion of the degenerate and trivial initial case $n = 0$ corresponding to a polygon reduced to a (double) edge having a unique triangulation involving no triangles.

Given a triangulation τ of $P_n \supset e_*$, the associated *checkerboard colouring* partitions the set of all $n = n_b(\tau) + n_w(\tau)$ triangles of τ into a subset of $n_b = n_b(\tau)$ black triangles containing the marked triangle Δ_* and a complementary subset of $n_w = n_w(\tau)$ white triangles such that edge-adjacent triangles have different colours. A *checkerboard triangulation* is a triangulation τ of $P_n \supset e_*$ whose n triangles are coloured by the unique checkerboard colouring associated to τ.

Endowing a checkerboard triangulation τ of $P_n \supset e_*$ with the weight $t^{n_b(\tau)-n_w(\tau)}$ and summing the monomials $t^{n_b(\tau)-n_w(\tau)}$ over the set $\mathcal{T}_n$ of all

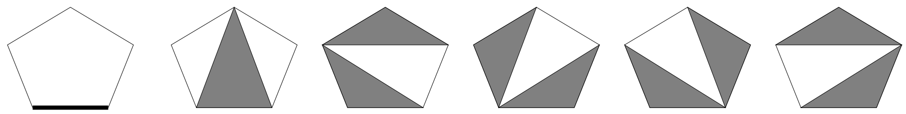

Figure 1: $P_3 \supset e_*$ with e_* given by the fat edge and its $C_3 = 5$ checkerboard triangulations.

$\binom{2n}{n}\frac{1}{n+1}$ checkerboard triangulations of $P_n \supset e_*$ we get Laurent-polynomials

$$
\begin{aligned}
&w_0(t) = 1\\
&w_1(t) = t\\
&w_2(t) = 2\\
&w_3(t) = t^{-1} + 4t\\
&w_4(t) = 12 + 2t^2\\
&w_5(t) = 12t^{-1} + 30t\\
&w_6(t) = 4t^{-2} + 100 + 28t^2\\
&w_7(t) = 140t^{-1} + 280t + 9t^3\\
&w_8(t) = 90t^{-2} + 980 + 360t^2\\
&w_9(t) = 22t^{-3} + 1680t^{-1} + 2940t + 220t^3\\
&w_{10}(t) = 1540t^{-2} + 10584 + 4620t^2 + 52t^4\\
&\quad\vdots\\
&w_n(t) = \sum_{\tau\in\mathcal{T}_n} t^{n_b(\tau)-n_w(\tau)} \in \mathbb{N}[t,t^{-1}]
\end{aligned}
$$

recursively defined by the following easy result.

Theorem 2.1. *(i) We have $w_0(t) = 1$ and*

$$w_{n+1}(t) = t\sum_{k=0}^{n} w_k(t^{-1})w_{n-k}(t^{-1}) \ .$$

(ii) The generating series $W = \sum_{n=0}^{\infty} w_n(t)x^n \in \mathbb{N}[[t,t^{-1},x]]$ satisfies the algebraic equation

$$t(1 + tx) - tW + 2tx^2W^2 + x^3W^4 = 0 \ .$$

Observe that we have $W \in \mathbf{R}$ with $\mathbf{R}$ denoting through the rest of the paper the algebra $\mathbf{R} = \left(\mathbb{Q}[t,t^{-1}]\right)[[x]]$ of formal power series in x with coefficients given by Laurent-polynomials in t.

We have moreover the following result for the Laurent-polynomials $w_n = w_n(t)$ and the associated generating series $W = \sum_{n=0}^{\infty} w_nx^n \in \mathbf{R}$.

Theorem 2.2. *We have the identity*

$$(n + 2 + 3j)(w_n, t^{-j}) = (n + 2 - 3j)(w_n, t^j)$$

where (w_n, t^j) denotes the coefficient of t^j in $w_n = \sum_{j\in\mathbb{Z}}(w_n, t^j)t^j \in \mathbb{N}[t,t^{-1}]$.

Setting $\overline{W} = \sum_{n=0}^{\infty} w_n(t^{-1})x^n \in \mathbf{R}$, Theorem 2.2 amounts to the equality

$$x\overline{W}_x + 2\overline{W} + 3t\overline{W}_t = xW_x + 2W - 3tW_t$$

where $W_x = \frac{\partial}{\partial x}W,\ W_t = \frac{\partial}{\partial t}W,\ \overline{W}_x = \frac{\partial}{\partial x}\overline{W},\ \overline{W}_t = \frac{\partial}{\partial t}\overline{W}$.

Corollary 2.3. *The generating series $W = \sum_{n=0}^{\infty} w_n x^n$ satisfies the partial differential equation*

$$2t(1-W) + x(2xW - t)W_x + 3t(t + 2xW)W_t = 0 \ .$$

Remark 2.4. *Partial derivations of W and $\overline{W}$ are formal. Proposition 4.3 (or a little work using assertion (ii) of Theorem 2.1) shows however that W and $\overline{W}$ define analytic functions in suitable open subsets containing $(1,0)$ of $\mathbb{C}^* \times \mathbb{C}$.*

Section 4 contains proofs for Theorem 2.1, 2.2, Corollary 2.3 and a few complements such as a homological interpretation for the coefficients of the Laurent polynomials w_n and features of the specialisations $W(-x,x) = 1$ and $W(-x^{-1},x) = 0$ of $W = W(t,x)$.

Section 5 contains the proof of our main Theorem giving the following closed formula for the coefficients of $w_n(t)$.

Theorem 2.5. *The Laurent-polynomials $w_n \in \mathbb{N}[t,t^{-1}]$ are given by the following formulae: For every $n \geq 0$,*

$$\begin{aligned}
w_{3n} &= \sum_{k=0}^{n} \binom{4n-2k}{n+k}\binom{2n+2k}{3k}\frac{t^{n-2k}}{3k+1} \ , \\
w_{3n+1} &= \sum_{k=0}^{n} \binom{4n+2-2k}{n+k}\binom{2n+2k}{3k}\frac{t^{n+1-2k}}{2n+1-k} \ , \\
w_{3n+2} &= \sum_{k=0}^{n} \binom{4n+2-2k}{n+1+k}\binom{2n+2+2k}{3k+1}\frac{t^{n-2k}}{3k+2} \ .
\end{aligned}$$

The outline of the proof for Theorem 2.5 is as follows: We display two partial differential equations which have at most a unique common solution. We show then that the series defined by the formulae of Theorem 2.5 and the algebraic function defined by assertion (ii) of Theorem 2.1 are both common solutions of the differential equations mentionned above. This implies that they coincide.

Remark 2.6. *Checkerboard triangulations of $P_n \supset e_*$ can be considered as the $\binom{2n}{n}\frac{1}{n+1}$ states of a spin model for the "energy" given by $t^{n_b - n_w}$. Theorem 2.5 shows that this spin model is exactly solvable or integrable.*

3 Main results for $d-$dissections

Dissections of polygons, also called cell-growth problems, generalise triangulations coinciding with $2-$dissections.

Let $n \geq 0$ and $d \geq 2$ be two natural integers. We denote by $t_{d,n}$ the number of dissections of $P_{(d-1)n}$ into n non-overlapping convex polygons such that every polygon of the dissection has $(d+1)$ vertices which are all contained in the set of vertices of $P_{(d-1)n}$. We call such a decomposition a $d-$*dissection* of $P_{(d-1)n}$. The generating function $g_d = \sum_{n=0}^{\infty} t_{d,n} x^n \in \mathbb{N}[[x]]$ encoding the numbers of all $d-$dissections satisfies the identity $g_d = 1 + x g_d^d$, as shown by arguments similar to those used in the proof of Theorem 2.1. The polynomial identity for g_d follows also from the equivalence between $d-$dissections and rooted plane $d-$regular trees. Section 4.2 explains this bijection in the case $d = 2$. The generalisation to arbitrary values of d is straightforward.

The coefficients $t_{d,n}$ of $g_d = \sum_{n=0}^{\infty} t_{d,n} x^n$ are given by the following well-known result (see for example Formula 2.3 in [3] or formula (5) in [6]).

Theorem 3.1. *The coefficients $t_{d,n}$ of $g_d = 1 + x + dx^2 + \cdots = \sum_{n=0}^{\infty} t_{d,n} x^n$ defined by the equation $g_d = 1 + x g_d^d$ are given by the formula*

$$t_{d,n} = \binom{dn}{n} \frac{1}{(d-1)n+1} \ .$$

For the sake of completeness, we include a short proof using Lagrange inversion at the beginning of Section 6.

The choice of a marked edge $e_* \subset \partial P_{(d-1)n}$ turns a $d-$dissection τ of $P_{(d-1)n}$ into a *checkerboard* $d-$*dissection* by partitioning all n polygons involved in τ into two disjoint subsets of n_b black and n_w white polygons such that the unique polygon $\Delta_* \in \tau$ containing the marked edge e_* is black and such that edge-adjacent polygons of τ have different colours. Associating the monomial $t^{n_b - n_w}$ to a checkerboard $d-$dissection involving n_b black and n_w white polygons and summing the monomials $t^{n_b - n_w}$ over all $t_{d,n}$ checkerboard $d-$dissections of $P_{(d-1)n} \supset e_*$ yields again Laurent-polynomials $w_{d,n} \in \mathbb{N}[t, t^{-1}]$.

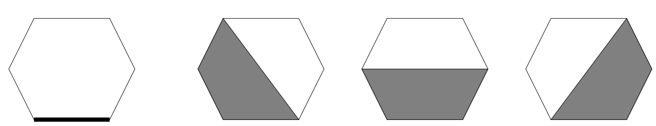

Figure 2: A hexagon with a marked edge and its $t_{3,2} = 3$ contributions to $w_{3,2} = 3$.

The proof of assertion (ii) in Theorem 2.1, modified suitably, shows the following result.

Theorem 3.2. *The generating series* $W = \sum_{n=0}^{\infty} w_{d,n} x^n \in \mathbf{R}$ *satisfies the algebraic equation*

$$W = 1 + tx\left(1 + t^{-1}xW^d\right)^d .$$

Theorem 2.2 and Corollary 2.3 have the following easy and straightforward generalisations.

Theorem 3.3. *We have the identity*

$$((d-1)n + 2 + (d+1)j)(w_{d,n}, t^{-j}) = ((d-1)n + 2 - (d+1)j)(w_{d,n}, t^j)$$

where $(w_{d,n}, t^j)$ *denotes the coefficient of* t^j *in* $w_{d,n} = \sum_{j\in\mathbb{Z}}(w_{d,n}, t^j)t^j \in \mathbb{N}[t, t^{-1}]$.

Theorem 3.3 is of course equivalent to

$$(d-1)x\overline{W}_x + 2\overline{W} + (d+1)t\overline{W}_t = (d-1)xW_x + 2W - (d+1)tW_t$$

where $W = \sum_{n=0}^{\infty} w_{d,n}(t)x^n$ and $\overline{W} = \sum_{n=0}^{\infty} w_{d,n}(t^{-1})x^n$.

Corollary 3.4. *The generating series* $W = \sum_{n=0}^{\infty} w_{d,n} x^n \in \mathbf{R}$ *satisfies the partial differential equation*

$$2t(1 - W) + (d-1)x(dxW^{d-1} - t)W_x + (d+1)t(t + dxW^{d-1})W_t .$$

Experimental observations suggest the following conjecture for the coefficients of the Laurent-polynomials $w_{d,n}$.

Conjecture 3.5. *The Laurent-polynomials* $w_{d,n} \in \mathbb{N}[t, t^{-1}]$ *are given by the following formulae: For every* $n \geq 0$,

$$w_{d,(d+1)n} = \sum_{k=0}^{(d-1)n} \binom{d^2n - dk}{n+k}\binom{d(n+k)}{(d+1)k}\frac{t^{(d-1)n-2k}}{((d+1)k+1)}$$

$$w_{d,(d+1)n+1} = \sum_{k=0}^{(d-1)n} \binom{d^2n + d - dk}{n+k}\binom{d(n+k)}{(d+1)k}\frac{t^{(d-1)n+1-2k}}{(dn+1-k)}$$

and

$$w_{d,(d+1)n+j} = \sum_{k=0}^{(d-1)n+j-2} \binom{d^2n + d(j-1) - dk}{n+1+k}\binom{dn+d+dk}{(d+1)k+d-j+1} \cdot \frac{t^{(d-1)n+j-2-2k}}{(d+1)k+d-j+2}$$

for j *such that* $2 \leq j \leq d$.

Most steps in the proof of Theorem 2.5 work in the context of Conjecture 3.5. Only a necessary computation involving larger and larger polynomials for increasing d fails. Maple was however able to complete it for a few small values of d and we have:

Theorem 3.6. *Conjecture 3.5 holds for* $2 \leq d \leq 6$.

4 A few complements and some easy proofs

4.1 Proof of Theorem 2.1

A checkerboard triangulation τ of $P_n \supset e_*$ with $n \geq 1$ has a unique decomposition into the marked black triangle $\Delta_* \in \tau$ containing e_* and into a pair τ_1, τ_2 of checkerboard triangulations with transposed colours of polygons P_k, P_{n-1-k} obtained by cutting τ along Δ_* with a knife marking all cuts. This construction is reversible and shows assertion (i).

Setting $\overline{W} = \sum_{n=0}^{\infty} w_n(t^{-1})x^n$, assertion (i) boils down to the equations

$$W = 1 + tx\overline{W}^2 \text{ and } \overline{W} = 1 + \frac{x}{t}W^2 .$$

This shows that $W = 1 + tx(1 + x/tW^2)^2$ and implies assertion (ii). □

4.2 Trees

Triangulations of $P_n \supset e_*$ are well-known to be in bijection with rooted plane $2-$regular trees, also called full binary trees, having n interior vertices of outdegree 2 and $n+1$ leaves, see for example the equivalence between the sets (i) and (vi) of Corollary 6.2.3 in [5]. The degenerate case $n = 0$ corresponds to a rooted leaf by convention. For $n \geq 1$, the rooted tree $T_*(\tau)$ associated to a triangulation τ of $P_n \supset e_*$ has interior vertices given by the n triangles of τ, the root-vertex v_* corresponding to the triangle $\Delta_* \supset e_*$, and leaves given by the $n+1$ unmarked edges of ∂P_n. Adjacency is given by pairs of vertices corresponding to subsets in τ which intersect in a common edge. More precisely, two interior vertices of T_* are adjacent if they correspond to edge-adjacent triangles of τ. An interior vertex v of T_* is adjacent to a leaf l if the triangle corresponding to v contains the unmarked edge of ∂P_n corresponding to l. The tree T_* has n_b interior vertices at even distance from its root vertex v_* and n_w interior vertices at odd distance from v_*. Otherwise stated, every tree is a bipartite graph and the numbers n_b, n_w count the interior vertices in the bipartite class of the root vertex v_* and in its complementary class.

4.3 A homological interpretation for the coefficients of w_n

Let τ be a checkerboard triangulation of $P_n \supset e_*$ where $n \geq 1$. We denote as always by $\Delta_* \in \tau$ the unique black triangle containing the marked edge $e_* \subset \partial P_n$. An easy induction on n shows that there exists a unique continuous piecewise affine map $\psi_\tau : P_n \longrightarrow \Delta_*$ which induces the identity-map on Δ_* and whose restriction to every triangle $\Delta \in \tau$ is an affine bijection. We call ψ_τ the *folding map* since it maps, up to piecewise affine transformations, the polygon P_n onto Δ_* by folding P_n along all $(n-1)$ interior edges of the triangulation τ (where an edge $e \subset \Delta \in \tau$ is interior if $e \not\subset \partial P_n$). It is easy to check that ψ_τ preserves the orientation on all black triangles and reverses the orientation on all white triangles of τ.

Orienting the boundaries $\partial P_n \subset P_n$ and $\partial\Delta_* \subset \Delta_*$ in the trigonometric counterclockwise sense yields canonical isomorphisms between the first homology groups $H_1(\partial P_n, \mathbb{Z}) \sim H_1(\partial\Delta_*, \mathbb{Z})$ and the cyclic group $\mathbb{Z}$. The elements of $H_1(\partial P_n, \mathbb{Z})$ or of $H_1(\partial\Delta_*, \mathbb{Z})$ can be interpreted as "winding coefficients" of closed loops contained in the boundary ∂P_n or $\partial\Delta_*$ with respect to interior points of the polygon P_n or of the triangle Δ_*.

The image $\psi_\tau(\partial\Delta) \subset \partial\Delta_*$ corresponds to the generator 1 of $H_1(\partial\Delta_*, \mathbb{Z})$ if Δ is a black triangle and to -1 if Δ is white. This shows that the homomorphism

$$(\psi_\tau)_* : H_1(\partial P_n, \mathbb{Z}) \longrightarrow H_1(\partial\Delta_*, \mathbb{Z})$$

induced by the folding map is given by $(\psi_\tau)_*(1) = n_b - n_w \in H_1(\partial\Delta_*, \mathbb{Z})$ where n_b and n_w are the numbers of black and white triangles in the checkerboard triangulation τ.

Since every boundary edge $e \subset \partial P_n$ is contained in a unique triangle (we assume $n \geq 1$) of a checkerboard triangulation, it inherits a well-defined colour according to its inclusion in the subset of black or white triangles. In particular, the marked edge e_* of ∂P_n is always black. The degenerate case $n = 0$ corresponds by convention to a double edge consisting of a marked black edge and an unmarked white edge.

Proposition 4.1. *The numbers e_b and e_w of black and white edges in a checkerboard triangulation τ of $P_n \supset e_*$ are given by the formulae*

$$e_b = \frac{n+2+3(n_b-n_w)}{2} = 2n_b - n_w + 1$$
$$e_w = \frac{n+2-3(n_b-n_w)}{2} = -n_b + 2n_w + 1$$

with n_b and n_w denoting the number of black and white triangles in τ.

Proof The homological interpretation of the coefficients in w_n shows the identity $3(e_b - e_w) = n_b - n_w$. This, together with the trivial equalities $e_b + e_w = n + 2$ and $n_b + n_w = n$, implies the result. □

4.4 The asymptotic mean-value of $n_b(\tau) - n_w(\tau)$

The following result gives the asymptotic mean value $n_b - n_w$ of a uniform random triangulation for $P_n \supset e_*$.

Proposition 4.2. *We have the equivalence*

$$w_n'(1) \sim \frac{3}{8}\binom{2n}{n}\frac{1}{n+1}$$

for $n \to \infty$.

In particular, uniform random triangulations have the asymptotic mean value $n_b - n_w = \frac{3}{8}$.

Proof Derivating the polynomial equation

$$t(1+tx) - tW + 2tx^2W^2 + x^3W^4 = 0$$

for W with respect to t, we get

$$1 + 2tx - W - tW_t + 2x^2W^2 + 4tx^2WW_t + 4x^3W^3W_t = 0 \ .$$

Eliminating W yields the equation

$$\begin{aligned}0 = & x(t+x)(8tx^2(2tx+1) - t + x) - t(8x^3(t^2+1) + 12tx^2 - t)W_t + \\ & + 2t^2x^2(64tx^3(t+x) + 60tx^2 - 6t + 48x^3 + x)W_t^2 + \\ & + t^3x^3(256x^3(t^2+tx+1) + 288tx^2 - 27t)W_t^4\end{aligned}$$

for the partial derivation $W_t = \frac{\partial}{\partial t}W$. The specialisation $t = 1$ of the minimal polynomial for W_t factorises and yields the minimal polynomial

$$x - \tilde{W} + x(4x+3)\tilde{W}^2 = 0$$

for

$$\tilde{W} = \sum_{n=0}^{\infty} w_n'(1)x^n = \frac{1 - (1+2x)\sqrt{(1-4x)}}{2x(4x+3)} = x + 3x^3 + 4x^4 + 18x^5 + \dots \ .$$

A straightforward computation using for example $\lim_{n\to\infty} C_{n+1}/C_n = 4$ for the Catalan numbers $C_n = \binom{2n}{n}\frac{1}{n+1}$ ends the proof. □

4.5 Analytical properties of W

Proposition 4.3. *The series $W = W(t,x)$ is absolutely convergent for $(t,x) \in \mathbb{C}^* \times \mathbb{C}$ such that* $\max(|t^{1/3}x|, \ |t^{-1/3}x|) < \frac{1}{4}$.

Proof Non-negativity of the integers $e_b = \frac{n+2+3(n_b-n_w)}{2}$ and $e_w = \frac{n+2-3(n_b-n_w)}{2}$ given by Proposition 4.1 shows the bound $|n_b - n_w| \leq \frac{n+2}{3}$. Positivity of all coefficients in w_n yields the majorations

$$|w_n(t)| \leq w_n(1)M_n = \binom{2n}{n}\frac{1}{n+1}M_n \leq 4^n M_n$$

where $M_n = M_n(t) = \max(|t|^{(n+2)/3}, |t^{-1}|^{(n+2)/3})$. This implies the result easily. □

Remark 4.4. *Analyticity of $W(t,x)$ in an open neighbourhood of $\mathbb{C}^* \times \{0\}$ follows also from assertion (ii) in Theorem 2.1. For fixed $t \in \mathbb{C}^*$, the algebraic equation for $W(t,x)$ defines a unique analytic extension $x \longmapsto W(t,x)$ of the evaluation $W(t,0) = 1$ (the three remaining branches are singular at $x = 0$).*

Remark 4.5. *Theorem 2.5. gives the exact bounds on the degrees of monomials involved in w_n: The Laurent polynomials w_{3n}, w_{3n+2} contain no monomial of degree $> n$ or $< -n$ and w_{3n+1} contains no monomial of degree $> n+1$ or $< 1-n$.*

4.6 Proof of Theorem 2.2 and Corollary 2.3

A *direct automorphism* of a triangulation τ for P_n is a piecewise affine map inducing an orientation-preserving homeomorphism of P_n which restricts to affine bijections between triangles of τ. The group of all such automorphisms is cyclic of order $\alpha^+(\tau) \leq 3$.

A checkerboard triangulation τ of $P_n \supset e_*$ yields, after unmarking the edge e_*, a contribution of $\frac{1}{\alpha^+(\tau)} e_b$ to the coefficient $t^{n_b - n_w}$ of w_n and a contribution of $\frac{1}{\alpha^+(\tau)} e_w$ to the coefficient $t^{-n_b+n_w}$ of w_n. Theorem 2.2. follows now easily from Proposition 4.1 for the numbers $e_b = \frac{n+2+3(n_b-n_w)}{2}$ and $e_w = \frac{n+2-3(n_b-n_w)}{2}$ of black and white edges in the checkerboard triangulation τ of $P_n \supset e_*$. $\square$

Corollary 2.3 follows from the equation $x\overline{W}_x + 2\overline{W} + 3t\overline{W}_t = xW_x + 2W - 3tW_t$, equivalent to Theorem 2.2, after elimination of $\overline{W}, \overline{W}_t, \overline{W}_x$ using the identity $\overline{W} = 1 + \frac{x}{t} W^2$ occuring in the proof of Theorem 2.1 and its partial derivations $W_t^- = -\frac{x}{t^2} W^2 + 2\frac{x}{t} W W_t$ and $W_x^- = \frac{1}{t} W^2 + 2\frac{x}{t} W W_x$.

4.7 Non-commutative edge polynomials

Labelling black and white edges in the boundary ∂P_n of a checkerboard triangulation by U and V, starting at the marked edge e_* and reading counterclockwise the labels of all $n+2$ edges in ∂P_n, we get a word of length $n+2$, starting with U, in the alphabet $\{U, V\}$. The sum over $n \in \mathbb{N}$ and over all checkerboard triangulations of $P_n \supset e_*$ of these words defines thus a unique non-commutative power-series $N \in \mathbb{N}\langle\langle U, V \rangle\rangle$ in two free non-commuting variables. The first few terms of N are

$$\begin{aligned} &UV + U^3 + UV^2U + U^2V^2 + UV^4 + UVU^3 + U^2VU^2 + U^3VU + U^4V + \\ &\quad +2U^6 + 2(U^3V^3 + U^2V^3U + UV^3U^2) + U^2VUV^2 + UV^2UVU + \\ &\quad +U^2V^2UV + UV^2U^2V + UVU^2V^2 + UVUV^2U + \dots \ . \end{aligned}$$

Denoting by $\overline{N} = N(V, U)$ the series obtained by transposing the variables U, V of $N = N(U, V)$ we have the identities

$$\begin{aligned} N &= UV + U\left(\tfrac{1}{V}\overline{N}\right)^2 \\ \overline{N} &= VU + V\left(\tfrac{1}{U}N\right)^2 \end{aligned}$$

which imply

$$N = UV + U\left(U + \left(\frac{1}{U}N\right)^2\right)^2 \ .$$

4.8 The specialisations $t = \pm x$ and $t = \pm x^{-1}$

Proposition 4.6. *The specialisations $W(-x, x)$ and $W(-x^{-1}, x)$ of $W(t, x) = \sum_{n=0}^{\infty} w_n(t) x^n$ are well defined and yield constant functions $W(-x, x) = 1$ and $W(-x^{-1}, x) = 0$.*

This proposition amounts to annullation of all alternate row-sums except the first one and of all alternate column sums in the array $\mathcal{A}$ given by

1										
1	2	1								
	4	12	12	4						
	2	30	100	140	90	22				
		28	280	980	1680	1540	728	140		
		9	360	2940	10584	20790	24024	16380	612	969

and defined by writing the coefficients of w_n suitably along antidiagonals.

Diagonal coefficients $1, 2, 12, 100, 980, \ldots$ of $\mathcal{A}$ are constant terms in the Laurent polynomials $w_{2n}(t)$. Their generating series is the hypergeometric function

$$\sum_{n=0}^{\infty}(w_{2n}, t^0)x^n = \sum_{n=0}^{\infty}\binom{2n}{n}^2 \frac{x^n}{n+1}$$

enumerating fair triangulations, see Theorem 1.1 of the introduction.

The generating functions of the extremal sequences

$$1, 1, 4, 22, 140, 969, 7084, \ldots \text{ and } 1, 2, 9, 52, 340, 2394, 17710, \ldots$$

of the array $\mathcal{A}$ are given by the algebraic hypergeometric functions

$$A = \sum_{n=0}^{\infty}\binom{4n}{n}\frac{x^n}{3n+1} = 1 + xB^2$$

and

$$B = \sum_{n=0}^{\infty}\binom{4n+2}{n}\frac{x^n}{2n+1} = A^2,$$

cf. Section 4.10 below.

Proof of Proposition 4.6 The upper bound $|n_b - n_w| \leq \frac{n+2}{3}$ on the degrees of monomials with non-zero coefficient in w_n implies that only a finite number of non-zero coefficients of W contribute to the monomial x^m of $W(-x^{\pm 1}, x)$. They imply also $W(-x^{\pm 1}, x) \in \mathbb{Z}[[x]]$. The factorisations

$$x(W-1)(1 - x^2 - x^2W + x^2W^2 + x^2W^3)$$

corresponding to the specialisation $t = -x$ in assertion (ii) of Theorem 2.1 and

$$\frac{1}{x}W(1 - 2x^2W + x^4W^3)$$

corresponding to the specialisation $t = -x^{-1}$ in assertion (ii) of Theorem 2.1, integrality of the ring $\mathbb{Z}[[x]]$ and a few easy verifications imply the result. $\Box$

The generating functions $R(x)$ and $C(x)$ of row and column sums of the array $\mathcal{A}$ considered above are given by

$$R(x) = W(\sqrt{x}, \sqrt{x}) = 1 + 4x + 32x^2 + 384x^3 + 5376x^4 + 82176x^5 + \dots$$

satisfying the equation $1 + x - R + 2xR^2 + xR^4 = 0$ and

$$C(x) = W(\frac{1}{\sqrt{x}}, \sqrt{x}) = 2 + 8x + 80x^2 + 1024x^3 + 14848x^4 + 231936x^5 + \dots$$

satisfying $2 - C + 2xC^2 + x^2C^4 = 0$. The evaluations $W(-x, x) = 1$ and $W(-x^{-1}, x) = 0$ show thus that row-sums or columns sums restricted to even elements of $\mathcal{A}$ (with indices starting at 0) are given by the coefficients of $\frac{1}{2}(R(x) + 1)$ and $\frac{1}{2}C(x)$.

Remark 4.7. *The equation $1 + x - R + 2xR^2 + xR^4 = 0$ for $R = R(x)$ is equivalent to $x = \frac{R-1}{(1+R^2)^2}$. The formal power series $R - 1$ is thus the reciprocal series of the power series of the rational function $y \longmapsto \frac{y}{(1+(1+y)^2)^2}$. Lagrange inversion (see for example Satz 2.4 in [4] or Theorem 5.4.2 in [5]) gives thus the formula*

$$R = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2n}{k} \binom{2n-k}{n-1-2k} \frac{1}{2^k} \right) \frac{(4x)^n}{n} \ .$$

4.9 Conjectural positivity properties

The coefficient of $u^\alpha v^\beta$ involved in the homogeneous polynomial $Q_n(u, v) = \sqrt{uv}^n \ w_n\left(\sqrt{\frac{u}{v}}\right) \in \mathbb{N}[u, v]$ of degree n counts the number of checkerboard triangulations of $P_n \supset e_*$ involving α black and β white triangles. Experimental observations suggest the following conjectural properties of the polynomials $Q_0, Q_1, \dots$.

Conjecture 4.8. *(i) The polynomials $Q_0(u, 1), Q_1(u, 1), \dots \in \mathbb{N}[u]$ have only real roots (which are ≤ 0 since all coefficients are positive) and the non-zero roots of $Q_n(u, 1)$ interleave the non-zero roots of $Q_{n+3}(u, 1)$.*

(ii) The symmetric polynomial $Q_n(u, v) + Q_n(v, u)$ is of the form $R_n(u + v, uv)$ where $R_n(e_1, e_2) \in \mathbb{Z}[e_1, e_2]$ involves only positive coefficients.

4.10 Eulerian triangulations

Theorem 2.5 or fairly elementary generating series manipulations show that $P_{3n+1} \supset e_*$ has exactly $\binom{4n+2}{n}\frac{1}{2n+1}$ checkerboard triangulations such that the boundary ∂P_{3n+1} of P_{3n+1} is contained in the subset of all black triangles. Such a triangulation τ could be called "Eulerian" since all vertices of P_{3n+1} have even degree in the planar graph defined by τ. Equivalently, a triangulation τ of P_n is Eulerian if every vertex of P_n belongs to an odd number of triangles in τ.

Remark 4.9. *The boundary $\partial P_{(d+1)n+1}$ of $P_{(d+1)n+1}$ seems to be contained in the subset of black polygons for exactly $\binom{d^2n+d}{n}\frac{1}{dn+1}$ checkerboard $d-$dissections. Such dissections define again Eulerian graphs and have the equivalent property that every vertex of $P_{(d+1)n+1}$ is contained in an odd number of dissecting polygons.*

5 Proof of Theorem 2.5

The idea for proving Theorem 2.5 is as follows:

We exhibit two partial differential operators $D = D_L - D_R$ and $\tilde{D} = \tilde{D}_L - \tilde{D}_R$ such that the two associated partial differential equations $DF = 0$ and $\tilde{D}F = 0$ have at most a unique common formal solution satisfying the initial condition $F \equiv 1 + tx \pmod{x^2}$ in the algebra $\mathbf{R} = \left(\mathbb{Q}[t, t^{-1}]\right)[[x]]$ of formal power series in x with coefficients in $\mathbb{Q}[t, t^{-1}]$.

We show then that we have $D\tilde{W} = \tilde{D}\tilde{W} = 0$ for the formal series $\tilde{W} \equiv 1 + tx \pmod{x^2}$ defined by the formulae of Theorem 2.5.

Finally, we consider the algebraic equation

$$t(1 + tx) - tW + 2tx^2W^2 + x^3W^4 = 0$$

for the generating series $W = \sum_{n=0}^{\infty} w_n x^n$ enumerating checkerboard triangulations. We show that this equation has a unique solution in $\mathbf{R}$ which is thus given by W satisfying the initial condition $W \equiv 1 + tx \pmod{x^2}$. We show then that W satisfies the equations $DW = \tilde{D}W = 0$. This implies $W = \tilde{W}$ and establishes Theorem 2.5.

Consider the four linear partial differential operators

$$\begin{aligned}
D_L &= 4tx\left(1 + t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\left(2 - 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\\
&= 4tx\left(2 - 4t\frac{\partial}{\partial t} + 4x\frac{\partial}{\partial x} - 3t^2\frac{\partial^2}{\partial t^2} - 2tx\frac{\partial^2}{\partial t\partial x} + x^2\frac{\partial^2}{\partial x^2}\right)\\
D_R &= \left(3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\left(-2 + 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\\
&= 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + 9t^2\frac{\partial^2}{\partial t^2} + 6tx\frac{\partial}{\partial t\partial x} + x^2\frac{\partial^2}{\partial x^2}\\
\tilde{D}_L &= 4x\left(1 - t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\left(3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\\
&= 4x\left(2x\frac{\partial}{\partial x} - 3t^2\frac{\partial^2}{\partial t^2} + 2tx\frac{\partial^2}{\partial t\partial x} + x^2\frac{\partial^2}{\partial x^2}\right)\\
\tilde{D}_R &= t\left(2 - 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\left(-3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\\
&= t\left(3t\frac{\partial}{\partial t} + 3x\frac{\partial}{\partial x} + 9t^2\frac{\partial^2}{\partial t^2} - 6tx\frac{\partial^2}{\partial t\partial x} + x^2\frac{\partial^2}{\partial x^2}\right)
\end{aligned}$$

Proposition 5.1. *The two partial differential equations*

$$D_L F = D_R F$$

and

$$\tilde{D}_L F = \tilde{D}_R F$$

have at most a unique common solution $F \in \mathbf{R} = (\mathbb{Q}[t, t^{-1}])[[x]]$ *which satisfies the initial condition* $F \equiv 1 + tx \pmod{x^2}$.

Proof The formulae

$$\begin{aligned} D_L(t^j x^m) &= 4(1+j+m)(2-3j+m)t^{j+1}x^{m+1} \\ D_R(t^j x^m) &= (3j+m)(-2+3j+m)t^j x^m \\ \tilde{D}_L(t^j x^m) &= 4(1-j+m)(3j+m)t^j x^{m+1} \\ \tilde{D}_R(t^j x^m) &= (2-3j+m)(-3j+m)t^{j+1}x^m \end{aligned}$$

show that a coefficient $(F, t^j x^m)$ of a common solution F is determined by the coefficients $(F, t^{j-1}x^{m-1})$ and $(F, t^{j+1}x^{m-1})$ except if $(j, m) \in \mathbb{Z} \times \mathbb{N}$ is among the four common roots $(0, 0), (\frac{1}{3}, -1), (\frac{1}{3}, 1), (\frac{2}{3}, 0)$ of the two polynomials

$$\begin{aligned} &(3j+m)(-2+3j+m) \ , \\ &(2-3j+m)(-3j+m) \ . \end{aligned}$$

This shows $(j, m) = (0, 0)$ and implies that such a solution F is either non-existent or uniquely defined by the initial condition $F \equiv 1 + tx \pmod{x^2}$. □

Proposition 5.2. *We have*

$$D_L \tilde{W} = D_R \tilde{W}$$

and

$$\tilde{D}_L \tilde{W} = \tilde{D}_R \tilde{W}$$

for the series $\tilde{W} = \sum_{n=0}^{\infty} \tilde{w}_n x^n \equiv 1 + tx \pmod{x^2}$ *defined by the formulae*

$$\begin{aligned} \tilde{w}_{3n} &= \sum_{k=0}^{n} \binom{4n-2k}{n+k} \binom{2n+2k}{3k} \frac{t^{n-2k}}{3k+1} \ , \\ \tilde{w}_{3n+1} &= \sum_{k=0}^{n} \binom{4n+2-2k}{n+k} \binom{2n+2k}{3k} \frac{t^{n+1-2k}}{2n+1-k} \ , \\ \tilde{w}_{3n+2} &= \sum_{k=0}^{n} \binom{4n+2-2k}{n+1+k} \binom{2n+2+2k}{3k+1} \frac{t^{n-2k}}{3k+2} \ . \end{aligned}$$

Lemma 5.3. *(i) We have*

$$D_L(x^{3n}\tilde w_{3n}(t)) = 8x^{3n+1}\sum_{k=0}^{n}\frac{(4n+1-2k)!\ (2n+2k)!\ t^{n+1-2k}}{(n+k)!\ (3n-3k)!\ (3k)!\ (2n-k)!}$$
$$D_L(x^{3n+1}\tilde w_{3n+1}(t)) =$$
$$8x^{3n+2}\sum_{k=1}^{n}\frac{(4n+1-2(k-1))!\ (2n+2+2(k-1))!\ t^{n-2(k-1)}}{(n+1+(k-1))!\ (3n-1-3(k-1))!\ (3(k-1)+2)!\ (2n-(k-1))!}$$
$$D_L(x^{3n+2}\tilde w_{3n+2}(t)) =$$
$$4x^{3(n+1)}\sum_{k=0}^{n}\frac{(4(n+1)-2k)!\ (2(n+1)+2k)!\ t^{n+1-2k}}{(n+1+k)!\ (3(n+1)-3k-2)!\ (3k+1)!\ (2(n+1)-k)!}$$

and

$$D_R(x^{3n}\tilde w_{3n}) = 4x^{3n}\sum_{k=0}^{n-1}\frac{(4n-2k)!\ (2n+2k)!\ t^{n-2k}}{(n+k)!\ (3n-3k-2)!\ (3k+1)!\ (2n-k)!}$$
$$D_R(x^{3n+1}\tilde w_{3n+1}(t)) = 8x^{3n+1}\sum_{k=0}^{n}\frac{(4n+1-2k)!\ (2n+2k)!\ t^{n+1-2k}}{(n+k)!\ (3n-3k)!\ (3k)!\ (2n-k)!}$$
$$D_R(x^{3n+2}\tilde w_{3n+2}(t)) = 8x^{3n+2}\sum_{k=0}^{n-1}\frac{(4n+1-2k)!\ (2n+2+2k)!\ t^{n-2k}}{(n+1+k)!\ (3n-1-3k)!\ (3k+2)!\ (2n-k)!}$$

(ii) We have

$$\tilde D_L(x^{3n}\tilde w_{3n}(t)) = 8x^{3n+1}\sum_{k=0}^{n-1}\frac{(4n-2k)!\ (2n+1+2k)!\ t^{n-2k}}{(n+k)!\ (3n-3k-1)!\ (3k+1)!\ (2n-k)!}$$
$$\tilde D_L(x^{3n+1}\tilde w_{3n+1}(t)) = 8x^{3n+2}\sum_{k=0}^{n}\frac{(4n+2-2k)!\ (2n+1+2k)!\ t^{n+1-2k}}{(n+k)!\ (3n+1-3k)!\ (3k)!\ (2n+1-k)!}$$
$$\tilde D_L(x^{3n+2}\tilde w_{3n+2}(t)) =$$
$$8x^{3(n+1)}\sum_{k=0}^{(n+1)-1}\frac{(4(n+1)-2-2k)!\ (2(n+1)+1+2k)!\ t^{(n+1)-1-2k}}{((n+1)+k)!\ (3(n+1)-3k-3)!\ (3k+2)!\ (2(n+1)-1-k)!}$$

and

$$\tilde D_R(x^{3n}\tilde w_{3n}) = 8x^{3n}\sum_{k=1}^{n}\frac{(4n-2-2(k-1))!\ (2n+1+2(k-1))!\ t^{n-1-2(k-1)}}{(n+(k-1))!\ (3n-3(k-1)-3)!\ (3(k-1)+2)!\ (2n-1-(k-1))!}$$
$$\tilde D_R(x^{3n+1}\tilde w_{3n+1}(t)) =$$
$$8x^{3n+1}\sum_{k=1}^{n}\frac{(4n-2(k-1))!\ (2n+1+2(k-1))!\ t^{n-2(k-1)}}{(n+(k-1))!\ (3n-3(k-1)-1)!\ (3(k-1)+1)!\ (2n-(k-1))!}$$
$$\tilde D_R(x^{3n+2}\tilde w_{3n+2}(t)) = 8x^{3n+2}\sum_{k=0}^{n}\frac{(4n+2-2k)!\ (2n+1+2k)!\ t^{n+1-2k}}{(n+k)!\ (3n+1-3k)!\ (3k)!\ (2n+1-k)!}$$

Proof Elementary and tedious verifications left to the reader. □

Proof of Proposition 5.2 Follows easily from Lemma 5.3. □

Consider the ideal $\mathcal{I} = (P, P_t, P_x, P_{tt}, P_{tx}P_{xx})$ generated by

$$P = t(1+tx) - tF + 2tx^2F^2 + x^3F^4,$$
$$P_t = 1 + 2tx - F - tF_t + 2x^2F^2 + 4tx^2FF_t + 4x^3F^3F_t,$$
$$P_x = t^2 - tF_x + 4txF^2 + 4tx^2FF_x + 3x^2F^4 + 4x^3F^3F_x,$$
$$P_{tt} = 2x - 2F_t - tF_{tt} + 8x^2FF_t + 4tx^2F_t^2 + 4tx^2FF_{tt} +$$
$$+12x^3F^2F_t^2 + 4x^3F^3F_{tt},$$
$$P_{tx} = 2t - F_x - tF_{tx} + 4xF^2 + 4x^2FF_x + 8txFF_t + 4tx^2F_tF_x +$$
$$+4tx^2FF_{tx} + 12x^2F^3F_t + 12x^3F^2F_tF_x + 4x^3F^3F_{tx},$$
$$P_{xx} = -tF_{xx} + 4tF^2 + 16txFF_x + 4tx^2F_x^2 + 4tx^2FF_{xx} + 6xF^4 +$$
$$+24x^2F^3F_x + 12x^3F^2F_x^2 + 4x^3F^3F_{xx}$$

of the free polynomial algebra $\mathbb{Q}[t, x, F, F_t, F_x, F_{tt}, F_{tx}, F_{xx}]$ in eight variables $t, x, F, F_t, F_x, F_{tt}, F_{tx}, F_{xx}$.

We introduce moreover the polynomials

$$\begin{aligned}
Q &= 4tx\left(2F - 4tF_t + 4xF_x - 3t^2F_{tt} - 2txF_{tx} + x^2F_{xx}\right) \\
&\quad -(3tF_t - xF_x + 9t^2F_{tt} + 6txF_{tx} + x^2F_{xx}) \\
\tilde{Q} &= 4x\left(2xF_x - 3t^2F_{tt} + 2txF_{tx} + x^2F_{xx}\right) \\
&\quad -t\left(3tF_t + 3xF_x + 9t^2F_{tt} - 6txF_{tx} + x^2F_{xx}\right)
\end{aligned}$$

and

$$K = \frac{\partial P}{\partial F} = -t + 4tx^2F + 4x^3F^3 \ .$$

Lemma 5.4. *We have $K^3Q \in \mathcal{I}$ and $K^3\tilde{Q} \in \mathcal{I}$.*

Proof Set

$$Q_1 = KQ - ((Q, F_{tt})P_{tt} + (Q, F_{tx})P_{tx} + (Q, F_{xx})P_{xx}) \equiv KQ \pmod{\mathcal{I}}$$

and

$$\tilde{Q}_1 = K\tilde{Q} - ((\tilde{Q}, F_{tt})P_{tt} + (\tilde{Q}, F_{tx})P_{tx} + (\tilde{Q}, F_{xx})P_{xx}) \equiv K\tilde{Q} \pmod{\mathcal{I}}$$

where $(R, F_{tt}), (R, F_{tx}), (R, F_{xx}) \in \mathbb{Q}[t, x, F, F_t, F_x]$ are the coefficients of F_{tt}, F_{tx}, F_{xx} of $R = Q$ or $R = \tilde{Q}$. Since the three polynomials $KF_{tt} - P_{tt}, KF_{tx} - P_{tx}, KF_{xx} - P_{xx}$ are elements of $\mathbb{Q}[t, x, F, F_t, F_x]$ and since Q and $\tilde{Q}$ are of degree 1 with respect to the variables F_{tt}, F_{tx}, F_{xx} we have the inclusions $Q_1, \tilde{Q}_1 \in \mathbb{Q}[t, x, F, F_t, F_x]$.

Similarly, we have $KF_t - P_t, KF_x - P_x \in \mathbb{Q}[t, x, F]$. Since $Q_1, \tilde{Q}_1$ are of degree 2 with respect to the variables F_t, F_x, substituting F_t by $F_t - \frac{1}{K}P_t$ and F_x by $F_x - \frac{1}{K}P_x$ in $K^2Q_1, K^2\tilde{Q}_1$ yields polynomials $Q_0 \equiv K^2Q_1 \equiv K^3Q \pmod{\mathcal{I}}$ and $\tilde{Q}_0 \equiv K^2\tilde{Q}_1 \equiv K^3\tilde{Q}$ such that $Q_0, \tilde{Q}_0 \in \mathbb{Q}[t, x, F]$. A computation (using for example Maple) shows that P divides Q_0 and $\tilde{Q}_0$ which ends the proof. $\Box$.

Proposition 5.5. *The algebraic equation*

$$t(1 + tx) - tW + 2tx^2W^2 + x^3W^4 = 0$$

has a unique solution W in $\mathbf{R} = (\mathbb{Q}[t, t^{-1}])[[x]]$.

This solution is a common solution of the two partial differential equations

$$\begin{aligned}
D_LW &= D_RW \\
\tilde{D}_LW &= \tilde{D}_RW
\end{aligned}$$

and satisfies the initial condition $W = 1 + tx \pmod{x^2}$.

Proof of Theorem 2.5 Follows from Propositions 5.1, 5.2 and 5.5. $\Box$

Proof of Proposition 5.5 The solution of

$$t(1 + tx) - tW + 2tx^2W^2 + x^3W^4 = 0$$

is the unique fixpoint in $\mathbf{R} = \left(\mathbb{C}[t, t^{-1}]\right)[[x]]$ of the attracting map $Z \longmapsto 1 + tx(1 + x/tZ^2)^2$.

Since the algebra $\mathbf{R}$ containing the solution W considered above is a differential algebra for both partial derivations $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, we can consider the homomorphism of algebras

$$\varphi : \mathbb{Q}[t, x, F, F_t, F_x, F_{tt}, F_{tx}, F_{xx}] \longrightarrow \mathbf{R}$$

defined by

$$\varphi(t) = t,\ \varphi(x) = x,\ \varphi(F_{t^\alpha x^\beta}) = \frac{\partial^{|\alpha+\beta|}}{\partial t^\alpha \partial x^\beta} W,\ \alpha + \beta \leq 2\ .$$

Since the polynomials $P_{t^\alpha x^\beta}$ are formally given by $\frac{\partial^{\alpha+\beta}}{\partial t^\alpha \partial x^\beta} P$, we have $\varphi(P_{t^\alpha x^\beta}) = 0$ for all $\alpha, \beta \in \mathbb{N}$. This implies the inclusion $\mathcal{I} \subset ker(\varphi)$ for the ideal $\mathcal{I} = (P, P_t, P_x, P_{tt}, P_{tx}, P_{xx})$.

The identities $\varphi(Q) = (D_L - D_R)W$ and $\varphi(\tilde{Q}) = (\tilde{D}_L - \tilde{D}_R)W$ and Lemma 5.4 imply thus the equalities $\varphi(K)^3(D_L - D_R)W = 0$ and $\varphi(K)^3(\tilde{D}_L - \tilde{D}_R)W = 0$. Since the algebra $\mathbf{R}$ has no zero divisors, and since $\varphi(K) \equiv -t \pmod{x^2}$, we have $(D_L - D_R)W = 0$ and $(\tilde{D}_L - \tilde{D}_R)W = 0$ with $W \equiv 1 + tx \pmod{x^2}$. □

Remark 5.6. *Lemma 5.4 can be replaced by the inclusions $Q^2, \tilde{Q} \in \mathcal{I}$ which can be checked by computing a Gröbner basis for $\mathcal{I}$. I thank Bernard Parisse who did the necessary computations using Xcas and CoCoa. The computation of a Gröbner basis is however typically quite long (several minutes in the above case) while the computations used for the previous proof of Lemma 5.4 are immediate.*

Remark 5.7. *Corollary 2.3 results also from the following computation which is analogous to the proof of Lemma 5.4: Replacing F_t by $F_t - \frac{1}{K}P_t$ and F_x by $F_x - \frac{1}{K}P_x$ in*

$$K\big(2t(1-F) + x(2xF - t)F_x + 3t(t + 2xF)F_t\big)$$

(with $K = -t + 4tx^2F + 4x^3F^3$ as above) we get $-(5t + 6xF)P$ which implies Corollary 2.3.

6 (Partial) proofs for $d-$dissections

Proof of Theorem 3.1 Setting $G = G(x) = xg_d(x^{d-1})$, the computation

$$G - G^d = x\left(g_d(x^{d-1}) - x^{d-1}g_d(x^{d-1})^{d-1}\right) = x$$

shows that $G(x) = x + \dots$ is the reciprocal function of $y \longmapsto y - y^d$. Since the coefficient γ_n of $g_d(x) = \sum_{n=0}^\infty \gamma_n x^n$ equals the coefficient of $x^{n(d-1)+1}$ in

$G(x) = xg_d(x^{d-1})$, Lagrange inversion (which states that $n\sigma_n \in \mathbb{C}$ is given by the coefficient of y^{n-1} in $(y/r(y))^n \in \mathbb{C}[[y]]$ if $\sum_{n=1}^\infty \sigma_n (r(y))^n = y$ for $r(y) = \sum_{n=1}^\infty \rho_n y^n \in \mathbb{C}[[y]]$, see eg. Satz 2.4 in [4] or Theorem 5.4.2 in [5]), shows that $(n(d-1)+1)\gamma_n$ is given by the coefficient of $y^{n(d-1)}$ in

$$\begin{aligned} \left(\frac{y}{y-y^d}\right)^{n(d-1)+1} &= \sum_{j=0}^\infty \binom{-n(d-1)-1}{j} (-1)^j y^{j(d-1)} \\ &= \sum_{j=0}^\infty \binom{n(d-1)+j}{j} y^{j(d-1)} \end{aligned}$$

and we have thus

$$\gamma_n = \binom{nd}{n} \frac{1}{n(d-1)+1}$$

as required. □

Proof of Theorem 3.2 Introducing $\tilde{W} = \tilde{W}(t,x) = W(t^{-1},x) \in \mathbf{R}$ and decomposing for $n \geq 1$ a $d-$dissection τ of $D_{(d-1)n} \supset e_*$ along the distinguished polygon $\Delta_* \in \tau$ containing e_*, we get the equations

$$\begin{aligned} W_d &= 1 + tx\tilde{W}_d^d \\ \tilde{W}_d &= 1 + t^{-1}xW_d^d \ . \end{aligned}$$

Eliminating $\tilde{W}$ yields the algebraic equation

$$W_d = 1 + tx\left(1 + t^{-1}xW_d^d\right)^d$$

of Theorem 3.2 for W. □

6.1 A partial "proof" for Conjecture 3.5

All steps but one in the proof of Theorem 2.5 work for arbitrary $d \geq 2$ and yield thus almost a proof of Conjecture 3.5. Failure occurs in the necessary machine computations which get more and more complicated for increasing values of d. We were however able to complete them for a few small values of d and thus to establish Theorem 3.6.

Given a constant $d \geq 2$, we consider the four following partial differential operators

$$D_L = dtx\left(1 - \frac{(d+1)}{2}t\frac{\partial}{\partial t} + \frac{(d-1)}{2}x\frac{\partial}{\partial x}\right) \prod_{j=1}^{d-1}\left(j + \frac{d}{2}t\frac{\partial}{\partial t} + \frac{d}{2}x\frac{\partial}{\partial x}\right)$$

$$D_R = \prod_{j=0}^{d-1}\left(-j + \frac{(d+1)}{2}t\frac{\partial}{\partial t} + \frac{(d-1)}{2}x\frac{\partial}{\partial x}\right)$$

$$\tilde{D}_L = dx\left(\frac{(d+1)}{2}t\frac{\partial}{\partial t} + \frac{(d-1)}{2}x\frac{\partial}{\partial x}\right)\prod_{j=1}^{d-1}\left(j - \frac{d}{2}t\frac{\partial}{\partial t} + \frac{d}{2}x\frac{\partial}{\partial x}\right)$$

$$\tilde{D}_R = t\prod_{j=0}^{d-1}\left(1 - j - \frac{(d+1)}{2}t\frac{\partial}{\partial t} + \frac{(d-1)}{2}x\frac{\partial}{\partial x}\right)$$

Proposition 6.1. *The two partial differential equations*

$$D_L F = D_R F$$

and

$$\tilde{D}_L F = \tilde{D}_R F$$

defined by the previous partial differential operators have at most a unique common solution $F \in \mathbf{R}$ satisfying the initial condition $F \equiv 1+tx \pmod{x^2}$.

Proof As in the proof of the special case $d = 2$ (see Proposition 5.1), a coefficient $(F, t^j x^m)$ of a common solution F is determined by the coefficients $(F, t^{j\pm 1}x^{m-1})$ except if $D_R(t^j x^m) = \tilde{D}_R(t^j x^m) = 0$. Such a pair of integers (j, m) satisfies the two linear equations

$$-a + \frac{d+1}{2}j + \frac{d-1}{2}m = 1 - b - \frac{d+1}{2}j + \frac{d-1}{2}m = 0$$

for some $a, b \in \{0, \ldots, d-1\}$. Adding these two linear equations we have

$$(d-1)m = a + b - 1 \leq 2d - 3$$

which shows $m \in \{0, 1\}$. A coefficient of t^j or of $t^j x$ in a solution F is however prescribed by the initial condition $F \equiv 1 + tx \pmod{x^2}$. □

Proposition 6.2. *We have*

$$D_L \tilde{W} = D_R \tilde{W}$$

and

$$\tilde{D}_L \tilde{W} = \tilde{D}_R \tilde{W}$$

for the series $\tilde{W} = \sum_{n=0}^{\infty} \tilde{w}_{d,n} x^n \equiv 1 + tx \pmod{x^2}$ defined by the formulae given in Conjecture 3.5.

Proof Follows from the formulae

$$D_L(x^{(d+1)n+1}\tilde{w}_{(d+1)n+1}) = dx^{(d+1)n+2}\sum_{k=1}^{(d-1)n} \frac{(d^2n+d-d(k-1)-1)!\ (dn+d+d(k-1))!\ t^{(d-1)n-2(k-1)}}{(n+(k-1)+1)!\ ((d^2-1)n-(d+1)(k-1)-1)!\ (d(k-1)+d+(k-1))!\ (dn-(k-1))!}$$

$$D_L(x^{(d+1)n+j}\tilde{w}_{(d+1)n+j}) = dx^{(d+1)n+(j+1)}\sum_{k=0}^{(d-1)n+j-2} \frac{(d^2n+dj-dk-1)!\ (dn+d(k+1))!\ t^{(d-1)n+(j+1)-2(k+1)}}{(n+(k+1))!\ ((d^2-1)n+dj-(d+1)(k+1))!\ ((d+1)(k+1)-j)!\ (dn+(j+1)-(k+1)-1)!}$$

$$D_R(x^{(d+1)n+1}\tilde{w}_{(d+1)n+1}) = x^{(d+1)n+1}\sum_{k=0}^{(d-1)n} \frac{(d^2n+d-dk)!\ (dn+dk)!\ t^{(d-1)n+1-2k}}{(n+k)!\ ((d^2-1)n-(d+1)k)!\ ((d+1)k)!\ (dn+1-k)!}$$

$$D_R(x^{(d+1)n+j}\tilde{w}_{(d+1)n+j}) = x^{(d+1)n+j}\sum_{k=0}^{(d-1)n+j-3} \frac{(d^2n+d(j-1)-dk)!\ (dn+d(k+1))!\ t^{(d-1)n+j-2-2k}}{(n+k+1)!\ ((d^2-1)n+d(j-2)-(d+1)k-1)!\ ((d+1)k+d-j+2)!\ (dn+j-k-1)!}$$

$$\tilde{D}_L(x^{(d+1)n+1}\tilde{w}_{(d+1)n+1}) = dx^{(d+1)n+2}\sum_{k=0}^{(d-1)n} \frac{(d^2n+d-dk)!\ (dn+d+dk-1)!\ t^{(d-1)n+1-2k}}{(n+k)!\ ((d^2-1)n+d-(d+1)k-1)!\ (d+1)k)!\ (dn+1-k)!}$$

$$\tilde{D}_L(x^{(d+1)n+j}\tilde{w}_{(d+1)n+j}) = dx^{(d+1)n+(j+1)}\sum_{k=0}^{\omega_d(j)} \frac{(d^2n+d(j-1)-dk)!\ (dn+2d+dk-1))!\ t^{(d-1)n+j-2-2k}}{(n+k+1)!\ ((d^2-1)n+dj-(d+1)(k+1)-1)!\ ((d+1)k+d-j+2)!\ (dn+j-k-1)!}$$

where $\omega_d(j) = (d-1)n+j-2$ for $j \in \{2,3,\ldots,d\}$ and $\omega_d(d+1) = (d-1)n+d-2$,

$$\tilde{D}_R(x^{(d+1)n+1}\tilde{w}_{(d+1)n+1}) = x^{(d+1)n+1}\sum_{k=1}^{(d-1)n} \frac{(d^2n+d-dk)!\ (dn+dk)!\ t^{(d-1)n+2-2k}}{(n+k)!\ ((d^2-1)n+d-(d+1)k)!\ ((d+1)k-d)!\ (dn+1-k)!}$$

$$\tilde{D}_R(x^{(d+1)n+j}\tilde{w}_{(d+1)n+j}) = x^{(d+1)n+j}\sum_{k=\alpha_d(j)}^{(d-1)n+j-2} \frac{(d^2n+d(j-1)-dk)!\ (dn+d+dk)!\ t^{(d-1)n+j-1-2k}}{(n+k+1)!\ ((d^2-1)n+d(j-1)-(d+1)k-1)!\ ((d+1)k-j+2)!\ (dn+j-k-1)!}$$

where $\alpha_d(2) = 0$ and $\alpha_d(j) = 1$ for $j \in \{3,\ldots,d+1\}$, corresponding to Lemma 5.3 and from the observation that the formula of Conjecture 3.5 for $\tilde{w}_{d,(d+1)n+j}$ with $j = d+1$ coincides with the formula for $\tilde{w}_{d,(d+1)(n+1)}$. $\square$

6.2 An algebraic reformulation

The integer $d \geq 2$ is again fixed in this Section. We denote by $\mathcal{A}_h = \mathbb{Q}[t,x,(F_{t^\alpha x^\beta})_{\alpha+\beta\leq h}]$ the free algebra generated by t, x and all partial derivations $F_{t^\alpha x^\beta} = \frac{\partial^{\alpha+\beta}}{\partial t^\alpha\ \partial x^\beta}F$ of order $\alpha+\beta \leq h$ of an unknown analytic function $F = F(t,x)$. We suppose that there are no algebraic relations among partial derivations of F.

We set

$$P = -t^{d-1}F + t^{d-1} + x(t+xF^d)^d$$

and consider the ideal $\mathcal{I} \subset \mathcal{A}_d$ generated by $P_{t^\alpha x^\beta} = \frac{\partial^{\alpha+\beta}P}{\partial t^\alpha \partial x^\beta} \in \mathcal{A}_d$ for $\alpha+\beta \leq d$.

Lemma 6.3. *The polynomial* $P = -t^{d-1}F + t^{d-1} + x(t + xF^d)^d$ *is irreducible over* $\mathbb{C}[[t, x, F]]$ *for every integer* $d \geq 1$.

Proof For $d \geq 1$ fixed, consider the Newton-polytope

$$\mathcal{N}(P) = Conv(\{(a, b, c) \in \mathbb{N}^3 \ |(P, t^a x^b F^c) \neq 0\})$$

of P defined as the convex hull of all exponents associated to monomials involved in P. A straightforward computation shows that $\mathcal{N}(P)$ is the 3−dimensional simplex with vertices

$$(d - 1, 0, 1), (d - 1, 0, 0), (d, 1, 0), (0, 1 + d, d^2) \in \mathbb{N}^3 .$$

A factorisation $P = P_1P_2$ of P implies the equality $\mathcal{N}(P) = \mathcal{N}(P_1)+\mathcal{N}(P_2)$. where $\mathcal{N}(P_i)$ is the Newton polytope of the factor P_i. Since $\mathcal{N}(P)$ is a simplex, the polytope $\mathcal{N}(P_i)$ is a of the form $\lambda_i\mathcal{N}(P)+\tau_i$ with $\lambda_i \in [0, 1]$ and $\tau_i \in \mathbb{Q}^3$. Since the simplex $\mathcal{N}(P)$ has edges without interior integral vertices and since $\mathcal{N}(P_i)$ are polytopes with integral vertices, we have $\{\lambda_1, \lambda_2\} = \{0, 1\}$. Suppose $\lambda_1 = 0$. The polynomial P_1 is thus of the form $\mu t^k x^l W^m$ for some $\mu \in \mathbb{C}^*$ and $(k, l, m) \in \mathbb{N}^3$ which implies $P_1 \in \mathbb{C}^*$ by inspection of P.$\Box$

We consider now the three elements

$$Q = (D_L - D_R)F, \ \tilde{Q} = (\tilde{D}_L - \tilde{D}_R)F$$

and

$$K = \frac{\partial P}{\partial F} = -t^{d-1} + d^2x^2(t + xF^d)^{d-1}F^{d-1}$$

of $\mathcal{A}_d$.

Theorem 6.4. *Conjecture 3.5 holds if and only if we have the two inclusions* $K^N Q \in \mathcal{I}$ *and* $K^N \tilde{Q} \in \mathcal{I}$ *for* $N = \sum_{j=1}^{d} \lfloor \frac{d}{j} \rfloor$.

Proof Set $Q_d = Q$. For h such that $1 \leq h \leq d$, define Q_{h-1} by the substitutions

$$F_{t^\alpha x^{h-\alpha}} \longmapsto F_{t^\alpha x^{h-\alpha}} - \frac{1}{K} P_{t^\alpha x^{h-\alpha}}, \ 0 \leq \alpha \leq h$$

in $K^{d_h}Q_h$ where $d_h = \lfloor d/h \rfloor$ is the degree of Q_h with respect to the variables $F_{t^\alpha x^{h-\alpha}}, \ 0 \leq \alpha \leq h$. One shows by descending induction on h that $Q_{h-1} \equiv K^{d_h}Q_h \pmod{\mathcal{I}}$ is an element of the algebra $\mathcal{A}_{h-1}$. Since P is irreducible by Lemma 6.3, we have $Q_0 \in \mathcal{I}$ if and only if Q_0 is divisible by P which shows that $K^N Q \in \mathcal{I}$ for $N = \sum_{j=1}^{d} d_j$.

The proof proceeds then as in the case $d = 2$. The inclusion $K^N Q \in \mathcal{I}$ (and $K \equiv -t^{d-1} \pmod{x^2}$) implies $\varphi(Q) = (D_L - D_R)W = 0$ where $\varphi : \mathcal{A}_d \longrightarrow \mathbf{R} = \left(\mathbb{Q}[t, t^{-1}]\right)[[x]]$ is the homomorphism of algebras defined by $\varphi(t) = t, \varphi(x) = x$ and $\varphi(F_{t^\alpha x^\beta}) = \frac{\partial^{\alpha+\beta}}{\partial t^\alpha \partial x^\beta} W$ for $W = \sum_{n=0}^{\infty} w_{d,n}(t)x^n$. It contains $\mathcal{I}$ in its kernel. Repeating the above arguments with $\tilde{Q}$ ends the proof. $\Box$

Proof of Theorem 3.6 Using Maple 8, we checked the inclusions $K^N Q, K^N \tilde{Q} \in \mathcal{I}$ of Theorem 6.4 up to $d = 6$. $\Box$

Remark 6.5. *The following trick avoids the use of rational fractions in the substitutions* $F_{t^\alpha x^{h-\alpha}} \longmapsto F_{t^\alpha x^{h-\alpha}} - \frac{1}{K} P_{t^\alpha x^{h-\alpha}},\ 0 \leq \alpha \leq h$*: Write* $Q_h = \sum_{j=0}^{d_h} q_{h,j}$ *where* $q_{h,j}$ *is homogeneous of degree* j *with respect to the variables* $F_{t^\alpha x^{h-\alpha}}$*. We have then* $Q_{h-1} = \sum_{j=0}^{d_h} K^{d_h - j} \tilde{q}_{h,j}$ *where* $\tilde{q}_{h,j}$ *is obtained from* $q_{h,j}$ *by the substitutions* $F_{t^\alpha x^{h-\alpha}} \longmapsto K F_{t^\alpha x^{h-\alpha}} - P_{t^\alpha x^{h-\alpha}},\ 0 \leq \alpha \leq d_h$*.*

This reduces the computations for proving Theorem 3.6 to elementary operations on polynomials, a domain of excellence for symbolic computer algebra systems.

Acknowledgement I am gratefull to Bernard Parisse for a Gröbner basis computation establishing the very first proof of Theorem 2.5 and to Didier Piau for many suggestions improving the exposition.

References

[1] D. Callan, Personal communication (e-mail), 17 september 2007.

[2] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/∼njas/sequences/index.html

[3] F. Harary, E.M. Palmer, R.C. Read, *On the cell-growth problem for arbitrary polygons*, Discrete Maths **11** (1975), 371–389.

[4] P. Henrici, *Die Lagrange-Bürmannsche Formel bei formalen Potenzreihen*, Jahresber. Deutsch. Math.-Verein. **86** (1984), 115–134).

[5] R.P. Stanley, Enumerative Combinatorics, Volume 2, Cambridge University Press.

[6] L. Takacs, *Enumeration of rooted trees and forests*, Math. Scientist **18** (1993) No. 1, 1–10.

Roland BACHER
INSTITUT FOURIER
Laboratoire de Mathématiques
UMR 5582 (UJF-CNRS)
BP 74
38402 St Martin d'Hères Cedex (France)

e-mail: Roland.Bacher@ujf-grenoble.fr