# TOTAL POSITIVITY FOR COMINUSCULE GRASSMANNIANS

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Abstract. In this paper we explore the combinatorics of the non-negative part $(G/P)_{\geq 0}$ of a cominuscule Grassmannian. For each such Grassmannian we define ⅃-diagrams – certain fillings of generalized Young diagrams which are in bijection with the cells of $(G/P)_{\geq 0}$. In the classical cases, we describe ⅃-diagrams explicitly in terms of pattern avoidance. We also define a game on diagrams, by which one can reduce an arbitrary diagram to a ⅃-diagram. We give enumerative results and relate our ⅃-diagrams to other combinatorial objects. Surprisingly, the totally non-negative cells in the open Schubert cell of the odd and even orthogonal Grassmannians are (essentially) in bijection with preference functions and atomic preference functions respectively.

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## 1. Introduction

The classical theory of total positivity concerns matrices in which all minors are non-negative. While this theory was pioneered in the 1930's, interest in this subject has been renewed on account of the work of Lusztig [9, 10]. Motivated by surprising connections he discovered between his theory of canonical bases for quantum groups and the theory of total positivity, Lusztig extended this subject by introducing the totally non-negative points $G_{\geq 0}$ in an arbitrary reductive group

2000 *Mathematics Subject Classification.* Primary 05Exx; Secondary 20G20, 14Pxx.
*Key words and phrases.* Total positivity, Grassmannian, CW complexes.
T. L. was partially supported by NSF DMS–0600677.

$G$ and the totally non-negative part $(G/P)_{\geq 0}$ of a real flag variety $G/P$. Lusztig conjectured a cell decomposition for $(G/P)_{\geq 0}$, which was proved by Rietsch [15]. Cells of $(G/P)_{\geq 0}$ correspond to pairs $(x, w)$ where $x, w \in W$, $x \leq w$ in Bruhat order, and $w$ is a minimal-length coset representative of $W^J = W/W_J$. Here $W_J \subset W$ is the parabolic subgroup corresponding to $P$.

Coming from a more combinatorial perspective, Postnikov [12] explored the combinatorics of the totally non-negative part of the type $A$ Grassmannian. He described and parameterized cells using certain fillings of Young diagrams by 0's and +'s which he called ⅃-*diagrams*, and which are defined using the avoidance of the ⅃-*pattern*. The ⅃-diagrams seem to have a great deal of intrinsic interest: they were independently discovered by Cauchon [3] in the context of primes in quantum algebras (see also [8]); they are in bijection with other combinatorial objects, such as decorated permutations [12]; and they are linked to the asymmetric exclusion process [5].

In this paper we use work of Stembridge [20] and of Proctor [14], to generalize ⅃-diagrams to the case of cominuscule Grassmannians. In this case the poset $W^J$ is a distributive lattice and hence can be identified with the lattice of order ideals of another poset $Q^J$. It turns out that the poset $Q^J$ can always be embedded into a two-dimensional square lattice. Each $w \in W^J$ corresponds to an order ideal $O_w \subset Q^J$ which can be represented by a generalized Young diagram. We then identify cells of the non-negative part of a cominuscule Grassmannian with certain fillings, called ⅃-diagrams, of $O_w$ by 0's and +'s. Arbitrary fillings of $O_w$ by 0's and +'s correspond to subexpressions of a reduced expression for $w$; the ⅃-diagrams correspond to *positive distinguished subexpressions* [11].

We give concise descriptions of ⅃-diagrams for type $B$ and $D$ cominuscule Grassmannians in terms of pattern avoidance. Unfortunately there does not seem to exist a concise description for the remaining $E_7$ and $E_8$ cominuscule Grassmannians. We also define a game (the ⅃-*game*) that one can play on diagrams filled with 0's and +'s, by which one can go from any such diagram to a ⅃-diagram.

We then explore the combinatorial properties of ⅃-diagrams. We define type $B$ decorated permutations and show that they are in bijection with ⅃-diagrams. We give some formulas and recurrences for the numbers of ⅃-diagrams. Finally, we show that there are twice as many type $(B_n, n)$ ⅃-diagrams in the open Schubert cell as *preference functions* of length $n$, while type $(D_n, n)$ ⅃-diagrams in the open Schubert cell are in bijection with *atomic preference functions* of length $n$.

**Organization.** In Section 2, we give the relevant background on total positivity for flag varieties, and in Section 3, we give background on cominuscule Grassmannians. In Section 4, we introduce ⅃-diagrams, ⅃-moves, and the ⅃-game. The following five sections are devoted to characterizing ⅃-diagrams for the cominuscule Grassmannians of types $A$, $B$ and $D$. In Section 10, we review type $A$ decorated permutations and describe type B decorated permutations, and in Section 11, we give enumerative results, including those on preference functions.

**Acknowledgements.** We are grateful to Frank Sottile and Alex Postnikov for interesting discussions.

## 2. Total positivity for flag varieties

We recall basic facts concerning the totally non-negative part $(G/P_J)_{\geq 0}$ of a flag variety and its cell decomposition.

2.1. **Pinning.** Let $G$ be a semisimple linear algebraic group over $\mathbb{C}$ split over $\mathbb{R}$, with split torus $T$. Identify $G$ (and related spaces) with their real points and consider them with their real topology. Let $\Phi \subset \mathrm{Hom}(T, \mathbb{R}^*)$ the set of roots and choose a system of positive roots $\Phi^+$. Denote by $B^+$ the Borel subgroup corresponding to $\Phi^+$. Let $B^-$ be the opposite Borel subgroup $B^-$ such that $B^+ \cap B^- = T$. Let $U^+$ and $U^-$ be the unipotent radicals of $B^+$ and $B^-$.

Denote the set of simple roots by $\Pi = \{\alpha_i \mid i \in I\} \subset \Phi^+$. For each $\alpha_i \in \Pi$ there is an associated homomorphism $\phi_i : \mathrm{SL}_2 \to G$, generated by 1-parameter subgroups $x_i(t) \in U^+$, $y_i(t) \in U^-$, and $\alpha_i^\vee(t) \in T$. The datum $(T, B^+, B^-, x_i, y_i; i \in I)$ for $G$ is called a *pinning*. Let $W = N_G(T)/T$ be the Weyl group and for $w \in W$ let $\dot{w} \in N_G(T)$ denote a representative for $w$.

2.2. **Totally non-negative parts of flag varieties.** Let $J \subset I$. The parabolic subgroup $W_J \subset W$ corresponds to a parabolic subgroup $P_J$ in $G$ containing $B^+$. Namely, $P_J = \sqcup_{w \in W_J} B^+ \dot{w} B^+$. Let $\pi^J : G/B^+ \to G/P_J$ be the natural projection.

The totally non-negative part $U^-_{\geq 0}$ of $U^-$ is defined to be the semigroup in $U^-$ generated by the $y_i(t)$ for $t \in \mathbb{R}_{\geq 0}$. The totally non-negative part $(G/P_J)_{\geq 0}$ of the partial flag variety $G/P_J$ is the closure of the image of $U^-_{\geq 0}$ in $G/P_J$.

2.3. **Cell decomposition.** We have the Bruhat decompositions

$$G/B^+ = \sqcup_{w \in W} B^+ \dot{w} B^+/B^+ = \sqcup_{w \in W} B^- \dot{w} B^+/B^+$$

of $G/B^+$ into $B^+$-orbits called *Bruhat cells*, and $B^-$-orbits called *opposite Bruhat cells*. For $v, w \in W$ define

$$R_{v,w} := B^+ \dot{w} B^+/B^+ \cap B^- \dot{v} B^+/B^+.$$

The intersection $R_{v,w}$ is non-empty precisely if $v \leq w$, and in that case is irreducible of dimension $\ell(w) - \ell(v)$. Here $\leq$ denotes the *Bruhat order* (or *strong order*) of $W$ [2]. For $v, w \in W$ with $v \leq w$, let

$$R_{v,w;>0} := R_{v,w} \cap (G/B^+)_{\geq 0}.$$

We write $W^J$ for the set of minimal length coset representatives of $W/W_J$. The Bruhat order of $W^J$ is the order inherited by restriction from $W$. Let $\mathcal{I}^J \subset W \times W^J$ be the set of pairs $(x, w)$ with the property that $x \leq w$. Given $(x, w) \in \mathcal{I}^J$, we define $P^J_{x,w;>0} := \pi^J(R_{x,w;>0})$. This decomposition of $(G/P_J)_{\geq 0}$ was introduced by Lusztig [10]. Rietsch showed that this is a cell decomposition:

**Theorem 2.1.** [15] *The sets $P^J_{x,w;>0}$ are semi-algebraic cells of dimension $\ell(w) - \ell(x)$, giving a cell decomposition of $(G/P_J)_{\geq 0}$.*

In fact the cell decomposition of Theorem 2.1 is a CW complex [13, 17].

## 3. (Co)minuscule Grassmannians

We keep the notation of Section 2. We say the parabolic $P_J$ is *maximal* if $J = I \setminus \{j\}$ for some $j \in I$. We may then denote the parabolic by $P_j := P_J$ and the partial flag variety by $G/P_j := G/P_J$, which we loosely call a Grassmannian. Similarly, we use the notation $\mathcal{I}^j$, $W^j$, $W_j$ and $W^j_{\max}$.

For a maximal parabolic subgroup $P_j$ we will call $P_j$, the flag variety $G/P_j$, and the simple root $\alpha_j$ *cominuscule* if whenever $\alpha_j$ occurs in the simple root expansion of a positive root $\gamma$ it does so with coefficient one. Similarly, one obtains the definition of *minuscule* by replacing roots with coroots. The (co)minuscule Grassmannian's have been classified and are listed below, with the corresponding Dynkin diagrams (plus choice of simple root) shown in Figure 1.

**Proposition 3.1.** *The maximal parabolic $P_j$, the flag variety $G/P_j$, and the simple root $\alpha_j$ are (co)minuscule if we are in one of the following situations:*

1. *$W = A_n$ and $j \in [1, n]$ is arbitrary*
2. *$W = B_n$ (or $C_n$) and $j = 1$ or $n$*
3. *$W = D_n$ (with $n \geq 4$) and $j = 1, n-1$ or $n$*
4. *$W = E_6$ and $j = 1$ or $6$*
5. *$W = E_7$ and $j = 1$.*

For more details concerning this classification we refer the reader to [1].

Besides the Bruhat (strong) order, we also have the *weak order* on a parabolic quotient (see [2] for details). An element $w \in W$ is *fully commutative* if every pair of reduced words for $w$ are related by a sequence of relations of the form $s_i s_j = s_j s_i$. The following result is due to Stembridge [20] (part of the statement is due to Proctor [14]).

**Theorem 3.2.** *If $(W, j)$ is (co)minuscule then $W^j$ consists of fully commutative elements. Furthermore the weak order $(W^j, \prec)$ and strong order $(W^j, <)$ of $W^j$ coincide, and this partial order is a distributive lattice.*

Since $(W^j, \prec)$ and $(W^j, <)$ coincide, we will just refer to this partial order as $W^j$. We indicate in Figure 2 (mostly taken from [7]) the posets $Q^j$ such that $W^j = J(Q^j)$, where $J(P)$ denotes the distributive lattice of order ideals in $P$. Note that the posets are drawn in "French" notation so that minimal elements are at the bottom left. The diagrams should be interpreted as follows: each box represents an element of the poset $Q^j$, and if $b_1$ and $b_2$ are two adjacent boxes such that $b_1$ is immediately to the left or immediately below $b_2$, we have a cover relation $b_1 \lessdot b_2$ in $Q^j$. The partial order on $Q^j$ is the transitive closure of $\lessdot$. (In particular the labeling of boxes shown in Figure 2 does not affect the poset structure.)

We now state some facts about $Q^j$ which can be found in [20]. Let $w_0^j \in W^j$ denote the longest element in $W^j$. The simple generators $s_i$ used in a reduced expression for $w_0^j$ can be used to label $Q^j$ in a way which reflects the bijection between the minimal length coset representatives $w \in W^j$ and (lower) order ideals $O_w \subset Q^j$. Such a labeling is shown in Figure 2; the label $i$ stands for the simple

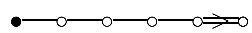
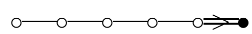
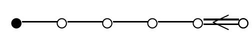
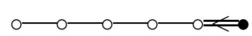
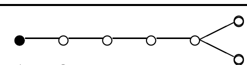
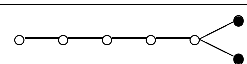
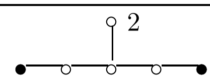
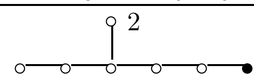
| Root system | Dynkin Diagram | Grassmannian |
|---|---|---|
| $A_n$ |  $1 \quad 2 \quad \cdots \quad j \quad \cdots \quad n$ | the usual Grassmannian $\mathrm{Gr}_{j,n+1}$ |
| $B_n, n \geq 2$ |  $1 \quad 2 \quad \cdots \quad \cdots \quad n$ | the odd dimensional quadric $\mathbb{Q}^{2n-1}$ |
| $B_n, n \geq 2$ |  $1 \quad 2 \quad \cdots \quad \cdots \quad n$ | odd orthogonal Grassmannian $\mathrm{OG}_{n,2n+1}$ |
| $C_n, n \geq 2$ |  $1 \quad 2 \quad \cdots \quad \cdots \quad n$ | the projective space $\mathbb{P}^{2n-1}$ |
| $C_n, n \geq 2$ |  $1 \quad 2 \quad \cdots \quad \cdots \quad n$ | the Lagrangian Grassmannian $\mathrm{LG}_{n,2n}$ |
| $D_n, n \geq 4$ |  $n$; $1 \quad 2 \quad \cdots \quad \cdots \quad n-1$ | the even dimensional quadric $\mathbb{Q}^{2n-2}$ |
| $D_n, n \geq 4$ |  $n$; $1 \quad 2 \quad \cdots \quad \cdots \quad n-1$ | even orthogonal Grassmannian $\mathrm{OG}_{n+1,2n+1}$ |
| $E_6$ |  $2$; $1 \quad 3 \quad 4 \quad 5 \quad 6$ | the real points of the Cayley plane $\mathbb{OP}^2$ |
| $E_7$ |  $2$; $1 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$ | the (real) Freudenthal variety Fr |

FIGURE 1. The (co)minuscule parabolic quotients

reflection $s_i$. If $b \in O_w$ is a box labelled by $i$, we denote the simple generator labeling $b$ by $s_b := s_i$; the corresponding index $i \in I$ is the *simple label* of $b$.

Given this labeling, if $O_w$ is an order ideal in $Q^j$, the set of linear extensions $\{e : O_w \to [1, \ell(w)]\}$ of $O_w$ are in bijection with the reduced words $R(w)$ of $w$: the reduced word (written down from right to left) is obtained by reading the labels of $O_w$ in the order specified by $e$. We will call the linear extensions of $O_w$ *reading orders*. (Alternatively, one may think of a linear extension of $O_w$ as a standard tableau with shape $O_w$.)

*Remark* 3.3. We use the following conventions: we do not distinguish between the root systems $B_n$ and $C_n$ since we are only interested in the posets $W^j$ (thus we refer to the Weyl group of both root systems as $B_n$); for $W = D_n$ we always pick $j = 1$ or $n$ since the case $j = n - 1$ is essentially the same as the case $j = n$; similarly for $W = E_6$ we always pick $j = 1$.

*Remark* 3.4. In the literature, the two cases *minuscule* and *cominuscule* are usually distinguished. This distinction will not be important for our applications.

| Parabolic quotient | $Q^j$ |
|---|---|
| $W = A_{n-1}$ | $\begin{array}{ccccc} 1&2&3&4&5\\ 2&3&4&5&6\\ 3&4&5&6&7 \end{array}$ $(n = 8 \text{ and } j = 3)$ |
| $W = B_n$ and $j = 1$ | $\begin{array}{ccccccc} 1&2&3&4&3&2&1 \end{array}$ $(n = 4)$ |
| $W = B_n$ and $j = n$ | $\begin{array}{cccc} 1&2&3&4\\ 2&3&4&\\ 3&4&&\\ 4&&& \end{array}$ $(n = 4)$ |
| $W = D_n$ and $j = 1$ | $\begin{array}{cccccc} &&5&3&2&1\\ 1&2&3&4&& \end{array}$ $(n = 5)$ |
| $W = D_n$ and $j = n$ | $\begin{array}{cccc} 1&2&3&4\\ 2&3&5&\\ 3&4&&\\ 5&&& \end{array}$ $(n = 5)$ |
| $W = E_6$ and $j = 1$ | $\begin{array}{cccccccc} &&&1&3&4&5&6\\ &&&3&4&2&&\\ &&2&4&5&&&\\ 1&3&4&5&6&&& \end{array}$ |
| $W = E_7$ and $j = 1$ | $\begin{array}{ccccccccc} &&&&&&&&7\\ &&&&&&&&6\\ &&&&&&&&5\\ &&&&&&&2&4\\ &&&&7&6&5&4&3\\ &&&&6&5&4&3&1\\ &&&&5&4&2&&\\ &&&2&4&3&&&\\ 7&6&5&4&3&1&&& \end{array}$ |

Figure 2. Underlying posets of parabolic quotients

## 4. Ⅎ-diagrams, Ⅎ-moves and the Ⅎ-game

4.1. **Positive distinguished subexpressions.** In this subsection we give background on distinguished and positive distinguished subexpressions; for more details, see [6] and [11]. Consider a reduced expression in $W$, say $s_3s_2s_1s_3s_2s_3$ in type $A_3$. We define a *subexpression* to be a word obtained from a reduced expression by replacing some of the factors with 1. For example, $s_3 s_2\, 1\, s_3 s_2\, 1$ is a subexpression of $s_3s_2s_1s_3s_2s_3$. Given a reduced expression $\mathbf{w} := s_{i_1} s_{i_2} \dots s_{i_n}$ for $w$, we set $w_{(k)} := s_{i_1} s_{i_2} \dots s_{i_k}$ if $k \geq 1$ and $w_{(0)} = 1$. The following definition was given in [11] and was implicit in [6].

**Definition 4.1** (Positive distinguished subexpressions)**.** *Let* $\mathbf{w} := s_{i_1}, \dots, s_{i_n}$ *be a reduced expression. We call a subexpression* $\mathbf{v}$ *of* $\mathbf{w}$ *positive distinguished if*

$$v_{(j-1)} < v_{(j-1)} s_{i_j} \tag{1}$$

*for all* $j = 1, \dots, n$.

Note that (1) is equivalent to $v_{(j-1)} \leq v_{(j)} \leq v_{(j-1)} s_{i_j}$. We will refer to a positive distinguished subexpression as a PDS for short.

**Lemma 4.2.** [11] *Given* $v \leq w$ *in* $W$ *and a reduced expression* $\mathbf{w}$ *for* $w$*, there is a unique PDS* $\mathbf{v}_+$ *for* $v$ *in* $\mathbf{w}$.

4.2. **⊕-diagrams and ⅃-diagrams.** The goal of this section is to identify the PDS's with certain fillings of the boxes of order ideals of $Q^j$.

Let $O_w$ be an order ideal of $Q^j$, where $w \in W^j$.

**Definition 4.3.** *An ⊕-diagram ("o-plus diagram") of shape* $O_w$ *is a filling of the boxes of* $O_w$ *with the symbols* $0$ *and* $+$.

Clearly there are $2^{\ell(w)}$ ⊕-diagrams of shape $O_w$. The value of an ⊕-diagram $D$ at a box $x$ is denoted $D(x)$. Let $e$ be a reading order for $O_w$; this gives rise to a reduced expression $\mathbf{w} = \mathbf{w}_e$ for $w$. The ⊕-diagrams $D$ of shape $O_w$ are in bijection with subexpressions $\mathbf{v}(D)$ of $\mathbf{w}$: we will make the seemingly unnatural specification that if a box $b \in O_w$ is filled with a 0 then the corresponding simple generator $s_b$ is present in the subexpression, while if $b$ is filled with a $+$ then we omit the corresponding simple generator. The subexpression $\mathbf{v}(D)$ in turn defines a Weyl group element $v := v(D) \in W$.

**Example 4.4.** *Consider the order ideal* $O_w$ *which is* $Q^j$ *itself for type* $A_4$ *with* $j = 2$*. Then* $Q^j$ *is the following poset*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline \end{array}.$$

*Let us choose the reading order (linear extension) indicated by the labeling below:*

$$\begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline 1 & 2 & 3 \\ \hline \end{array}.$$

*Then the ⊕-diagrams*

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 0 & + & 0 \\ \hline 0 & 0 & + \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline + & 0 & + \\ \hline \end{array}$$

*correspond to the expressions* $s_3s_2s_1s_4s_3s_2$, $s_3 1 s_1 1 s_3 s_2$ *and* $s_3s_2s_1 1 s_3 1$. *The first and the last are PDS's while the second one is not, since it is not reduced.*

We next show that $v(D)$ does not depend on the linear extension $e$. The following statement can be obtained by inspection.

**Lemma 4.5.** *If $b, b' \in O_w$ are two incomparable boxes, $s_b$ and $s_{b'}$ commute.*

Lemma 4.5 implies the following statement.

**Proposition 4.6.** *Let $D$ be an $\oplus$-diagram. Then*

(1) *the element $v := v(D)$ is independent of the choice of reading word $e$.*
(2) *whether $\mathbf{v}(D)$ is a PDS depends only on $D$ (and not $e$).*

*Proof.* For part (1), note that two linear extensions of the same poset (viewed as permutations of the elements of the poset) can be connected via transpositions of pairs of incomparable elements. By Lemma 4.5, $v(D)$ is therefore independent of the choice of reading word.

Suppose $D$ is an $\oplus$-diagram of shape $O_w$, and consider the reduced expression $\mathbf{w} := \mathbf{w}_e = s_{i_1} \dots s_{i_n}$ corresponding to a linear extension $e$. Suppose $\mathbf{v}(D)$ is a PDS of $\mathbf{w}$. For part (2), it suffices to show that if we swap the $k$-th and $(k+1)$-st letters of both $\mathbf{w}$ and $\mathbf{v}(D)$, where these positions correspond to incomparable boxes in $O_w$, then the resulting subexpression $\mathbf{v}'$ will be a PDS of the resulting reduced expression $\mathbf{w}'$. If we examine the four cases (based on whether the $k$-th and $(k+1)$-st letters of $\mathbf{v}(D)$ are 1 or $s_{i_k}$) it is clear from the definition that $\mathbf{v}'$ is a PDS. □

Proposition 4.6 allows us to make the following definition.

**Definition 4.7.** *A ⅃-diagram of shape $O_w$ is an $\oplus$-diagram $D$ of shape $O_w$ such that $\mathbf{v}(D)$ is a PDS.*

The following statement follows immediately from Lemma 4.2 and Theorem 2.1.

**Proposition 4.8.** *The cells of $(G/P_j)_{\geq 0}$ defined in Theorem 2.1 are in bijection with pairs $(D, O_w)$ where $O_w$ is an order ideal in $Q^j$ and $D$ is a ⅃-diagram of shape $O_w$. Furthermore, the cell labeled by $(D, O_w)$ is isomorphic to $(\mathbb{R}^+)^s$ where $s$ is the number of $+$'s in $D$.*

Let us now state one of the main aims of this work.

**Problem 4.9.** *Give a compact description of ⅃-diagrams.*

4.3. **The ⅃-game.** Let $D$ be an $\oplus$-diagram of shape $O_w$ corresponding to an element $v(D) \in W$. By Lemma 4.2 and Proposition 4.6 there is a unique ⅃-diagram $D_+$ with $v(D_+) = v(D)$. We call $D_+$ the ⅃-ification of $D$.

**Problem 4.10.** *Describe how to produce $D_+$ from $D$.*

Our solution to Problem 4.10 will be algorithmic, involving a series of game-like moves. Suppose $C \subset O_w$ is a convex subset: that is, if $x$ and $y$ are in $C$ then any $z$ such that $x < z < y$ must also be in $C$. We may extend the definition of $\oplus$-diagrams to $C$. In addition Proposition 4.6 still holds for $\oplus$-diagrams of shape $C$. If $D$ is an $\oplus$-diagram of shape $C$ we again denote by $v(D) \in W$ the corresponding

Weyl group element. If $S : C \to \{0, +, ?\}$ is a filling of $C$ with the symbols 0, + and ?, we say that an $\oplus$-diagram $D$ is compatible with $S$ if for every $x \in C$

(1) $D(x) = 0 \implies S(x) \in \{0, ?\}$, and
(2) $D(x) = + \implies S(x) \in \{+, ?\}$.

If $x, y \in O_w$ are two boxes we let $(x, y) = \{z \in O_w \mid x < z < y\}$ be the open interval between $x$ and $y$. Similarly, define the half open intervals $(x, y]$ and $[x, y)$.

**Definition 4.11.** *A ⅃-move $M$ is a triple $(x, y, S)$ consisting of a pair $x < y \in O_w$ of comparable, distinct boxes together with a filling of the open interval $S : (x, y) \to \{0, +, ?\}$ such that*

$$v(D \cup x) = v(D \cup y) \tag{2}$$

*for every $\oplus$-diagram $D$ of shape $(x, y)$ compatible with $S$. Here $D \cup x$ ($D \cup y$) is the $\oplus$-diagram of shape $[x, y)$ ($(x, y]$) obtained from $D$ by placing a 0 in $x$ ($y$). We say that $(x, y, S)$ is a ⅃-move from $y$ to $x$ via $S$.*

*Now if $D$ is an $\oplus$-diagram whose shape contains $[x, y]$, we say that a ⅃-move $M = (x, y, S)$ can be performed on $D$ if $D(y) = 0$ and $D|_{(x,y)}$ is compatible with $S$. The result of $M$ on $D$ is then the $\oplus$-diagram $D'$ obtained from $D$ by setting $D(y) = +$ and switching the entry of $D(x)$ (that is, $D'(x) = 0$ if $D(x) = +$ and $D'(x) = +$ if $D(x) = 0$).*

*Remark* 4.12. Let the simple generator corresponding to the box $x$ (resp. $y$) be the simple root $\alpha$ (resp. $\beta$). Then (2) is equivalent to $v(D)s_\alpha = s_\beta v(D)$ which in turn is equivalent to

$$v(D)^{-1} \cdot \beta = \alpha. \tag{3}$$

For two $\{0, +, ?\}$-fillings $S, S'$ of the same shape let us say that $S'$ is a specialization of $S$ (and $S$ a generalization of $S'$) if $S'$ is obtained from $S$ by changing some ?'s to 0's or +'s. It is then clear from the definition that if $S'$ is a specialization of $S$ and $(x, y, S)$ is a ⅃-move then so is $(x, y, S')$.

The following lemma is immediate from the definitions.

**Lemma 4.13.** *If $D'$ is obtained from $D$ by a sequence of ⅃-moves, $v(D') = v(D)$.*

Performing a ⅃-move on an $\oplus$-diagram $D$ either reduces the number of 0's or moves a 0 to a box which is smaller in the partial order (and the + to a bigger box). Thus any sequence of ⅃-moves must eventually terminate.

**Proposition 4.14.** *No ⅃-moves can be performed on a ⅃-diagram. Every $\oplus$-diagram $D$ can be ⅃-ified by a finite sequence of ⅃-moves.*

*Proof.* Let us assume that a reading order has been fixed for $O_w$ and let $n = \ell(w)$. It is known ([11, Lemma 3.5]) that the unique PDS $\mathbf{v}_+ = t_1 t_2 \ldots t_n$ for $v$ can be constructed greedily from the right. More precisely, we have that $v_{(n)} = v$, and once we have determined $t_i \ldots t_n$ we can determine $v_{(i-1)}$; to construct $\mathbf{v}_+$ we set

$$t_j = \begin{cases} s_{i_j} & \text{if } v_{(j)} s_{i_j} < v_{(j)} \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

The application of a ⅃-move shifts simple generators to the right in the corresponding word. Since $\mathbf{v}_+$ already corresponds to the rightmost word, we deduce that no ⅃-moves can be performed on a ⅃-diagram.

Now suppose an $\oplus$-diagram $D$ is not a ⅃-diagram. Let $D$ differ from its ⅃-ification $D_+$ at a box $b$ where $b$ is chosen to be as early as possible in the reading order. By the greedy property of a PDS, $D(b) = +$ and $D_+(b) = 0$. Denote the set of boxes occurring after $b$ in the reading order by $A \subset O_w$. Then $\mathbf{v}(D_+)$ has the form $\mathbf{v}((D_+)|_A)s_b\mathbf{v}'$ for some $\mathbf{v}'$ and $\mathbf{v}(D)$ has the form $\mathbf{v}((D|_A)\mathbf{v}'$, which implies that $v(D|_A)s_b = v((D_+)|_A)$ and $v((D_+)|_A) < v(D|_A)$. Thus by the exchange axiom, $v(D|_A)s_b$ is obtained by omitting a simple generator from $\mathbf{v}(D|_A)$. Let $b'$ be the box corresponding to this simple generator; then the ⅃-move $(b, b', v(D|_{(b,b')}))$ can be performed on $D$. Repeating this, we eventually obtain $D_+$. $\square$

We say that a set $\mathbb{S}$ of ⅃-moves is *complete* if every $\oplus$-diagram $D$ can be ⅃-ified using ⅃-moves in $\mathbb{S}$ only.

**Problem 4.15.** *Describe a complete set of* ⅃*-moves.*

## 5. Type $A_{n-1}$

In this section we will give a compact description of ⅃-diagrams in type $A_{n-1}$ and observe that they are the same as the ⅃-diagrams defined by Postnikov [12]. Let $(W, j) = (A_{n-1}, j)$ so that any $O_w$ can be identified with a Young diagram within a $j \times (n-j)$ rectangle.

**Theorem 5.1.** *An $\oplus$-diagram of shape $O_w$ in type $A_{n-1}$ is a ⅃-diagram if and only if there is no* 0 *which has a* $+$ *below it and a* $+$ *to its left.*

In Theorem 5.1, "below" means below and in the same column, while "to its left" means to the left and in the same row. If an $\oplus$-diagram satisfies these condition, we say that it possesses the ⅃-condition. Theorem 5.1 can be proved using the wiring-diagram argument from [12, Theorem 19.1]. This is similar to the proof of the (much) more difficult Theorem 8.1 below. Instead, our proof below will appeal to the fact that the cells of the type $A_{n-1}$ Grassmannians have previously been enumerated.

Let $x < y$ be two distinct, comparable boxes in $O_w$. Then $[x, y]$ is a rectangle (or as a poset, a product of chains). Given $x < y$, let $S_0$ denote the following $\{0, +, ?\}$ filling of $(x, y)$:

$$\begin{array}{|c|c|c|c|c|} \hline + & 0 & 0 & 0 & y \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline x & 0 & 0 & 0 & + \\ \hline \end{array} \tag{5}$$

That is, $S_0$ is filled with 0's except for the top left and bottom right corners, where it is filled with $+$'s.

**Proposition 5.2.** *The triples $(x, y, S_0)$ defined above are* ⅃*-moves.*

We will call the ⅃-moves $(x, y, S_0)$ the *rectangular* ⅃-moves.

*Proof.* For simplicity and concreteness let us suppose that the top left hand $+$ lies on the diagonal with corresponding simple generator $s_1$, and that the rectangle $[x, y]$ has $r \geq 2$ rows and $c \geq 2$ columns. We use the criterion for a ⅃-move described in Remark 4.12. Note that $\alpha = \alpha_r$ and $\beta = \alpha_c$.

Since $S_0$ has no ?'s we need only check (3) for $D = S_0$. Furthermore we pick the reading order obtained by reading the rows from left to right starting from the bottom row:

| | | | | |
|---|---|---|---|---|
| 15 | 16 | 17 | 18 | |
| 10 | 11 | 12 | 13 | 14 |
| 5 | 6 | 7 | 8 | 9 |
| | 1 | 2 | 3 | 4 |

We calculate using the notation $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$,

$$\begin{aligned}
&v(D)^{-1} \cdot \alpha_c \\
&= (s_{r+1} \cdots s_{r+c-1})(s_{r-1} s_r \cdots s_{r+c-2}) \cdots (s_2 s_3 \cdots s_{c+1})(\hat{s}_1 s_2 s_3 \cdots s_{c-1})\alpha_c \\
&= (s_{r+1} \cdots s_{r+c-1})(s_{r-1} s_r \cdots s_{r+c-2}) \cdots (s_2 s_3 \cdots s_{c+1})\alpha_{2,c} \\
&= (s_{r+1} \cdots s_{r+c-1})(s_{r-1} s_r \cdots s_{r+c-2}) \cdots \alpha_{3,c+1} \\
&= \cdots \\
&= (s_{r+1} \cdots s_{r+c-2})\alpha_{r,r+c-2} \\
&= \alpha_r.
\end{aligned}$$

This proves that $(x, y, S_0)$ is indeed a ⅃-move. □

**Theorem 5.3.** *These* ⅃*-moves form a complete system of* ⅃*-moves.*

*Proof of Theorems 5.1 and 5.3.* Let $D$ be an $\oplus$-diagram which does not satisfy the ⅃-condition. Let $y$ be one of the boxes closest to the bottom left which contains a 0 violating the ⅃-condition. Let $z_1$ ($z_2$) be the box to the left of (below) $y$ containing a $+$ which is closest to $y$. Let $x$ be the box which forms a rectangle with $y$, $z_1$, and $z_2$. We claim that $D|_{(x,y)} = S_0$ as in (5).

| | | | | |
|---|---|---|---|---|
| $z_1$ | 0 | 0 | 0 | $y$ |
| | | | | 0 |
| | | | | 0 |
| $x$ | | | | $z_2$ |

Otherwise there is a box $t \in (x, y) - \{z_1, z_2\}$ containing a $+$. We pick $t$ closest to $y$. If $t$ is not below $z_1$ then above $t$ is a box $y'$ in the same row as $y$ such that $D(y') = 0$. This $y'$ thus violates the ⅃-condition and is closer to the bottom left than $y$, a contradiction. A similar argument holds if $t$ is not to the left of $z_2$. We conclude that $t$ does not exist.

Thus the rectangular ⅃-move $(x, y, S_0)$ can be performed on $D$. Therefore the ⅃-diagrams must be a subset of those $\oplus$-diagrams which satisfy the ⅃-condition, that is, such that there is no 0 which has a $+$ below it and a $+$ to its left. But in fact it has been shown that the $\oplus$-diagrams satisfying the ⅃-condition are in bijection with pairs $(x, w)$ where $x \in W$, $w \in W^J$, and $x \leq w$ [12, 22]. Therefore the ⅃-diagrams must be exactly those $\oplus$-diagrams satisfying the ⅃-condition. This proves Theorems 5.1 and 5.3. $\square$

In [12], Postnikov studied the totally non-negative part of the type A Grassmannian $(Gr_{k,n})_{\geq 0}$, and showed that it has a cell decomposition where cells are in bijection with certain combinatorial objects he called ⅃-diagrams. Postnikov's ⅃-diagrams are obtained from ours by reflecting in a horizontal axis. Since Postnikov was using the English convention for Young diagrams whereas we are using French, Theorem 5.1 shows that our definition of ⅃-diagrams is consistent with Postnikov's definition.

## 6. Type $(B_n, n)$

Now let $(W, j) = (B_n, n)$ so that $O_w \subset Q^j$ can be identified with a shape (a lower order ideal) within a staircase of size $n$. We refer to the $n$ boxes along the diagonal of $Q^j$ as the diagonal boxes.

**Theorem 6.1.** *A type $(B_n, n)$ ⅃-diagram is an $\oplus$-diagram $D$ of shape $O_w$ such that*

(1) *if there is a $0$ above (and in the same column as) a $+$ then all boxes to the left and in the same row as that $0$ must also be $0$'s.*
(2) *any diagonal box containing a $0$ must have only $0$'s to the left of it.*

If an $\oplus$-diagram $D$ satisfies the conditions above we will say that it satisfies the ⅃-conditions.

We now provide some ⅃-moves which will turn out to be complete. Let $x < y$ be two distinct, comparable boxes in $O_w$ such that $[x, y]$ is a rectangle. Denote by $S_0$ the filling of $(x, y)$ as in (5). The following result is proved in the same manner as Proposition 5.2.

**Proposition 6.2.** *The triples $(x, y, S_0)$ defined above are ⅃-moves.*

We will call the ⅃-moves $(x, y, S_0)$ the *rectangular* ⅃-moves.

Now let $x < y$ be two distinct diagonal boxes, so that $[x, y]$ is itself a staircase. Denote by $S_1$ the following filling of $(x, y)$:

$$\begin{array}{|c|c|c|c|c|} \hline + & 0 & 0 & 0 & y \\ \hline 0 & 0 & 0 & 0 & \\ \hline 0 & 0 & 0 & & \\ \hline 0 & 0 & & & \\ \hline x & & & & \\ \hline \end{array}$$

In other words, $S_1$ is filled with 0's with the exception of the top-left corner box.

**Proposition 6.3.** *The triples $(x, y, S_1)$ defined above are ⅃-moves.*

We call the ⅃-moves $(x, y, S_1)$ diagonal ⅃-moves.

*Proof.* We follow the same general strategy as in the proof of Proposition 5.2, again using the row reading order. Let us assume that the top-left corner box of $(x, y)$ is labeled by simple generator $s_k$. We calculate, using the notation $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$,

$$\begin{aligned}
&v(D)^{-1} \cdot \alpha_n \\
&= (s_{n-1}s_n) \cdots (s_{k+1} \cdots s_{n-1}s_n)(s_{k+1} \cdots s_{n-2}s_{n-1})\alpha_n \\
&= (s_{n-1}s_n) \cdots (s_{k+1} \cdots s_{n-1}s_n)\alpha_{k+1,n} \\
&= (s_{n-1}s_n) \cdots \alpha_{k+2,n} \\
&= (s_{n-1}s_n)(\alpha_{n-1} + \alpha_n) \\
&= \alpha_n.
\end{aligned}$$

This proves that $(x, y, S_1)$ is indeed a ⅃-move. □

**Theorem 6.4.** *The ⅃-moves $(x, y, S_0)$ and $(x, y, S_1)$ form a complete system of ⅃-moves.*

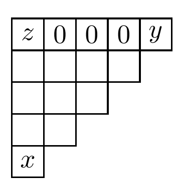
Before we prove Theorems 6.1 and 6.4, we recall the basic facts concerning the representation of $B_n$ as signed permutations (see [2]). Let us identify the type $A_{2n-1}$ Weyl group with the symmetric group $S_{\{\pm 1,\ldots,\pm n\}}$. There is a homomorphism $\iota$ from the $B_n$ Weyl group with generators $s_1, \ldots, s_n$ to $S_{\{\pm 1,\ldots,\pm n\}}$, which sends $s_n$ to $(-1, 1)$ and $s_i$ to the "signed transposition" $(n-i, n-i+1)(-(n-i), -(n-i+1))$. This map is bijective onto the set of $\pi \in S_{\{\pm 1,\ldots,\pm n\}}$ such that $\pi(i) = -\pi(-i)$, called *signed permutations*. The Bruhat order on $B_n$ agrees with the order on signed permutations inherited from type $A_{2n-1}$ Bruhat order.

The embedding $\iota : B_n \to S_{\{\pm 1,\ldots,\pm n\}}$ allows us to identify a type $(B_n, n)$ $\oplus$-diagram $D$ of shape $O_w$ with the type $(A_{2n-1}, n)$ $\oplus$-diagram $\iota(D)$ of shape $O_{\iota(w)}$ obtained by reflecting $D$ over the diagonal $y = x$. The following observation is clear from the definitions.

(6) If $\mathbf{v}(\iota(D))$ is a PDS of $\iota(w)$ then $\mathbf{v}(D)$ is a PDS of $w$.

*Proof of Theorems 6.1 and 6.4.* Let $D$ be an $\oplus$-diagram. If $D$ violates condition (1) of Theorem 6.1 then a rectangular ⅃-move can be performed on it, as in the proof of Theorem 5.1. Otherwise, suppose $D$ violates condition (2) of Theorem 6.1.

Let $y$ be the diagonal box containing the 0 violating condition (2) closest to the bottom left and let $z$ be the box in the same row as $y$ containing a $+$ and closest to $y$. Let $x$ be the diagonal box in the same column as $z$. We claim that $D|_{(x,y)} = S_1$. Using the fact that $D$ satisfies condition (1) of Theorem 6.1 we deduce that $D|_{(x,y)}$ contains only 0's along the diagonal. Using the assumption that $y$ was chosen closest to the bottom left we then deduce that $D_{(x,y)} = S_1$.



This shows that $(x, y, S_1)$ can be performed on $D$. Thus after a finite sequence of the moves $(x, y, S_0)$ and $(x, y, S_1)$, the $\oplus$-diagram $D$ can be made to satisfy the ⅃-conditions. In particular, a ⅃-diagram must satisfy the ⅃-conditions.

Conversely, suppose an $\oplus$-diagram $D$ satisfies the ⅃-conditions of Theorem 6.1. A comparison of the ⅃-conditions of Theorems 5.1 and 6.1 implies that $\iota(D)$ (obtained by reflecting $D$ in the diagonal) is a type $(A_{2n-1}, n)$ ⅃-diagram. Thus $\mathbf{v}(\iota(D))$ is a PDS, hence by (6), $\mathbf{v}(D)$ is a PDS. Therefore $D$ is a type $(B_n, n)$ ⅃-diagram. □

## 7. Type $(B_n, 1)$

Now let $(W, j) = (B_n, 1)$ so that $O_w \subset Q^j$ can be identified with a single row. We call the box labeled $n$ (if contained in $O_w$) the *middle* box, and any two boxes with the same simple label *conjugate*. The conjugate of the middle box is itself.

**Theorem 7.1.** *A type $(B_n, 1)$ ⅃-diagram is an $\oplus$-diagram $D$ of shape $O_w$ such that if there is a $0$ to the right of the middle box, then the box $b$ immediately to the left of this $0$ and the conjugate $b'$ to $b$ cannot both contain $+$'s.*

*Proof.* Suppose $D$ and $D'$ are two $\oplus$-diagrams of shape $O_w$ so that $v(D) = v(D')$. Then the words corresponding to $\mathbf{v}(D)$ and $\mathbf{v}(D')$ are related by relations of the form $s_i s_j = s_j s_i$ and $s_i^2 = 1$; that is, no braid relation is required. This readily implies the description stated. □

Let $x < y$ be a pair of conjugate boxes in $O_w$. Let $S_0$ denote the following filling of $(x, y)$:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | $+$ | ? | ? | ? | ? | ? | $+$ | $y$ |

The following claim is immediate.

**Proposition 7.2.** *The triples $(x, y, S_0)$ defined above are ⅃-moves.*

**Theorem 7.3.** *The ⅃-moves $(x, y, S_0)$ form a complete system of ⅃-moves.*

## 8. Type $(D_n, n)$

Now let $(W, j) = (D_n, n)$ so that $O_w \subset Q^j$ can be identified with a shape contained inside a staircase. We refer to the $n$ boxes along the diagonal of $Q^j$ as the diagonal boxes. The distance of a box $b$ from the diagonal is the number of boxes that $b$ is on top of, so that a diagonal box has distance $0$ from the diagonal.

In the following we will say that a box $b$ is to the left or right (above or below) another $b'$ if and only if they are also in the same row (column). We will use compass directions when the same row or column condition is not intended.

**Theorem 8.1.** *A type $(D_n, n)$ ⅃-diagram is an $\oplus$-diagram $D$ of shape $O_w$ such that*

(1) *if there is a $0$ above a $+$ then all boxes to the left of that $0$ must also be $0$'s.*
(2) *if there is a $0$ with distance $d$ from the diagonal to the right of a $+$ in box $b$ then there is no $+$ strictly southwest of $b$ and $d+1$ rows south of the $0$.*

(3) *one cannot find a box $c$ containing a $0$ and three distinct boxes $b_1, b_2, b_3$ containing $+$'s so that $c$ has distance $d$ from the diagonal and is to the right of $b_1$, the box $b_2$ is the box $d+1$ rows below $b_1$, and finally $b_3$ is strictly northwest of $b_2$ and strictly south of $b_1$.*

An $\oplus$-diagram $D$ satisfying the conditions of Theorem 8.1 is said to satisfy the ⅃-conditions.

We now provide a complete set of ⅃-moves. Let $x < y$ be two distinct, comparable boxes in $O_w$ such that $[x, y]$ is a rectangle. Denote by $S_0$ the filling of $(x, y)$ as in (5). The following result is proved in the same manner as Proposition 5.2.

**Proposition 8.2.** *The triples $(x, y, S_0)$ defined above are ⅃-moves.*

We will call the ⅃-moves $(x, y, S_0)$ the *rectangular* ⅃-moves.

Now let $x < y$ be two distinct boxes so that $x$ is $c$ columns west of $y$ and $r$ rows south. Let $y$ be distance $d$ from the diagonal. We suppose that $r > d+1$ and set $k = r - (d+1)$. Denote by $S_1$ the following $\{0, +, ?\}$-filling of $(x, y)$:

| | | | | | |
|---|---|---|---|---|---|
| ? | ? | + | 0 | 0 | $y$ |
| ? | ? | 0 | 0 | 0 | 0 |
| ? | ? | 0 | 0 | 0 | 0 |
| + | 0 | ? | 0 | 0 | |
| 0 | 0 | 0 | 0 | | |
| 0 | 0 | 0 | | | |
| $x$ | 0 | | | | |

where

(1) the $+$ in the row of $y$ is $k$ boxes to the left of $y$,
(2) the $+$ in the column of $x$ is $k$ boxes above $x$. Our assumptions imply that this $+$ is southwest of the first $+$ and is $d+1$ rows south,
(3) the box below the first $+$ and to the right of the second $+$ is a ?, and
(4) the remaining boxes are filled with 0's except for the boxes both west of the first $+$ and north of the second $+$.

**Proposition 8.3.** *The triples $(x, y, S_1)$ defined above are ⅃-moves.*

*Proof.* We follow the same general strategy as in the proof of Proposition 5.2, again using the row reading order. Let us assume that the top-left corner box of $(x, y)$ is labeled 0 (for readability) and that the diagonal box below $y$ is labeled by $n$ rather than $n-1$. We lose no generality here since there is an automorphism of the $D_n$ Weyl group swapping $s_n$ and $s_{n-1}$ and fixing all other generators. Our assumptions give the picture:

| | | | | | | |
|---|---|---|---|---|---|---|
| $0$ | | | $c-k$ | | | $c$ |
| $1$ | | | | | | |
| | | | | | $n-2$ | $n$ |
| $d+1$ | | | $?$ | | $n-1$ | |
| | | | | $n$ | | |
| | | | | | | |
| $r$ | | $m$ | | | | |

where the label $m$ of the diagonal box to the right of $x$ depends on the parity of $k$, and the central ? is labeled $c-k+d+1$. Let $m^*$ denote $n$ if $m = n-1$ and vice versa. Note also that $n = c+d+1$.

In the following we use the notation $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$ (with $\alpha_{i+1,i} = 0$), the notation $S_a^b = s_a s_{a+1} \cdots s_b$ and also $\overline{s_j}$ to indicate a simple generator which may or may not be present. We assume $k \geq 2$; otherwise the calculation is even simpler.

$$
\begin{aligned}
&v(D)^{-1} s_{\alpha_c} \\
&= (S_{r+1}^{n-2} s_m)(S_{r-1}^{n-2} s_{m^*}) \cdots (S_{d+2}^{n-2} s_n)(S_{d+2}^{c-k+d} \overline{s_{c-k+d+1}} S_{c-k+d+2}^{n-2} s_{n-1}) \\
&\qquad (\overline{S_d^{c-k+d-1}} S_{c-k+d}^{n-2} s_n) \cdots (\overline{S_1^{c-k}} S_{c-k+1}^{c+1})(\overline{S_0^{c-k-1}} S_{c-k+1}^{c-1}) \alpha_c \\
&= \cdots (\overline{S_1^{c-k}} S_{c-k+1}^{c+1}) \alpha_{c-k+1,c} \\
&= \cdots (\overline{S_d^{c-k+d-1}} S_{c-k+d}^{n-2} s_n) \cdots \alpha_{c-k+2,c+1} \\
&= \cdots (\overline{S_d^{c-k+d-1}} S_{c-k+d}^{n-2} s_n) \alpha_{c+d+1-k,n-2} \\
&= \cdots (S_{d+2}^{c-k+d} \overline{s_{c-k+d+1}} S_{c-k+d+2}^{n-2} s_{n-1}) \alpha_{c+d+1-k,n-2} + \alpha_n \\
&= \cdots (S_{d+2}^{n-2} s_n) \alpha_{d+2,n} + \alpha_{c+d+2-k,n-2} \\
&= \cdots (S_{r-1}^{n-2} s_{m^*}) \cdots \alpha_{d+3,n} + \alpha_{c+d+3-k,n-2} \\
&= \cdots (S_{r-1}^{n-2} s_{m^*}) \alpha_{r-1,n} \quad \text{since } c-k+r-1 = n-1 \text{ we have } \alpha_{c-k+r-1,n-2} = 0 \\
&= (S_{r+1}^{n-2} s_m) \alpha_{r,n-2} + \alpha_m \\
&= \alpha_r.
\end{aligned}
$$

This proves that the triples $(x, y, S_1)$ are indeed Ⅎ-moves. □

Now we define a third kind of Ⅎ-move $(x, y, S_2)$. We keep the same assumptions and notation for $x$ and $y$ as for $(x, y, S_1)$. However, now given $x$ and $y$ there is more than one choice for $S_2$. Denote by $S_2$ (one of) the following $\{0, +, ?\}$-fillings of $(x, y)$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ? | ? | ? | ? | + | 0 | 0 | $y$ |
| ? | ? | ? | ? | 0 | 0 | 0 | 0 |
| + | 0 | 0 | 0 | + | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | ? | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | + | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | 0 | | |
| 0 | 0 | 0 | 0 | 0 | | | |
| $x$ | 0 | 0 | 0 | | | | |

where

(1) the + (called $z_1$) in the row of $y$ is $k$ boxes to the left of $y$,
(2) the lower + (called $z_2$) below $z_1$ is $k$ rows north of $x$ or alternatively $d+1$ rows south of $y$,
(3) the remaining two +'s are chosen on the same but *any* row strictly south of $z_1$ and north of $z_2$: one of these (called $z_4$) is in the same column as $z_1$ and $z_2$ while the other (called $z_3$) is in the same column as $x$, and
(4) the remaining boxes are filled with 0's except for: the boxes which are strictly west of $z_1$ and strictly north of $z_3$; and the boxes between (and in the same column as) $z_2$ and $z_4$.

The following result is proved in the same manner as Proposition 8.3. In fact the half of the calculation below $z_2$ is identical.

**Proposition 8.4.** *The triples $(x, y, S_2)$ defined above are ⅃-moves.*

**Theorem 8.5.** *The ⅃-moves $(x, y, S_0)$, $(x, y, S_1)$ and $(x, y, S_2)$ form a complete system of ⅃-moves.*

*Proof of Theorems 8.1 and 8.5.* We first show that the ⊕-diagrams satisfying the ⅃-conditions correspond to PDS's. It is well known [2] that $D_n$ Weyl group elements can be identified as signed permutations on the $2n$ letters $\{\pm 1, \pm 2, \ldots, \pm n\}$ which are *even*: that is, have an even number of signs in positions 1 through $n$. This is achieved by the map $\delta$ which sends $s_n \mapsto (1, -2)(2, -1)$ and $s_i \mapsto (n-i, n-i+1)(i-n, i-n-1)$ for $1 \le i \le n-1$. Note that $\delta$ does *not* preserve Bruhat order.

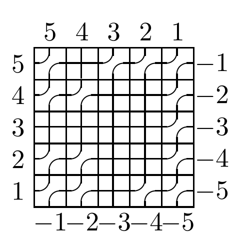
Using $\delta$, we obtain a type $(A_{2n-1}, n)$ ⊕-diagram from a type $(D_n, n)$ ⊕-diagram $D$. The type $(A_{2n-1}, n)$ ⊕-diagram can be converted to a wiring diagram wire$(D)$ in a $n \times n$ square (+'s become elbows and 0's become crosses). For example:

| | | | |
|---|---|---|---|
| + | 0 | + | + |
| + | + | 0 | |
| 0 | 0 | | |
| + | | | |

$\longrightarrow$

| | | | | |
|---|---|---|---|---|
| + | 0 | + | + | + |
| + | + | 0 | 0 | + |
| 0 | 0 | 0 | 0 | + |
| + | + | 0 | + | 0 |
| + | + | 0 | + | + |

$\longrightarrow$




Note that most boxes are replaced by an elbow or a cross in the same position and the diagonal-symmetric position. However, boxes corresponding to the simple generator $s_n$ are replaced by a $2 \times 2$ square of boxes all containing either elbows or crosses.

The condition for a wiring diagram to be the wiring diagram of a PDS is the following: two wires $p, q$ which cross in a square corresponding to $b \in D$ are not allowed to touch or cross again (as we read from northwest to southwest), except when that touching/crossing happens in one of the two by two squares corresponding to $s_n$. If $p, q$ both enter a two by two square corresponding to a diagonal square $b' \in D$, then the requirement is instead that the effect on wire$(D)$ of changing $b$ from a 0 to a + is not the same as the effect of changing $b'$ between a + and a 0.

We allow touching/crossing again in that two by two square as long as not all four boxes are touching/crossing.

Now suppose $D$ is an $\oplus$-diagram satisfying all three conditions of the theorem. If $D$ does not correspond to a PDS then by Proposition 4.14, a ⅃-move can be performed. Let us, as in Proposition 4.14, pick the southwestern-most such ⅃-move. Thus we have two boxes $x$ and $y$, where $y$ is filled with a 0 and $x$ is southwest of $y$.

For the ⅃-move to be valid – i.e. for the signed permutation to be unchanged by the ⅃-move – the wires which cross in box $y$ of wire$(D)$ must cross or touch again in box $x$. Here if $x$ or $y$ corresponds to a generator $s_n$ then one must consider the entire $2 \times 2$ square of wires. Suppose first that $y$ corresponds to a simple generator $s_i$ for $i \neq n$, and let wires $a, b$ cross in $y$ (we use $y$ to refer to the box in $D$ and also wire$(D)$). Say $y$ is in column $c$, using always the labeling of the wiring diagram.

For $a$ and $b$ to cross again, there must be a + to the left of $y$, so by (1) there is no + below. Suppose the closest + to the left of $y$ is in column $c'$. Let us suppose first that the wire $a$ travels down and passes straight through the diagonal, while the wire $b$ travels leftwards before turning at the first +. In this case, by (1) and (2), wire $a$ must make a turn in row $c'$, resulting in a + in position $(c, c')$. However, using conditions (2) and (3) we see that it is not possible for wire $b$ to travel below row $c$, and so can never meet $a$ again. Now suppose that wire $a$ does not cross the diagonal. This is only possible if the diagonal square $b$ of $D$ below $y$ corresponds to simple generator $s_{n-1}$ and the diagonal square $z$ immediately southwest of $b$ is a +. Using the conditions (1),(2) and (3) we obtain a picture similar to

| | | | |
|---|---|---|---|
| ? | ? | + | $y$ |
| 0 | 0 | ? | 0 |
| 0 | 0 | ? | **0** |
| 0 | 0 | **+** | |

where diagonal boxes are in bold. The wires $a$ and $b$ can only touch at the box $z$. But setting $x = z$ is not a valid ⅃-move, since the effect of changing $z$ is to swap $a$ and $-b$, not to swap $a$ and $b$.

Finally, suppose $y$ corresponds to the simple generator $s_n$. Then again there must be a + to the left of $y$, and automatically we deduce that one wire (say $a$) travels down through the diagonal while wire $b$ travels to the left and turns at the closest +. The argument for this case is the same as before: the wires $a$ and $b$ never touch again. Thus if $D$ satisfies the ⅃-conditions it must be a ⅃-diagram.

Let $D$ be an $\oplus$-diagram. We shall show that if $D$ does not satisfy the ⅃-conditions then one of the ⅃-moves $(x, y, S_0)$, $(x, y, S_1)$ and $(x, y, S_2)$ can be applied to it, which will complete the proof. If $D$ violates the ⅃-condition (1) of Theorem 8.1 then a rectangular ⅃-move can be performed on it, as in the proof of Theorem 5.1. Otherwise, suppose $D$ violates either condition (2) or (3) of Theorem 8.1.

Let $y$ be the box containing the 0 violating condition (2) or (3) closest to the bottom left. Suppose $y$ is distance $d$ from the diagonal. Let $z_1$ be the box to the left of $y$ containing a + which is closest to $y$.

Suppose first that there is a box $z_2$ such that $(y, z_1, z_2)$ violates condition (2). Pick $z_2$ rightmost with this property. Let $z_1$ be $k$ boxes to the left of $y$ and let $x$ be the box $k$ boxes below $z_2$. We claim that $D|_{(x,y)}$ is compatible with $S_1$ and shall explain the claim pictorially. Using condition (1) and the rightmost property of $z_2$ we may deduce at least the following information:

$$\begin{array}{cccccc}
? & ? & z_1 & 0 & 0 & y \\
? & ? & ? & 0 & 0 & 0 \\
? & ? & ? & 0 & 0 & 0 \\
z_2 & 0 & ? & 0 & 0 & \\
? & 0 & ? & 0 & & \\
? & 0 & ? & & & \\
x & 0 & & & &
\end{array}$$

To deduce the location of the remaining 0's we need to use the assumption that $y$ is the bottom leftmost box containing a 0 violating conditions (2) or (3). The 0's to the left of $y$ allow us to deduce that the ?'s in the rows between $x$ and $z_2$ are 0's. The 0's below $y$ allow us to deduce that the ?'s north of $z_2$ and below $z_1$ are also 0's. This shows that $D|_{(x,y)}$ is compatible with $S_1$ and so the ⅃-move $(x, y, S_1)$ can be performed on $D$.

If $y$ does not participate in a pattern of type (2) but *does* participate in a pattern of type 3, then there is no + southwest of $z_1$ and $(d+1)$ rows south. Using condition (1), there must be a box $z_2$ containing + which is $(d+1)$ rows below $z_1$, and there is a $z_3$ so that $(y, z_1, z_2, z_3)$ violates condition (3). We assume $z_3$ is chosen as south and as east as possible (there may be more than one choice). Let $x$ be the box $k$ rows south of $z_2$ and in the same column as $z_3$, where $z_1$ is $k$ boxes to the left of $y$. We claim that $D|_{(x,y)}$ is compatible with $S_2$ and shall explain the claim pictorially. The following information can be deduced using the eastmost-ness of $z_1$ and $z_3$ and condition (1).

$$\begin{array}{cccccccc}
? & ? & ? & ? & z_1 & 0 & 0 & y \\
? & ? & ? & ? & ? & 0 & 0 & 0 \\
z_3 & 0 & 0 & 0 & z_4 & 0 & 0 & 0 \\
? & 0 & 0 & 0 & ? & 0 & 0 & 0 \\
? & 0 & 0 & 0 & z_2 & 0 & 0 & \\
? & 0 & 0 & 0 & ? & 0 & & \\
? & 0 & 0 & 0 & ? & & & \\
x & 0 & 0 & 0 & & & &
\end{array}$$

The 0's to the right of $z_2$ allow us to deduce that the ?'s above $x$ and south of $z_2$ are 0's. The southmost-ness of $z_3$ and the assumption that $y$ and $z_1$ are not involved in a violation of condition (2) gives us the remaining 0's between $z_3$ and $x$. The 0's between $y$ and $z_1$ and the assumption on $y$ being as southwest as possible allows us to deduce that the ?'s below $z_2$ are 0's. Finally, the southwest assumption on $y$ allows us to deduce that the ?'s between $z_1$ and $z_4$ are 0's. This shows that $D|_{(x,y)}$ is compatible with $S_2$ and so the ⅃-move $(x, y, S_2)$ can be performed on $D$.

□

*Remark* 8.6. Theorems 5.1 and 6.1 can also be proved using wiring diagrams.

## 9. Type $(D_n, 1)$

Now let $(W, j) = (D_n, 1)$ so that $O_w \subset Q^j$ can be identified with a shape contained inside the doubled tail diamond of size $n$. The analysis for this case is nearly identical to type $(B_n, 1)$.

We call the boxes labeled $n-1$ and $n$ (if contained in $O_w$) the *middle* boxes, and any two boxes with the same simple label *conjugate*. If $b$ is a middle box then the conjugate of $b$ is the other middle box. The proof of the following statement is the same as for Theorem 7.1.

**Theorem 9.1.** *A type $(D_n, 1)$ ⅃-diagram is an $\oplus$-diagram $D$ of shape $O_w$ such that if there is a 0 in a box $c$ greater than the middle boxes, then for any box $b \lessdot c$ covered by $c$ the conjugate $b'$ and the box $b$ itself cannot both contain $+$'s.*

Let $x < y$ be a pair of conjugate and comparable boxes in $O_w$. Let $S_0$ denote the following filling of $(x, y)$:

$$\begin{array}{cccccccc} & & & \boxed{?} & \boxed{?} & \boxed{?} & \boxed{+} & \boxed{y} \\ \boxed{x} & \boxed{+} & \boxed{?} & \boxed{?} & \boxed{?} & & & \end{array}$$

if $(x, y)$ is not adjacent to the middle boxes and

$$\begin{array}{cc} \boxed{+} & \boxed{y} \\ \boxed{x} & \boxed{+} \end{array}$$

otherwise.

The following claim is immediate.

**Proposition 9.2.** *The triples $(x, y, S_0)$ defined above are ⅃-moves.*

**Theorem 9.3.** *The ⅃-moves $(x, y, S_0)$ form a complete system of ⅃-moves.*

## 10. Decorated permutations for types $A$ and $B$

In [12], Postnikov defined decorated permutations, proved that they are in bijection with type $A$ ⅃-diagrams, and described the partial order on totally positive Grassmannian cells in terms of decorated permutations. We will review decorated permutations in the type $A$ case and then describe type B decorated permutations.

10.1. **Type A decorated permutations.** In this section we will fix $W = S_n$, the symmetric group, with standard generators $\{s_i\}$, $j \in \{1, \dots n\}$, and $W_J = \{s_1, \dots, \hat{s_j}, \dots, s_n\}$. Recall from Section 2 that $R^j$ denotes the poset of cells of the corresponding Grassmannian.

A *decorated permutation* $\tilde{\pi} = (\pi, d)$ is a permutation $\pi$ in the symmetric group $S_n$ together with a coloring (decoration) $d$ of its fixed points $\pi(i) = i$ by two colors. Usually we refer to these two colors as "clockwise" and "counterclockwise", for reasons which the partial order will make clear. When writing a decorated permutation in one-line notation, we will put a bar over the clockwise fixed points. A nonexcedance of a (decorated) permutation $\tilde{\pi}$ is an index $i \in [n]$ such that either $\pi(i) < i$ or $\pi(i) = i$ and $i$ is labeled clockwise.

Let $\text{⅃}(j, n)$ denote the set of type $(A_{n-1}, j)$ ⅃-diagrams and let $\mathcal{D}_{j,n}$ denote the set of decorated permutations on $n$ letters with $j$ nonexcedances. Let us refer to the maximal order ideal in $Q^j$ as the *maximal shape*. So for example in type $A_n$ the maximal shape for $Q^j$ is a $j \times (n - j)$ rectangle.

Much of the content of the following result can be found in [12, 22].

**Theorem 10.1.** *There exist maps $\Phi_1, \Phi_2$, and $\Phi_3$, such that the following diagram commutes.*

$$\begin{tikzcd} \mathcal{I}^j \arrow[dr, "\Phi_1"'] & & \text{⅃}(j,n) \arrow[ll, "\Phi_2"'] \arrow[dl, "\Phi_3"] \\ & \mathcal{D}_{j,n} & \end{tikzcd}$$

10.1.1. *From pairs of permutations to decorated permutations.* The bijection $\Phi_1$ was stated (with slightly different conventions) in the appendix of [22]. Let $(v, w) \in \mathcal{I}^j$. Then $\Phi_1((v, w)) = (\pi, d)$ where $\pi = vw^{-1}$. To define the decoration $d$, we make any fixed point that occurs in one of the positions $w(1), w(2), \dots, w(k)$ a *clockwise loop* – a nonexcedance – and we make any fixed point that occurs in one of the positions $w(k + 1), \dots, w(n)$ a *counterclockwise loop* – a weak excedance. The fact that $\Phi_1$ is a bijection will be established in Section 10.1.4.

10.1.2. *From pairs of permutations to ⅃-diagrams (and back).* To define $\Phi_2$, we may simply take a ⅃-diagram $D$ of shape $O_w$ to the pair $(v(D), w)$. It follows from Proposition 4.8 that this is a bijection.

The map $\Phi_2$ can also be described as follows (see also [12]). View an $\oplus$-diagram $D$ within the maximal shape and label the unit steps of the northeast border of the maximal shape with the numbers from 1 to $n$ (from northwest to southeast); we label the southwest border with the numbers from 1 to $n$ (from northwest to southeast). The map $\Phi_2$ is defined by interpreting an $\oplus$-diagram $D$ as a wiring diagram; replace each 0 with a ┼ and each + with a ╯╭. By starting from the southwest labels of the border and traveling northeast we can read off a permutation $v$. If we replace all boxes with a $+$ and perform the above procedure we can read off a permutation which we will call $w$. Then $\Phi_2(D) = (v, w)$.

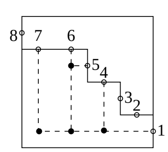
10.1.3. *From ⅃-diagrams to decorated permutations.* Now we describe $\Phi_3$. Given a ⅃-diagram $D$ of shape $O_w$, we label the northeast border of $O_w$ with the numbers 1 to $n$ from northwest to southeast, just as in the definition of $\Phi_2$. We ignore the 0's in the ⅃-diagram and replace each + with a vertex as well as a "hook" which extends north and east (ending at the boundary of $O_w$). We let $G(D)$ denote the graph which is the union of the hooks and the vertices. Now define a permutation $\pi$ as follows. If $i$ is the label of a horizontal unit step, then we start at $i$, and travel along $G(D)$, first traveling as far south as possible, and then zigzagging east and north, turning wherever possible (at each new vertex). Then $\pi(i)$ is defined to be the endpoint of this path. Clearly $\pi(i) \geq i$. Similarly, if $i$ is the label of a vertical unit step, then we start at $i$, and travel along $G(D)$, first traveling as far west as possible, and then zigzagging north and east, turning wherever possible. As before, $\pi(i)$ is defined to be the endpoint of this path, and clearly $\pi(i) \leq i$. If $i$ is the step which cannot travel anywhere then $i$ becomes a counterclockwise fixed point (weak excedance) if the step is horizontal and$i$ becomes a clockwise fixed point (nonexcedance) if the step is vertical. This map was proved to be a bijection in [19] and is a simplification of a map found by Postnikov.

*Remark* 10.2. If we consider a clockwise fixed point to be a nonexcedance and a counterclockwise fixed point to be a weak excedance, then it is clear that $\Phi_3$ maps ⅃-diagrams contained in a $j \times n - j$ rectangle to permutations in $S_n$ with precisely $j$ nonexcedences. Clearly even more is true; the shape of the Young diagram $O_w$ determines the positions of the nonexcedences and weak excedances.

**Example 10.3.** *Consider the following ⅃-diagram $D$ (viewed inside of the $j$ by $n - j$ rectangle associated to the corresponding Grassmannian $Gr_{j,n}$).*

| | | | |
|---|---|---|---|
| | | | |
| 0 | + | | |
| 0 | 0 | 0 | |
| + | + | + | 0 |

*Then $\Phi_2(D) = ((1, 3, 6, 2, 4, 5, 8, 7), (1, 4, 6, 8, 2, 3, 5, 7))$. To compute $\Phi_3(D)$, we construct the following graph $G(D)$. Following the procedure explained above, the*




*resulting permutation is $(\overline{1}, 4, 5, 3, 8, 6, 7, 2)$.*

10.1.4. *Proof of Theorem 10.1.* If we compare the definition of $\Phi_3$ to $\Phi_1$ and $\Phi_2$, it is clear that what $\Phi_3$ does is to interpret the ⅃-diagram as two wiring diagrams (one for $v$ and one for $w$) and then to compute $w^{-1}$ followed by $v$. Thus $\Phi_3 = \Phi_1 \circ \Phi_2$, proving the commutativity. Since $\Phi_2$ and $\Phi_3$ are bijections, so is $\Phi_1$.

10.2. **Type B decorated permutations and type B permutation tableaux.** We now describe the type $(B_n, n)$ analogue of Theorem 10.1. Let $\tilde{s}_0, \tilde{s}_1, \dots, \tilde{s}_{n-1}$ denote the Coxeter generators of the type $B_n$ Coxeter group, where $\tilde{s}_0$ labels the special node of the Dynkin diagram. *Note that in this section only we are departing from the earlier notation set up in Section 3 by using* $0$ *rather than* $n$ *for the special generator.* To avoid confusion, we refer to all objects as "type $B_n$" objects without specifying the cominuscule node.

Recall from Section 7 that $B_n$ Weyl group elements can be thought of as signed permutations via the embedding $\iota$ into $S_{\{\pm 1,\dots,\pm n\}}$. We will use the notation $(a_1, \dots, a_n)$ to denote the signed permutation $\pi$ such that $\pi(i) = a_i$.

We define a $B_n$ *decorated permutation* to be a signed permutation in which fixed points are either labeled clockwise or counterclockwise, and such that $\pi(i)$ is a clockwise fixed point if and only if $\pi(-i)$ is a counterclockwise fixed point. When writing a type $B$ decorated permutation in list notation, we will indicate a clockwise fixed point by putting a bar over the corresponding letter.

In this section our parabolic subgroup $W_J$ will be $\{\tilde{s}_1, \dots, \tilde{s}_{n-1}\}$. The pairs $(v, w)$ of Theorem 2.1 will be denoted $\mathcal{I}^B$. Let $\reflectbox{L}^B(n)$ denote the set of type $B$ $\reflectbox{L}$-diagrams contained in the maximal shape (this time an inverted staircase), and let $\mathcal{D}_n^B$ denote the set of type $B$ decorated permutations on the letters $\{\pm 1, \dots, \pm n\}$.

**Theorem 10.4.** *There exist bijections* $\Phi_1^B, \Phi_2^B$*, and* $\Phi_3^B$*, such that the following diagram commutes.*

$$\begin{tikzcd} \mathcal{I}^B \arrow[dr, "\Phi_1^B"'] & & \reflectbox{L}^B(n) \arrow[ll, "\Phi_2^B"'] \arrow[dl, "\Phi_3^B"] \\ & \mathcal{D}_n^B & \end{tikzcd}$$

*Proof.* Let $\theta_{\reflectbox{L}} : \reflectbox{L}(n, 2n) \to \reflectbox{L}(n, 2n)$ denote the involution of type $(A_{2n-1}, n)$ $\reflectbox{L}$-diagrams obtained by reflection in the diagonal. Reflecting a type $B$ $\reflectbox{L}$-diagram in the diagonal defines an embedding $\iota_{\reflectbox{L}} : \reflectbox{L}^B(n) \to \reflectbox{L}(n, 2n)$ such that the image of $\iota_{\reflectbox{L}}$ is the fixed points of $\theta_{\reflectbox{L}}$.

Let $\mathcal{I}^n$ denote the parametrising set of Theorem 2.1 for $(A_{2n-1}, n)$ where we take the symmetric group to be $S_{\{\pm 1,\dots,\pm n\}}$. Define $\iota_{\mathcal{I}} : \mathcal{I}^B \to \mathcal{I}^n$ by $\iota_{\mathcal{I}}(x, w) = (\iota(x), \iota(w))$ where on the right hand side we use $\iota : B_n \to S_{\{\pm 1,\dots,\pm n\}}$. This map makes sense since $\iota$ is a Bruhat embedding [2]. The image of $\iota_{\mathcal{I}}$ is the set of fixed points of the map $\theta_{\mathcal{I}}$, obtained by applying $\theta : S_{\{\pm 1,\dots,\pm n\}} \to S_{\{\pm 1,\dots,\pm n\}}$ given by $\theta(\pi)(i) = -\pi(-i)$, to each of a pair of permutations.

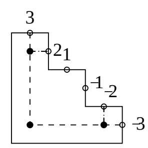
Similarly, define $\iota_{\mathcal{D}} : \mathcal{D}_n^B \to \mathcal{D}(n, 2n)$. Again the image is the set of fixed points of $\theta_{\mathcal{D}} : \mathcal{D}(n, 2n) \to \mathcal{D}(n, 2n)$ the map induced by $\theta$. Here if $\pi(i) = i$ is a fixed point labeled clockwise (resp. counterclockwise) then $\theta_{\mathcal{D}}(\pi)(-i) = -i$ is a fixed point labeled counterclockwise (resp. clockwise).

By the diagonal reflection invariance of the definitions of $\Phi_1, \Phi_2, \Phi_3$ in Theorem 10.1 we see that the three bijections commute with the respective involutions $\theta_{\reflectbox{L}}, \theta_{\mathcal{D}}$, and $\theta_{\mathcal{I}}$. We may thus restrict Theorem 10.1 to the fixed points of $\theta_{\reflectbox{L}}, \theta_{\mathcal{D}}$, and $\theta_{\mathcal{I}}$, giving the stated result. $\square$

**Example 10.5.** *Consider the following $\reflectbox{L}$-diagram $D$.*

$$\begin{array}{|c|c|c|}\hline + & & \\ \hline 0 & 0 & \\ \cline{1-2} + & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

*This diagram corresponds to the element $(v, w) = (\tilde{s}_0\tilde{s}_1, \tilde{s}_2\tilde{s}_0\tilde{s}_1\tilde{s}_0)$ in $\mathcal{I}^0$, which in list notation is equal to $((2, -1, 3), (-3, -1, 2))$. To compute the corresponding decorated permutation we construct from $D$ the following graph $G(D)$. Then the*




*resulting decorated permutation is $(\overline{1}, 3, -2)$.*

10.2.1. *Type $B$ permutation tableaux.* We define a type $B_n$ *permutation tableau* to be a type $(B_n, n)$ $\reflectbox{L}$-diagram $D$ of shape $O_w$ which contains no all-zero column. In other words, if a column of $O_w$ has at least one box, not all boxes in that column may be 0 in $D$. Since $\Phi_3^B$ is a bijection, the type $B_n$ permutation tableaux are in bijection with the set of type $B_n$ decorated permutations such that all fixed points in positions 1 through $n$ are counterclockwise. Thus the type $B_n$ permutation tableaux are in bijection with the set of $B_n$ permutations; in particular, there are $2^n n!$ of them. Later, Proposition 11.4 will give a more refined count of the type $B_n$ permutation tableaux.

Note that if a type $B_n$ $\reflectbox{L}$-diagram contains an all-zero column, then the diagonal box in that column contains a 0, which implies that all boxes to its left are 0. Deleting this "hook" we obtain a type $B_{n-1}$ $\reflectbox{L}$-diagram, and if we repeat this procedure, we will eventually obtain a type $B$ permutation tableau.

Permutation tableaux in type $A$ were studied in [19]. They are simpler than $\reflectbox{L}$-diagrams but can be applied to the study of permutations. Rather surprisingly, type $A$ permutation tableaux are related to the asymmetric exclusion process [5].

10.2.2. *Partial order on $\mathcal{I}^B$.* Rietsch [16] has given a concrete description of the order relation on cells: $P^J_{x,w;>0} \subset \overline{P^J_{x',w';>0}}$ precisely if there exists $z \in W_J$ such that $x' \leq xz \leq wz \leq w'$. This poset has nice combinatorial properties: it is thin and EL-shellable, and hence is the poset of cells of a regular CW complex [22].

Postnikov [12] described this poset in the case of the type $A$ Grassmannian in terms of decorated permutations. One draws decorated permutations on a circle; the cover relation involves uncrossing two chords emanating from $i$ and $j$ that form a "simple crossing". Similarly, one can describe the partial order on $\mathcal{I}^B$ in the case of the type $B$ Grassmannian in terms of type $B$ decorated permutations. The cover relation is exactly the same, except that each time we uncross the pair of chords $i$ and $j$, we must also uncross the pair of chords $-i$ and $-j$ (which will necessarily also form a simple crossing).

## 11. Enumeration of cells

The cells in the totally non-negative Grassmannian for type $A$ were enumerated in [21, 12]. In this section we will give some formulae and recurrences for the number of totally non-negative cells in Grassmannians of types $(B_n, 1)$, $(D_n, 1)$, $(B_n, n)$, and $(D_n, n)$. We will often count the cells which lie inside the open Schubert cell, or in other words, we will count Ⅎ-diagrams of maximal shape.

11.1. **Enumeration of type $(B_n, 1)$ and $(D_n, 1)$ Ⅎ-diagrams.** Let $\hat{b}_n$ be the number of type $(B_n, 1)$ Ⅎ-diagrams of maximal shape, and let $\hat{b}_n(q)$ be the $q$-generating function of $(B_n, 1)$ Ⅎ-diagrams of maximal shape, where we weight Ⅎ-diagrams according to the number of +'s. Similarly define $\hat{d}_n$ and $\hat{d}_n(q)$. Below, $[i]$ denotes the $q$-analog of $i$.

**Proposition 11.1.** *$\hat{b}_n(q)$ is the coefficient of $x^n$ in*

$$\frac{1 - (q + q^2)x + (1 - q^2)x^2}{1 - [2]^2 x + [2]x^2}.$$

*In particular, the numbers $\hat{b}_n$ are equal to sequence A006012 in the Sloane Encyclopedia of Integer Sequences* [18]*, and have generating function $\frac{1-2x}{1-4x+2x^2}$.*

*Proof.* It is easy to check that $\hat{b}_0(q) = 1$, $\hat{b}_1(q) = [2]$, and $\hat{b}_2(q) = 1 + 2q + 2q^2 + q^3$. We will prove that for $n \geq 3$, $\hat{b}_n(q) = (1 + q)^2 \hat{b}_{n-1}(q) - (1 + q)q^2 \hat{b}_{n-2}(q)$, using the description of Theorem 7.1. A maximal-shape $(B_n, 1)$ Ⅎ-diagram $D'$ can be obtained from a maximal-shape $(B_{n-1}, 1)$ Ⅎ-diagram $D$ by adding two boxes, one to the left and one to the right of $D$. Each of these two boxes can contain a 0 or a +, except that we may not put a 0 into the new box to the right of $D$ if $D$ has a + in its leftmost and rightmost boxes. This gives the stated recursion. □

Now let us consider type $(D_n, 1)$ Ⅎ-diagrams. For $n \geq 4$, we have the same recurrence as above: $\hat{d}_n(q) = (1 + q)^2 \hat{d}_{n-1}(q) - (1 + q)q^2 \hat{d}_{n-2}(q)$. We have initial conditions $\hat{d}_0(q) = 1$, $\hat{d}_1(q) = [2]$, $\hat{d}_2(q) = [2]^2$, $\hat{d}_3(q) = [2]^4 - q^2[2]$. This implies the following result.

**Proposition 11.2.** *$\hat{d}_n(q)$ is the coefficient of $x^n$ in*

$$\frac{1 - (q + q^2)x + (1 - 2q^2 - q^3)x^2 + (1 + 2q - q^3)x^3}{1 - [2]^2 x + [2]x^2}.$$

*In particular, the numbers $\hat{d}_n$ (for $n \geq 3$) are given by sequence A007070 in the Sloane Encyclopedia of Integer Sequences* [18]*.*

11.2. **Enumeration of $(B_n, n)$ Ⅎ-diagrams and permutation tableaux.**

11.2.1. *The total number of* $(B_n, n)$ Ⅎ*-diagrams.* Let $B(n)$ be the number of type $(B_n, n)$ Ⅎ-diagrams; equivalently, it is the number of type $B$ decorated permutations on $\{\pm 1, \ldots, \pm n\}$. Let $b(n)$ be the number of elements in the Coxeter group of type $B_n$. That is, $b(n) = 2^n n!$.

**Proposition 11.3.** *The sequence of numbers* $B(n)$ *is sequence A010844 from the Sloane Encyclopedia of Integer Sequences* [18]. *The numbers are given by the recurrence* $B(0) = 1$ *and* $B(n+1) = 2(n+1)B(n) + 1$.

*Proof.* A type $B$ decorated permutation is chosen via the following procedure: first choose a number $k$ (where $0 \leq k \leq n$), which will be the number of clockwise fixed points; then choose their location in $\binom{n}{k}$ ways; finally, choose the structure of the remainder of the permutation by choosing a normal type $B$ permutation of size $n-k$ in $b(n-k)$ ways. Therefore

$$B(n) = b(n) + \binom{n}{1} b(n-1) + \binom{n}{2} b(n-2) + \cdots + \binom{n}{n} b(0).$$

Since $\binom{n+1}{k} b_{n+1-k} = 2(n+1)\binom{n}{k} b_{n-k}$, we have $B(n+1) = 2(n+1)B(n) + 1$. □

11.2.2. *The total number of* $B_n$ *permutation tableaux.* We say that a 0 in a Ⅎ-diagram is *restricted* if it is on the diagonal or if there is a + below it in the same column. We say that a row is restricted if it contains a restricted 0. Let $t_{n,k,j}$ be the number of type $B_n$ permutation tableaux with $k$ unrestricted rows and exactly $j$ +'s on the diagonal. Let $T_n(x, y) = \sum_{k,j} t_{n,k+1,j} x^k y^j$.

**Proposition 11.4.** $T_n(x, y) = (y + 1)^n (x + 1)(x + 2) \ldots (x + n - 1)$.

The strategy of this proof comes from an idea in [4].

*Proof.* We will show that $T_n(x, y) = (y+1)(x+1)T_{n-1}(x+1, y)$. To prove this, let us consider the process of building a type $B_n$ permutation tableau $D'$ from a type $B_{n-1}$ permutation tableau $D$ of shape $O_w$. Let $k$ be the number of unrestricted rows of $D$. There are two possibilities: either we can add a new (empty) row of length $n$ to $D$ (adding a north step to the border of $O_w$), or we can add a new column $c$ of length $n$ to $D$ (adding a step west to the border of the Young diagram). The first possibility is easy to analyze: $D'$ contains the same number of +'s on the diagonal as $D$ and has one additional unrestricted row.

For the second possibility there are two cases: either we will fill the bottom (diagonal) box of $c$ with a 0 or we will fill it with a +. In each case all boxes in a restricted row must be filled with 0's, and the other boxes may be filled with 0 or +. We need to compute how many ways there are to fill the boxes of $c$ such that the resulting tableau $D'$ has $i$ unrestricted rows, where $i \leq k + 1$.

In the first case, note that every 0 above the bottom-most + in $c$ is restricted. Also the 0 is the bottom (diagonal) box of $c$ is restricted. Summing over the position $a$ of the bottom-most +, the number of ways to fill the boxes of $c$ such that $D'$ has $i$ unrestricted rows is $\sum_{a=1}^{k} \binom{a-1}{k-i} = \binom{k}{i-1}$.

In the second case, since the bottom (diagonal) box of $c$ is a $+$, all 0's that we place in column $c$ are restricted. Therefore there are $\binom{k}{i-1}$ ways to fill the boxes of $c$ such that the $D'$ has $i$ unrestricted rows.

Our arguments show that

$$t_{n,i,j} = t_{n-1,i-1,j} + \sum_{k \geq i} \binom{k}{i-1} t_{n-1,k,j} + \sum_{k \geq i-1} \binom{k}{i-1} t_{n-1,k,j-1},$$

which implies that $t_{n,i,j} = \sum_{k \geq i-1} \binom{k}{i-1} (t_{n-1,k,j} + t_{n-1,k,j-1})$. It follows that $T_n(x, y) = (y + 1)(x + 1)T_{n-1}(x + 1, y)$. □

11.2.3. *Recurrences for type $(B_n, n)$ cells of maximal shape.* Let $b(n)$ be the number of type $(B_n, n)$ Ⅎ-diagrams of maximal (staircase) shape. Let $[i] = 1+q+\cdots+q^{i-1}$ denote the $q$-analogue of $i$ and let $[i]^{(j)}$ denote the $j$-th derivative (with respect to $q$) of $[i]$. We have the following recurrence for $b(n)$.

**Proposition 11.5.** *We have $b(0) = 1$ and*

$$b(n) = [n + 1]b(n - 1) + q^2 \sum_{i=1}^{n-2} \frac{[n - 1]^{(i)}}{i!} b(n - i - 1).$$

*Proof.* This result is proved by considering the various possibilities for the top row of the Ⅎ-diagram. Whenever there is a 0 in the top row which is to the right of some $+$ (let us call such a 0 *restricted*), then every box below that 0 must also be a 0. In a type B Ⅎ-diagram, whenever there is a 0 on the diagonal, all boxes to its left must also be 0's. Therefore whenever we have a restricted 0 in the top row, say in column $i$, then the $i$-th column and the $n+1-i$-th row of the Ⅎ-diagram contain 0's. If we delete this column and this row, the resulting diagram is a Ⅎ-diagram of (inverted) staircase shape of type $B_{n-1}$.

If we $q$-count the possible configurations for the top row of a type $B_n$ Ⅎ-diagram which have no restricted 0's, we will get $[n + 1]$. Deleting the top row of such a diagram leaves us with a type $B_{n-1}$ Ⅎ-diagram.

If we $q$-count the possible configurations for the top row of a type $B_n$ Ⅎ-diagram which have precisely $i$ restricted 0's, where $i \geq 1$, we get $q^2 \frac{[n-1]^{(i)}}{i!}$. Deleting the top row of the Ⅎ-diagram as well as the $i$ columns and rows corresponding to the $i$ restricted 0's leaves us with a type $B_{n-i-1}$ Ⅎ-diagram. □

11.2.4. *Preference Functions.* Let $\mathcal{B}_n$ denote the set of $(B_n, n)$ Ⅎ-diagrams of maximal (staircase) shape such that the bottom square contains a $+$. The set of $(B_n, n)$ Ⅎ-diagrams of maximal shape has twice the cardinality of $\mathcal{B}_n$, since the bottom square of a Ⅎ-diagram imposes no restrictions on any other square.

**Definition 11.6.** *A preference function of $n$ is a word of length $n$ where all the numbers 1 through $k$ occur at least once for some $k \leq n$.*

In other words, a preference function of $n$ lists the possible ways that $n$ candidates may rank in a tournament, allowing ties.

**Theorem 11.7.** *The set $\mathcal{B}_n$ is in bijection with the set of preference functions of length $n$. Therefore the number of maximal type $(B_n, n)$ Ⅎ-diagrams has twice the cardinality of the preference functions of length $n$. This is sequence A000629 in the Sloane Encyclopedia of Integer Sequences* [18].

Theorem 11.7 follows from Lemmata 11.8 and 11.10 below. First recall that $\Phi_3^B$ gives a bijection between type $B_n$ Ⅎ-diagrams of maximal shape and type $B_n$ decorated permutations such that the nonexcedances are in positions $1, \dots, n$. So in particular, any fixed points that occur are clockwise. (Since they are all oriented the same way, we will ignore this orientation from now on.) Let $J_n$ denote the set of type $B_n$ decorated permutations such that the nonexcedances are in positions $1, \dots, n$, and such that there is never a fixed point in position $n$. Restricting $\Phi_3^B$ to $\mathcal{B}_n$ gives the following result.

**Lemma 11.8.** *$\Phi_3^B$ is a bijection from $\mathcal{B}_n$ to $J_n$.*

We now define a bijection $\alpha$ from $J_n$ to preference functions of length $n$. Let $\pi \in J_n$. The preference function $p = (p_1, \dots, p_n) := \alpha(\pi)$ is defined as follows. Let $I^+$ be the set of indices $i$ of $\pi$ where $\pi(i) > 0$. The entries of $\pi$ in positions $I^+$ will tell us about the "repeated" entries in the preference function. Let $K$ be the complement of the set $I^+ + 1 := \{i + 1 \mid i \in I^+\}$ in $\{1, \dots, n\}$. Let $S_\pi$ be the sequence that we obtain if we take the sequence of negative entries of $\pi$, forget the signs, and then use the relative order of the entries to extract a permutation on $\{1, \dots, m\}$ for $m \leq n$. We now put the entries of $S_\pi$ (in order) into the entries $p_k$ where $k \in K$. Finally, looking at each $i + 1 \in I^+ + 1$ in increasing order, we define $p_{i+1} := p_{\pi(i)}$.

**Example 11.9.** *Suppose $n = 9$ and $\pi = (-6, -8, -3, -1, -9, 5, -7, 4, -2)$. Then $I^+ = \{6, 8\}$, $K = \{1, 2, 3, 4, 5, 6, 8\}$, and $S_\pi = (4, 6, 3, 1, 7, 5, 2)$, and the preference function is $\alpha(\pi) = (4, 6, 3, 1, 7, 5, 7, 2, 1)$.*

**Lemma 11.10.** *The map $\alpha$ is a bijection from $J_n$ to the set of preference functions of length $n$.*

*Proof.* Since no permutation in $J_n$ has a fixed point in position $n$, $I^+ + 1$ is a subset of $\{1, \dots, n\}$ as it should be. Also, the definition $p_{i+1} := p_{\pi(i)}$ makes sense because $\pi(i) \leq i$ by the condition on nonexcedances of $\pi$. Therefore $\alpha$ is well-defined.

To show $\alpha$ is a bijection we will define its inverse. Let $K$ be the set of indices corresponding to the first occurrence of each positive integer in the preference function $p$. $K$ includes 1, so we can reconstruct the set $I^+$ as $K^c - 1 = \{k - 1 \mid k \in K^c\}$, where $K^c$ is the complement of $K$ in $\{1, \dots, n\}$. Now for each entry $a$ in $p$ (say in position $i + 1$) which is not the first occurrence of $a$, we look at the closest occurrence of $a$ to the left of position $i + 1$. Say it occurs in position $i' \leq i$. Then we set $\pi(i) = i'$; note that the nonexcedance condition is satisfied. Let $T$ be the set of all such $i'$ and let $T^c$ be the complement of $T$. We now complete our reconstruction of $\pi$ by placing the elements of $T^c$ in the unfilled positions of $\pi$ in the same relative order as the first occurrences of entries of $p$, and then negating their signs. $\square$

11.3. **Enumeration of $(D_n, n)$ ⅃-diagrams.** In this section we show that the set $\mathcal{D}_n$ of maximal type $(D_n, n)$ ⅃-diagrams is in bijection with a distinguished subset of preference functions, the *atomic* preference functions. A preference function is *atomic* if no strict leading subword consists of the only occurrences in the word of the letters 1 through $j < k$.

**Theorem 11.11.** *Atomic preference functions of length $n$ are in bijection with maximal type $(D_n, n)$ ⅃-diagrams. Therefore the cardinality of the set of maximal type $(D_n, n)$ ⅃-diagrams is given by sequence A095989 from the Sloane Encyclopedia of Integer Sequences* [18].

Let $\mathcal{A}_n$ be the set of atomic preference functions of length $n$. For $D \in \mathcal{D}_n$ we let $0_R(D)$ denote the set of indices $i$ such that row $i$ of $D$ is completely filled with 0's.

We will prove Theorem 11.11 by describing two maps between these sets and showing that they are inverse to each other.

First we describe $\Phi : \mathcal{D}_n \to \mathcal{A}_n$, and then $\Psi : \mathcal{A}_n \to \mathcal{D}_n$ which will be the inverse.

$\Phi$ is defined as follows. Embed the ⅃-diagram $D$ into a staircase shape with $n$ rows by adding a diagonal to $D$ which is filled entirely with $*$'s. Label the west side (north side) of the staircase with the numbers from $1_W$ to $n_W$ ($1_N$ to $n_N$), as in the following diagram:

$$\begin{array}{c|c|c|c|c|c|}
 & 5_N & 4_N & 3_N & 2_N & 1_N \\ \cline{2-6}
1_W & & & & & * \\ \cline{2-6}
2_W & & & & * \\ \cline{2-5}
3_W & & & * \\ \cline{2-4}
4_W & & * \\ \cline{2-3}
5_W & * \\ \cline{2-2}
\end{array}$$

Turn each $+$ and each $*$ into an and each 0 into a crossing . This will turn our diagram into a wiring diagram which gives a permutation $\pi = \pi(D)$ (where paths $i \mapsto \pi(i)$ travel from the west to the north border). We now add signs to $\pi$, making the $i$th entry positive if $i \in 0_R(D)$; and otherwise negative. Clearly this signed permutation is an element of the set $J_n$ defined in Section 11.2.4; in fact, the map we have described is essentially the map $\Phi_3^B$. We now define $\Phi(D) := \alpha(\pi(D))$, where $\alpha$ is the map used in Section 11.2.4.

**Proposition 11.12.** *$\Phi(D)$ is an atomic preference function for $D \in \mathcal{D}_n$.*

*Proof.* Suppose that $\Phi(D)$ is not atomic. This means that there is a proper leading subword (say of length $j$) of $\Phi(D)$ which consists of all occurrences of the numbers 1 through $r$ for some positive $r$. Recalling the definition of $\alpha$, this means that any negative entry of $\pi := \pi(D)$ after position $j$ has greater absolute value than any negative entry of $\pi(D)$ in the first $j$ positions. Furthermore, if for any $i \in I^+$ we have $\pi(i) \le j$ then $i + 1 \le j$. This implies that if we ignore signs, the first $j$ entries of $\pi(D)$ form a permutation of $S_j$, and $\pi(j)$ is negative.

Now note that since $|\pi(1)| \le j$, the first $n - j$ entries in the first row of $D$ must be zero. Similarly, since $|\pi(2)| \le j$, the first $n - j$ entries of the second row of $D$ must be zero. Continuing, since $|\pi(j)| \le j$, the first $n - j$ entries of the $j$th row of

$D$ must be zero, i.e. all entries in the $j$th row of $D$ are zero. But this means that $\pi(j) > 0$, a contradiction. $\square$

We remark that this proof did not use the forbidden patterns of type $(D_n, n)$ Ⅎ-diagrams in any way. In fact, we can define $\Phi$ for *any* $\oplus$-diagram $D$, and $\Phi(D)$ will be an atomic preference function; this is a many-to-one map. What we need to prove next is that when we restrict $\Phi$ to the set of type $(D_n, n)$ Ⅎ-diagrams, we get a bijection to the set of atomic preference functions.

11.3.1. *Inverse bijection.* We shall refer to the type $(D_n, n)$ Ⅎ-conditions as the first, second and four-box pattern. We remark that a consequence of our labeling of the columns and rows is that if the 0 involved in the second or four-box Ⅎ-pattern is in column $k$ then the lowest $+$ involved in the Ⅎ-pattern is in row $k$.

We will construct the inverse map $\Psi : \mathcal{A}_n \to \mathcal{D}_n$ recursively. Recall from Lemma 11.10 that given a preference function $f$, we can already construct the signed permutation $w(f) := \alpha^{-1}(f)$. Let $\mathbf{w} := |w(f)|$.

We will first construct the path $P_n = n_N \to \mathbf{w}^{-1}(n)_W$, then the path $P_{n-1} = (n-1)_N \to \mathbf{w}^{-1}(n-1)_W$, then $(n-2)_N \to \mathbf{w}^{-1}(n-2)_W$, and so on. The general idea here is that each path $P_i$ will travel as close to the northwest border of the staircase as possible. This idea will be made precise in the form of an algorithm in the following paragraphs.

Let $D_i = \cup_{j=i}^{n} P_j$ denote the set of boxes used by $P_i, P_{i+1}, \ldots, P_n$ so that $D_1$ is completely filled in. Abusing notation, we also use $D_i$ to denote the corresponding staircase shape partially filled with 0's and $+$'s (diagonal boxes are always filled in with $*$'s). Let $C_b(D_i), R_a(D_i)$ denote the $b$-th column and $a$-th row of $D_i$. We say a row or a column is *complete* if all its boxes have been filled in.

Let $i^* = \mathbf{w}^{-1}(i)$ so that $P_i$ goes from $i$ to $i^*$. A path $P_i$ is completely determined by its set $P_i^+$ of boxes containing $+$'s which are linearly ordered according to the order in which they are visited. Our construction of $P_i^+$ will always have the form

$$P_i^+ = (c_1, c_1', c_2, c_2', \ldots, c_k, c_k', c_{k+1}', c_{k+\ell-1}')$$

where $c_j \in P_i^+ \cap (D_i - D_{i+1})$ and $c_j' \in P_i^+ \cap D_{i+1}$. In other words, the primed boxes are old, while the unprimed ones are newly added. In our notation it is possible for $\ell = 0$, or in other words, $c_k$ is the last $+$ on $P_i^+$.

We now give the construction of $P_i$ by describing $c_1, c_2, \ldots, c_k$. It may be helpful for the reader to look at Example 11.4 alongside the description of this algorithm.

Given $c_1, c_2, \ldots, c_{j-1}$, it is clear that $c_{j-1}'$ is determined. Suppose $c_{j-1}' = (a, b)$ is in row $a$ and column $b$. If $j = 1$ we set $c_{j-1}' = (a, b) = (0, i)$. If all rows below row $a$ have been filled in then we are already done: the path $P_i$ is determined.

Otherwise, let $a'$ (if it exists) be the highest row (smallest number) below $a$ and above row $b$ which contains a $+$ and let $c^* = (a', b')$ (if it exists) be the rightmost $+$ in row $a'$. We have a number of mutually exclusive cases:

(Z) Suppose one of the following holds:
  (a) $c^*$ does not exist and $i^* \geq b$.
  (b) all the rows below row $a$ have been filled in $D_{i+1}$.

(c) $c^*$ exists and is equal to $(i^*, b)$.
(d) $c^*$ exists and $i^* > a'$ and all the rows below and including $a'$ are complete in $D_{i+1}$.

Then $c_{j-1}$ is already the last new $+$ in $D_i$ and the rest of the $+$'s on $P_i$ are determined by $D_{i+1}$.

(A) If $c^*$ exists and $i^* < a'$, or $c^*$ does not exist and $i^* < b$ we set $c_j = (i^*, b)$, and

$$c_j \text{ will be the last } + \text{ of } P_i. \tag{7}$$

(B) Suppose $c^*$ exists, $i^* = b$ and either $w^{-1}(i) > 0$ or $R_b(D_{i+1})$ is filled with 0's. Then $c_{j-1}$ is already the last new $+$ on $P_i$, and:

$$\text{the path } P_i \text{ will visit the diagonal square } (i^*, i^*) \text{ and exit at } i^*_W. \tag{8}$$

(C) Suppose $c^*$ exists. If $i^* > b$, $w(b) < 0$ and $R_b(D_{i+1})$ is filled with 0's apart from one box then we set $c_j = (b, b')$. Then:

$$\text{the path } P_i \text{ will visit the diagonal square } (b, b) \text{ and then turn at } c_j; \tag{9}$$

$$\text{also } c_j \text{ is the last new } + \text{ of } D_i. \tag{10}$$

(D) In all other cases we set $c_j = (a', b)$. Then:

$$\text{the path } P_i \text{ will turn at } c_j \text{ and head to } c'_j = c^*. \tag{11}$$

Note that only in Case D does one have to continue constructing $c_{j+1}$. The construction of $P_i$ typically involves multiple instances of Case D, followed by one instance of another case.

We will call a $+$ placed via Case C a *special* $+$ and a $+$ placed via Case D a *normal* $+$. Let us call a $+$ inside some $D_i$ a *corner* $+$ if (a) its row is not yet complete, and (b) it is the rightmost $+$ in its row. We claim that

**Proposition 11.13.** *The algorithm described above is well-defined. More precisely,*

(1) *the construction gives paths $P_i$ which go from $i_N$ to $i^*_W$,*
(2) *the positions of the new $+$'s $c_j$ are empty in $D_{i+1}$,*
(3) *no $+$'s are encountered while going from $c'_{j-1}$ to $c_j$,*
(4) *the stated facts* (7), (8), (9), (10) *and* (11) *hold.*

*Furthermore, in Cases A,B,C,D each $c'_j$ of the form $c^*$ in the algorithm is a corner $+$ of $D_{i+1}$.*

Proposition 11.13 will be proved simultaneously with the following propositions.

**Proposition 11.14.** *Let $C = C_b(D_i)$ be a column of an intermediate diagram $D_i$.*

(1) *If $b < i$, then $C$ is empty.*
(2) *If $C$ contains a corner $+$, say $c$, then this corner $+$ is unique. Every filled square of $C$ below $c$ belongs to a complete row. Every filled square to the right of $c$ belongs to a complete column.*
(3) *If $b \geq i$ and $C$ does not contain a corner $+$, then $C$ is completely filled in.*

**Proposition 11.15.** *Let $R = R_a(D_i)$ be a row of an intermediate diagram $D_i$.*

(1) *If $R$ is complete then $R$ is either completely filled with 0's or all rows below $R$ are also complete.*
(2) *If $R$ contains a corner $+$, then the exit $a_W$ has been used by a path $P_j$ for some $j \geq i$.*

**Proposition 11.16.** *In any intermediate diagram $D_i$,*

(1) *all boxes weakly to the north-west of a corner $+$ or a normal $+$ are filled in; and all boxes strictly northwest of any $+$ are filled in,*
(2) *corner $+$'s are arranged from north-east to south-west, forming the corners of a (English notation) Young diagram.*

**Proposition 11.17.** *The set of new squares $P_i \cap (D_i - D_{i+1})$ contains at most one box in each row, except in Case B when one has a row completely filled with 0's.*

**Proposition 11.18.** *There are no violations of the type $(D_n, n)$ $\reflectbox{L}$-condition in any intermediate diagram $D_i$.*

*Proof of Propositions 11.13, 11.14, 11.15, 11.16, 11.17 and 11.18.* All the claims hold when no paths have been added. Let us assume that $D_{i+1}$ has been constructed and that all statements hold. We shall show that $D_i$ satisfies all these conditions.

We first note:

(1) Proposition 11.14(1) is obvious.
(2) A corner $+$ is always normal.
(3) Proposition 11.15(1) follows from Proposition 11.17 and the following wire-counting argument. Let $n$ be the number of rows below and including $R$. If $R$ is not completely filled with 0's then $n-1$ wires will travel from the north through row $R$. Let $R'$ be any row below $R$, say of length $\ell - 1$. Then there are $\ell$ exits below and including $R'$ of which at least $\ell - 1$ must have been used, so $\ell - 1$ wires exit to the left below $R'$ and hence must enter and occupy every square of $R'$.

Suppose $P_i$ has been constructed up to $c'_{j-1} = (a, b)$ where $c'_{j-1}$ might mean the "entrance" $i_N = (0, i)$. In our explanations we will assume that $c'_{j-1} \neq (0, i)$ – i.e. that $j > 1$. (Note that the special case $c'_{j-1} = (0, i)$ is easier.) We may assume (inductively) that $i^* > a$ and that $c'_{j-1}$ is a corner $+$ in $D_{i+1}$.

Case 1: Suppose that $c^* = (a', b')$ exists. By Proposition 11.16(2), $c^*$ is either a corner $+$ or its row is filled in. Furthermore, if $b' < b$ then by Proposition 11.16(1) all the rows below row $a$ are complete, so we are in Case Z(b), and nothing needs to be proved (the fact that $P_i$ will exit correctly follows from counting wires). Suppose $b' = b$ so that the row $a'$ is complete. If $i^* \geq a'$ then we are in either Case Z(c) or Z(d). The only new boxes we fill are with 0's. It is straightforward to verify the claimed properties.

Now suppose that $i^* < a'$. Then we are in Case A. Consider $R = R_{i^*}(D_{i+1})$. By Proposition 11.14(2), the box $(i^*, b)$ is either empty or $R$ is complete. If $R$ is complete, then because of the way we chose $a'$, it must be filled with 0's. But this can be shown to be impossible by considering the wire that passed through $(i^*, b)$. (The wire $P'$ passing through $(i^*, b)$ did not exit at $i^*_W$, since the current wire $P$

needs to use this exit. So this wire traveled down column $b$ through $(i^*, b)$. But there aren't any $+$'s between $(i^*, b)$ and $(a, b)$ so $P'$ turns at $(a, b)$ which means it is the same wire as $P$, a contradiction.) By Proposition 11.16, Proposition 11.14(3) and the way we chose $a'$, we see that all the boxes to the left of $(i^*, b)$ have been filled with 0's in $D_{i+1}$. It is easy to see that Propositions 11.13, 11.14, 11.15, 11.16, and 11.17 continue to be satisfied in $D_i$. Since the boxes in $\{(c, d) \mid a < c \leq i^*, b < d \leq n\}$ are all filled with 0's the first and four-box ⅃-conditions are immediate (for the four-box condition one also uses Proposition 11.16). Suppose the second ⅃-condition is violated by the new $+$ in box $(i^*, b)$. There are two possibilities: (i) the $+$ in $(i^*, b)$ is the lower $+$ in the ⅃-pattern, or (ii) $+$ in $(i^*, b)$ is the higher $+$ in the ⅃-pattern. In case (i), the 0 in the violating pattern is in column $i^*$ say at $(x, i^*)$ with a $+$ at $(x, y)$ where $b > y > i^*$. Since $(i^*, b)$ is empty in $D_{i+1}$, by Proposition 11.15 and Proposition 11.14, column $i^*$ is filled in so all the squares below $(x, i)$ contain 0's. But it follows from the definition of the algorithm that row $i^*$ is already complete in $D_{i+1}$, a contradiction. In case (ii), let the 0 in the ⅃-pattern be in box $(i^*, j)$. Then $C_j(D_{i+1})$ is complete with 0's under $(i^*, j)$ and one deduces that $R_j(D_{i+1})$ is complete and has a single $+$, which must then be in a column to the left of column $b$. Consider the wire $P'$ which passed vertically through the 0 in box $(j, b)$. This wire cannot turn somewhere between $(j, b)$ and $(a, b)$ for then the second ⅃-condition is already violated in $D_{i+1}$. But this means $P' = P_i$, a contradiction. This completes the verification of the properties in Case A (when $c^*$ exists).

Now suppose $i^* \geq a'$. We have already treated the case $b' < b$ so we assume $b' \geq b$. If $R_{a'}(D_{i+1})$ is complete, then by Proposition 11.15(1) all rows below are complete so we are in Case Z(d). The argument is again straightforward.

Otherwise, if $R_{a'}(D_{i+1})$ is not complete, then $c^*$ is a corner $+$, $b' \geq b$ and $(a', b)$ is empty. The squares between $c'_{j-1}$ and $(a', b)$ are either unfilled or contain 0's (this comes from Proposition 11.14(2) and the way we chose $a'$). Similarly, the squares below $(a', b)$ are either unfilled or belong to complete rows. Note that these squares do not contain $+$'s for otherwise either $(a', b)$ would be filled (if the closest such $+$ is normal) or $c^*$ could not be a $+$ (if the closest such $+$ is special – this follows from the description of Case C below). It is also clear from the definitions and Proposition 11.14(2,3) that

$$\text{there are 0's between } (a', b) \text{ and } c^*. \tag{12}$$

Now suppose $i^* \geq b$. Suppose first that $R_b(D_{i+1})$ is complete. If $R_b(D_{i+1})$ is filled with 0's then since there are no $+$'s under $(a', b)$ we deduce that all the exits $(b+1)_W, (b+2)_W, \ldots, n_W$ have been used. Thus automatically we have $i^* = b$ and we are in Case B. Since we are only adding 0's the claimed properties are easy to verify, except perhaps the ⅃-condition. But row $R_b$ is also filled with 0's so there are no possibilities of any ⅃-patterns.

We claim with our assumptions that $R_b(D_{i+1})$ cannot be complete but not filled with 0's. Suppose this is the case. Then by Proposition 11.17, $n - b + 1$ wires have already been drawn passing through row $b$. Consider the wire $P_j$ which passes through $(b, b')$. This wire cannot also pass through $c^*$ so by assumptions there is $x \in (a', b)$ so that $(x, b')$ contains a $+$. This $+$ must be special, so $R_x(D_{i+1})$ is complete. But then there is a wire $P_k$ passing through vertically through box $(x, b)$, which contradicts our assumptions.

Thus we suppose that $R_b(D_{i+1})$ is not filled in (but still $i^* \geq b$). If $i^* = b$ and $w^{-1}(i) > 0$ we are again in Case B and it is easy to verify all the claimed properties.

Now consider Case C, so $R_b(D_{i+1})$ is filled with 0's apart from one square. Since exit $b_W$ has not been used, $n-b-1$ of the exits $(b+1)_W, \ldots, n_W$ have been used in $D_{i+1}$. By a counting argument, all rows below $R_b$ are filled in, so the $+$ in $(b, b')$ is the last new $+$ on path $P_i$. This proves Proposition 11.13. The other properties are straightforward to establish except Proposition 11.18. It follows from Proposition 11.14(2,3) and Proposition 11.15(1) that the region $\{(c, d) \mid b \geq c > a, n \geq d > b\}$ is filled with 0's in $D_{i+1}$. Using the 0's in (12) we see that the $+$ in $(b, b')$ is not involved in any Ⅎ-conditions. Finally we consider the new 0's placed in column $b$. Only $(a', b)$ has a $+$ to the left, so Proposition 11.18 follows.

In all other situations we are in Case D. The new $+$ in $(a', b)$ becomes the new corner $+$ in column $b$, unless the row $a'$ becomes complete. If row $a'$ becomes complete, then column $b$ is also complete by Proposition 11.15(1). Again the stated properties are immediate except Proposition 11.18. This last property follows from the definition of $c^*$ (minimality of row) and arguments similar to those in Case A.

Case 2: $c^*$ does not exist. If $i^* < b$, then the argument is exactly the same as in Case A when $c^*$ does exist. So we may suppose $i \geq b$ and we are in Case Z(a). Consider the columns $C_k(D_{i+1})$ for $k > b$. By Proposition 11.14 they are either completely filled, or contain a corner $+$. Any corner $+$'s in these rows must be below or on row $b$. But a counting argument shows that there is not enough space to fit corner $+$'s, and thus all rows below and including row $b$ are complete. The argument is now the same as the other Case Z arguments. □

We have shown that $\Psi$ maps atomic preference functions to Ⅎ-diagrams. Recall that $0(D)$ denotes the set of rows of $D$ which are completely filled with 0's.

**Lemma 11.19.** *Let $f$ be an atomic preference function with corresponding signed permutation $w = w(f)$. Then $0(\Psi(f)) = \{j \mid w(j) > 0\}$.*

*Proof.* Let $D = \Psi(f)$. Suppose $i^*$ is such that $w^{-1}(i) > 0$; then setting $j = i^*$, we have $w(j) > 0$. Then in particular $i^* \geq i$ since $w$ does not have an excedance at $i^*$. By construction, up till $D_{i+1}$ no $+$'s have been placed in row $i^*$. If $i^* = i$ we are done. Otherwise $i^* > i$, and if column $C_{i^*}(D_{i+1})$ has a corner $+$ we are done since it must be encountered by the path $P_i$. Suppose otherwise, so $C_{i^*}$ is complete (by Proposition 11.14) and must contain a $+$ in say $(x, y)$. But then $R_x(D_{i+1})$ is complete, so by Proposition 11.15(1) so is $R_{i^*}(D_{i+1})$.

Conversely, suppose $i^*$ is such that $w^{-1}(i) < 0$; in other words, setting $j = i^*$, we have $w(j) < 0$. If $i^* < i$ then we are done since the construction of $P_i$ will place a $+$ in row $R_{i^*}$ before it is complete. So suppose $i^* \geq i$ and that $R_{i^*}(D_{i+1})$ is completely filled with 0's. If $i^* = i$ then $f$ is not atomic so we suppose $i^* > i$. Let us pick $j \in (i, i^*]$ such that $R_{i^*}(D_j)$ is complete but $R_{i^*}(D_{j+1})$ is not. Again with the same argument as above, we conclude that $C_{i^*}(D_{j+1})$ has a corner $+$, say $c^*$.

Thus the path $P_j$ travels through $c^*$, at which time it will enter Case C and create a $+$ in row $i^*$. $\square$

**Theorem 11.20.** *The map $\Psi : \mathcal{A}_n \to \mathcal{D}_n$ is a bijection.*

*Proof.* It follows from the construction (Proposition 11.13(1)) that $\Phi \circ \Psi$ is the identity, so it suffices to show that $\Phi$ is injective. For an atomic preference function $f$, we will show that there are no choices in the construction of $\Psi(f)$ if we require that $\Psi(f)$ is a Ⅎ-diagram $D$ satisfying $\Phi(\Psi(f)) = f$ and satisfying the condition of Lemma 11.19 that all 0 rows correspond exactly to the $j$ such that $w(j) > 0$.

We may suppose by induction that there are no choices for the construction of $D_{i+1} = \cup_{j=i+1}^{n} P_j$. Now suppose we have constructed the $+$'s of $P_i^+$ up to $c'_{j-1}$ as in the stated algorithm. We will show that the stated algorithm is the only possible way to extend $P_i$, using the notation and explicit descriptions given in the proof of the Propositions.

In Cases Z and A we have no choice if we require $P_i$ exits at $i^*_W$. So we may assume we are in Cases B, C, or D and that $c^*$ exists. In particular $(a', b)$ is empty. For $P_i$ to exit at $i^*_W$ we must fill any unfilled boxes between $c'_{j-1}$ and $(a', b)$ with 0's. If $i^* = b$ and $w^{-1}(i) < 0$ then the only way for row $i$ to be completely filled with 0's is for $P_i$ to go to the diagonal and then go straight to $i^*_W$ without turning. Alternatively, if $R_b(D_{i+1})$ is complete and $i^* \geq b$ then by the proof of the algorithm we must have $i^* = b$ and $R_b(D_{i+1})$ filled with 0's. This shows that Case B is forced.

Otherwise we are in Cases C or D. The first choice is thus $(a', b)$. Suppose we place a 0 in $(a', b)$. Then using the first Ⅎ-condition we see that all boxes below $(a', b)$ must also be filled with 0. Since the boxes between $c^*$ and $(a', b)$ are filled with 0's we see using the first and second Ⅎ-conditions that the only place for a $+$ in row $b$ is in box $(b, b')$. But we must not have row $i^*$ being completely filled with 0's, otherwise we would be in Case B. Thus we must turn at $(b, b')$. We claim that this is exactly Case C. It is clear that $i^* > b$. We need to show that row $i^*$ in $D_{i+1}$ is filled (necessarily with 0's) except the box $(b, b')$ which is empty. The columns $C_k(D_{i+1})$ for $b' > k > b$ do not contain corner $+$'s so by Proposition 11.14 they are complete. The columns $C_k(D_{i+1})$ for $k < b'$ cannot contain $+$'s in the rows between $a'$ and $b$, so they are either complete or contain a corner $+$ below row $b$. Thus row $b$ contains 0's in all boxes except $(b, b')$ in $D_{i+1}$. If row $b$ is complete in $D_{i+1}$, then a wire-counting argument shows that exit $i^*_W$ has been used. But the wire passing through $i^*_W$ cannot have gone straight from $c^*$ to $(b, b')$ for otherwise $(a', b)$ would not be empty in $D_{i+1}$. Thus there must be a complete row between rows $a'$ and $b$, which contradicts either Proposition 11.17 or the fact that the current wire will travel down column $b$ to the diagonal from $(a', b)$.

Thus when there is a choice, a 0 is placed in $(a', b)$ only in Case C. However, if the diagram satisfies the conditions of Case C and we place a $+$ in $(a', b)$ instead then the wire $P_i$ will exit in row $b$, contradicting the estimate $i^* > b$. Thus Case C is forced by our assumptions. In all other cases, we will perform Case D.

$\square$

11.4. **Example.** Suppose $n = 9$ and $f = (4, 6, 3, 1, 7, 5, 7, 2, 1)$. Then $\mathbf{w} = w(f) = -6\ -8\ -3\ -1\ -9\ 5\ -7\ 4\ -2$. The construction of $\Psi(f)$ proceeds as follows:

First $i = 9$ and $i^* = 5$. When $j = 1$, we have $(a, b) = (0, 9)$ and $c^*$ does not exist. We are in Case A so $c_1 = (5, 9)$ and $D_9$ is as shown below.

| | $9_N$ | $8_N$ | $7_N$ | $6_N$ | $5_N$ | $4_N$ | $3_N$ | $2_N$ | $1_N$ |
|---|---|---|---|---|---|---|---|---|---|
| $1_W$ | $\boxed{0}$ | | | | | | | | $*$ |
| $2_W$ | $\boxed{0}$ | | | | | | | $*$ | |
| $3_W$ | $\boxed{0}$ | | | | | | $*$ | | |
| $4_W$ | $\boxed{0}$ | | | | | $*$ | | | |
| $5_W$ | $\boxed{+}$ | | | | $*$ | | | | |
| $6_W$ | | | | $*$ | | | | | |
| $7_W$ | | | $*$ | | | | | | |
| $8_W$ | | $*$ | | | | | | | |
| $9_W$ | $*$ | | | | | | | | |

Now $i = 8$ and $i^* = 2$. When $j = 1$, we have $(a, b) = (0, 8)$, $a' = 5$, and $c^* = (5, 9)$. We are in Case A so $c_1 = (2, 8)$ and $D_8$ is below.

| | $9_N$ | $8_N$ | $7_N$ | $6_N$ | $5_N$ | $4_N$ | $3_N$ | $2_N$ | $1_N$ |
|---|---|---|---|---|---|---|---|---|---|
| $1_W$ | 0 | $\boxed{0}$ | | | | | | | $*$ |
| $2_W$ | 0 | $\boxed{+}$ | | | | | | $*$ | |
| $3_W$ | 0 | | | | | | $*$ | | |
| $4_W$ | 0 | | | | | $*$ | | | |
| $5_W$ | + | | | | $*$ | | | | |
| $6_W$ | | | | $*$ | | | | | |
| $7_W$ | | | $*$ | | | | | | |
| $8_W$ | | $*$ | | | | | | | |
| $9_W$ | $*$ | | | | | | | | |

Now $i = 7$ and $i^* = 7$. When $j = 1$, $(a, b) = (0, 7)$, $a' = 2$, and $c^* = (2, 8)$. This is Case D so $c_1 = (2, 7)$ and $c'_1 = (2, 8)$. When $j = 2$, $c'_1 = (2, 8) = (a, b)$, $a' = 5$, and $c^* = (5, 9)$. This is Case D so $c_2 = (5, 8)$ and $c'_2 = (5, 9)$. When $j = 3$, $(a, b) = (5, 9)$. Neither $a'$ nor $c^*$ exist so we are in Case A, $c_3 = (7, 9)$, and $D_7$ is below.

| | $9_N$ | $8_N$ | $7_N$ | $6_N$ | $5_N$ | $4_N$ | $3_N$ | $2_N$ | $1_N$ |
|---|---|---|---|---|---|---|---|---|---|
| $1_W$ | 0 | 0 | $\boxed{0}$ | | | | | | $*$ |
| $2_W$ | 0 | + | $\boxed{+}$ | | | | | $*$ | |
| $3_W$ | 0 | $\boxed{0}$ | | | | | $*$ | | |
| $4_W$ | 0 | $\boxed{0}$ | | | | $*$ | | | |
| $5_W$ | + | $\boxed{+}$ | | | $*$ | | | | |
| $6_W$ | $\boxed{0}$ | | | $*$ | | | | | |
| $7_W$ | $\boxed{+}$ | | $*$ | | | | | | |
| $8_W$ | | $*$ | | | | | | | |
| $9_W$ | $*$ | | | | | | | | |

Now $i = 6$ and $i^* = 1$. When $j = 1$, $(a, b) = (0, 6)$, $a' = 2$, and $c^* = (2, 7)$. We are in Case A so $c_1 = (1, 6)$ and $D_6$ is below.

| | $9_N$ | $8_N$ | $7_N$ | $6_N$ | $5_N$ | $4_N$ | $3_N$ | $2_N$ | $1_N$ |
|---|---|---|---|---|---|---|---|---|---|
| $1_W$ | 0 | 0 | 0 | $\boxed{+}$ | | | | | $*$ |
| $2_W$ | 0 | + | + | | | | | $*$ | |
| $3_W$ | 0 | 0 | | | | | $*$ | | |
| $4_W$ | 0 | 0 | | | | $*$ | | | |
| $5_W$ | + | + | | | $*$ | | | | |
| $6_W$ | 0 | | | $*$ | | | | | |
| $7_W$ | + | | $*$ | | | | | | |
| $8_W$ | | $*$ | | | | | | | |
| $9_W$ | $*$ | | | | | | | | |

Now $i = 5$ and $i^* = 6$. When $j = 1$, we have $(a, b) = (0, 5)$, $a' = 1$, and $c^* = (1, 6)$. This is Case D so $c_1 = (1, 6)$ and $c'_1 = (1, 6)$. When $j = 2$ we have $(a, b) = (1, 6)$, $a' = 2$, and $c^* = (2, 7)$. This is Case B so there are no new +'s; $D_5$ is below.

| | $9_N$ | $8_N$ | $7_N$ | $6_N$ | $5_N$ | $4_N$ | $3_N$ | $2_N$ | $1_N$ |
|---|---|---|---|---|---|---|---|---|---|
| $1_W$ | 0 | 0 | 0 | + | $\boxed{+}$ | | | | $*$ |
| $2_W$ | 0 | + | + | $\boxed{0}$ | | | | $*$ | |
| $3_W$ | 0 | 0 | | $\boxed{0}$ | | | $*$ | | |
| $4_W$ | 0 | 0 | | $\boxed{0}$ | | $*$ | | | |
| $5_W$ | + | + | | $\boxed{0}$ | $*$ | | | | |
| $6_W$ | 0 | $\boxed{0}$ | $\boxed{0}$ | $*$ | | | | | |
| $7_W$ | + | | $*$ | | | | | | |
| $8_W$ | | $*$ | | | | | | | |
| $9_W$ | $*$ | | | | | | | | |

Now $i = 4$ and $i^* = 8$. When $j = 1$, $(a, b) = (0, 4)$, $a' = 1$, and $c^* = (1, 5)$. This is Case D so $c_1 = (1, 4)$ and $c'_1 = (1, 5)$. When $j = 2$, $(a, b) = (1, 5)$, $a' = 2$, and $c^* = (2, 7)$. This is again Case D so $c_2 = (2, 5)$ and $c'_2 = (2, 7)$. When $j = 3$, $(a, b) = (2, 7)$, $a' = 5$, and $c^* = (5, 8)$. This is still Case D so $c_3 = (5, 7)$ and $c'_3 = (5, 8)$. When $j = 4$, $(a, b) = (5, 8)$, $a' = 7$, and $c^* = (7, 9)$. This is Case B so there are no new +'s; $D_4$ is below.

| | $9_N$ | $8_N$ | $7_N$ | $6_N$ | $5_N$ | $4_N$ | $3_N$ | $2_N$ | $1_N$ |
|---|---|---|---|---|---|---|---|---|---|
| $1_W$ | 0 | 0 | 0 | + | + | $\boxed{+}$ | | | $*$ |
| $2_W$ | 0 | + | + | 0 | $\boxed{+}$ | | | $*$ | |
| $3_W$ | 0 | 0 | $\boxed{0}$ | 0 | | | $*$ | | |
| $4_W$ | 0 | 0 | $\boxed{0}$ | 0 | | $*$ | | | |
| $5_W$ | + | + | $\boxed{+}$ | 0 | $*$ | | | | |
| $6_W$ | 0 | 0 | 0 | $*$ | | | | | |
| $7_W$ | + | $\boxed{0}$ | $*$ | | | | | | |
| $8_W$ | $\boxed{0}$ | $*$ | | | | | | | |
| $9_W$ | $*$ | | | | | | | | |

Now $i = 3$ and $i^* = 3$. When $j = 1$, $(a, b) = (0, 3)$, $a' = 1$, and $c^* = (1, 4)$. This is Case D so $c_1 = (1, 3)$ and $c'_1 = (1, 4)$. When $j = 2$, $(a, b) = (1, 4)$, $a' = 2$, and $c^* = (2,5)$. This is Case D so $c_2 = (2,4)$ and $c'_2 = (2,5)$. When $j = 3$, $(a,b) = (2,5)$, $a' = 5$, and $c^* = (5,7)$. This is Case A so $c_3 = (3,5)$; $D_3$ is below.

| | $9_N$ | $8_N$ | $7_N$ | $6_N$ | $5_N$ | $4_N$ | $3_N$ | $2_N$ | $1_N$ |
|---|---|---|---|---|---|---|---|---|---|
| $1_W$ | 0 | 0 | 0 | + | + | + | $\boxed{+}$ | | $*$ |
| $2_W$ | 0 | + | + | 0 | + | $\boxed{+}$ | | $*$ | |
| $3_W$ | 0 | 0 | 0 | 0 | $\boxed{+}$ | | $*$ | | |
| $4_W$ | 0 | 0 | 0 | 0 | | $*$ | | | |
| $5_W$ | + | + | + | 0 | $*$ | | | | |
| $6_W$ | 0 | 0 | 0 | $*$ | | | | | |
| $7_W$ | + | 0 | $*$ | | | | | | |
| $8_W$ | 0 | $*$ | | | | | | | |
| $9_W$ | $*$ | | | | | | | | |

Finally $i = 2$ and $i^* = 9$. When $j = 1$, $(a,b) = (0,2)$, $a' = 1$, and $c^* = (1,3)$. This is Case D so $c_1 = (1,2)$ and $c' = (1,3)$. When $j = 2$, $(a,b) = (1,3)$, $a' = 2$, and $c^* = (2,4)$. This is Case D so $c_2 = (2,3)$ and $c'_2 = (2,4)$. When $j = 3$, $(a,b) = (2,4)$, $a' = 3$, and $c^* = (3,5)$. This is Case C so $c_3 = (4,5)$; $D_2$ is below.

| | $9_N$ | $8_N$ | $7_N$ | $6_N$ | $5_N$ | $4_N$ | $3_N$ | $2_N$ | $1_N$ |
|---|---|---|---|---|---|---|---|---|---|
| $1_W$ | 0 | 0 | 0 | + | + | + | + | $\boxed{+}$ | $*$ |
| $2_W$ | 0 | + | + | 0 | + | + | $\boxed{+}$ | $*$ | |
| $3_W$ | 0 | 0 | 0 | 0 | + | $\boxed{0}$ | $*$ | | |
| $4_W$ | 0 | 0 | 0 | 0 | $\boxed{+}$ | $*$ | | | |
| $5_W$ | + | + | + | 0 | $*$ | | | | |
| $6_W$ | 0 | 0 | 0 | $*$ | | | | | |
| $7_W$ | + | 0 | $*$ | | | | | | |
| $8_W$ | 0 | $*$ | | | | | | | |
| $9_W$ | $*$ | | | | | | | | |

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