Decomposition into $weight \times level + jump$ and application to a new classification of primes

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Abstract

In this paper we introduce an Euclidean decomposition of elements a_n of an increasing sequence of natural numbers $(a_n)_{n \in \mathbb{N}^*}$ into weight \times level + jump which we use to classify the numbers a_n either by weight or by level. We then show that this decomposition can be seen as a generalization of the sieve of Eratosthenes (which is the particular case of the whole sequence of natural numbers). We apply this decomposition to prime numbers in order to obtain a new classification of primes, we analyze a few properties of this classification and we make a series of conjectures based on numerical data. Finally we show how composite numbers and 2-almost primes behave under the decomposition.

We introduce an algorithm whose input is an increasing sequence of positive integers $(a_n)_{n \in \mathbb{N}^*}$ and whose output is a sequence of unique triplets of positive integers $(k_n, L_n, d_n)_{n \in \mathbb{N}^*}$.

We define the *jump* (first difference, gap) of a_n by

$$d_n := a_{n+1} - a_n$$

Then let l_n be defined by

$$l_n := \begin{cases} a_n - d_n \text{ if } a_n - d_n > d_n; \\ 0 \text{ otherwise.} \end{cases}$$

The *weight* of a_n is defined to be

$$k_n := \begin{cases} \min\{k \in \mathbb{N}^* \text{ s.t. } k > d_n, k | l_n\} \text{ if } l_n \neq 0; \\ 0 \text{ otherwise.} \end{cases}$$

Finally we define the *level* of a_n by

$$L_n := \begin{cases} \frac{l_n}{k_n} \text{ if } k_n \neq 0; \\ 0 \text{ otherwise.} \end{cases}$$

We then have a decomposition of a_n into weight \times level + jump: $a_n = l_n + d_n = k_n \times L_n + d_n$ when $l_n \neq 0$.

In the Euclidean division of a_n by its weight k_n , the quotient is the level L_n , and the remainder is the jump d_n .

Lemma 1.1. A necessary and sufficient condition for the decomposition of a number a_n belonging to an increasing sequence of positive integers $(a_n)_{n \in \mathbb{N}^*}$ into weight \times level + jump to hold is that

$$a_{n+1} < \frac{3}{2}a_n.$$

Proof. The decomposition is possible if $l_n \neq 0$, that is if $a_n - d_n > d_n$, which can be rewritten as $a_{n+1} < \frac{3}{2}a_n$.

In order to use this algorithm to classify the numbers a_n we introduce the following rule (whose meaning will become clearer in the next section): if for a_n we have $k_n = L_n = l_n = 0$ then a_n is not classified; if for a_n we have $k_n > L_n$ then a_n is classified by *level*, if not then a_n is classified by *weight*.

2 Application of the algorithm to the sequence of natural numbers

In this situation we have $a_n = n$ et $d_n = 1$. The decomposition is impossible for n = 1 and n = 2 ($l_1 = l_2 = 0$). Apart from those two cases, we have the decomposition of n into weight \times level + jump: $n = k_n \times L_n + 1$ when n > 2 and we also have the following relations

$$L_n = 1$$

$$\Leftrightarrow \quad k_n > L_n$$

$$\Leftrightarrow \quad k_n = l_n = n - 1$$

$$\Leftrightarrow \quad l_n = n - 1 \text{ is prime,}$$

$$L_n > 1$$

$$\Leftrightarrow \quad k_n \le L_n$$

$$\Leftrightarrow \quad k_n \times L_n = l_n = n - 1$$

$$\Leftrightarrow \quad l_n = n - 1 \text{ is composite.}$$

The weight of n is the smallest prime factor of n-1 and the level of n is the largest proper divisor of n-1. We can characterize the fact that a number $l_n = n-1$ is prime by the fact that n is classified by level (or equivalently here by the fact that n is of level 1).

Since there is an infinity of prime numbers, there is an infinity of natural numbers of *level* 1. Similarly there is an infinity of natural numbers with a *weight* equal to k with k prime. The algorithm allows to separate prime numbers (l_n or *weights* of natural numbers classified by *level*) from composite numbers (l_n of natural numbers classified by *weight*), and is then indeed a reformulation of the sieve of Eratosthenes. Apply this algorithm to any other increasing sequence of positive integers, for example to the sequence of prime numbers itself, can be seen as an generalization of that sieve.

n <u>A000027</u>	$k_n \underline{A020639}(n-1)$	$L_n \underline{A032742}(n-1)$	d_n	l_n
1	0	0	1	0
2	0	0	1	0
3	2	1	1	2
4	3	1	1	3
5	2	2	1	4
6	5	1	1	5
7	2	3	1	6
8	7	1	1	7
9	2	4	1	8
10	3	3	1	9
11	2	5	1	10
12	11	1	1	11
13	2	6	1	12

Table 1: The 13 first terms of the sequences of *weights*, *levels*, *jumps*, and l_n in the case of the sequence of natural numbers.

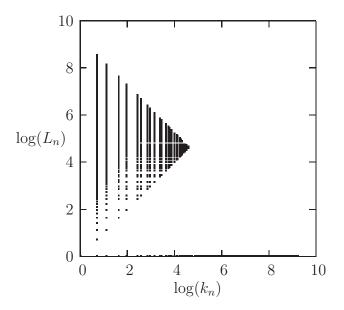


Figure 1: Plot of natural numbers in $\log(k_n)$ vs. $\log(L_n)$ coordinates (with $n \leq 10000$). The sieve of Eratosthenes.

3 Application of the algorithm to the sequence of primes

We can wonder what happens if we try to apply the decomposition to the sequence of primes itself: for any $n \in \mathbb{N}^*$ we have $a_n = p_n$ and $d_n = g_n$ (the prime gap). The algorithm of section 1 can then be rewritten with these new notations as follows.

The *jump* (first difference, gap) of p_n is

$$g_n := p_{n+1} - p_n$$

Let l_n be defined by

$$l_n := \begin{cases} p_n - g_n \text{ if } p_n - g_n > g_n; \\ 0 \text{ otherwise.} \end{cases}$$

The weight of p_n is then

$$k_n := \begin{cases} \min\{k \in \mathbb{N}^* \text{ s.t. } k > g_n, k | l_n\} \text{ if } l_n \neq 0; \\ 0 \text{ otherwise.} \end{cases}$$

The *level* of p_n is

$$L_n := \begin{cases} \frac{l_n}{k_n} & \text{if } k_n \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

In the Euclidean division of p_n by its weight k_n , the quotient is the level L_n , and the remainder is the jump g_n .

So we have the decomposition of p_n into $weight \times level + jump$ reads $p_n = k_n \times L_n + g_n$ when $l_n \neq 0$. So one should investigate for which n we have $l_n \neq 0$, which is provided by the following result.

Theorem 3.1. This decomposition is always possible except for $p_1 = 2$, $p_2 = 3$ and $p_4 = 7$ (i.e., $p_{n+1} \ge \frac{3}{2}p_n$ holds only for n = 1, n = 2 and n = 4).

Proof. The decomposition if possible if, and only if, l_n is not equal to zero. But $l_n \neq 0$ if and only if $p_{n+1} < \frac{3}{2}p_n$, that is $p_n - g_n > g_n(*)$ by lemma 1.1. Let us now apply results of Pierre Dusart on the prime counting function π to show that this is always true except for n = 1, n = 2 and n = 4.

Indeed this last equation (*) can be rewritten in terms of π as $\pi(\frac{3}{2}x) - \pi(x) > 1$ (i.e., there is always a number strictly included between x and $\frac{3}{2}x$ for any $x \in \mathbb{R}^+$). But Dusart has shown [1, 2] that on the one hand for $x \geq 599$ we have

$$\pi(x) \ge \frac{x}{\log x} \left(1 + \frac{1}{\log x}\right)$$

and on the other hand for x > 1 we have

$$\pi(x) \le \frac{x}{\log x} \left(1 + \frac{1.2762}{\log x} \right)$$

So for $x \ge 600$ we have

$$\pi(\frac{3}{2}x) - \pi(x) > \frac{900}{\log 900} \left(1 + \frac{1}{\log 900}\right) - \frac{600}{\log 600} \left(1 + \frac{1.2762}{\log 600}\right)$$

and since the right hand side of this inequality is approximately equal to 39.2 we indeed have that $\pi(\frac{3}{2}x) - \pi(x) > 1$, so the inequality (*) holds for any prime greater than 600. We check numerically that it also holds in the remaining cases when x < 600, except for the aforementioned exceptions n = 1, n = 2 and n = 4 which ends the proof. Let us now state a few direct results. For any p_n different from $p_1 = 2$, $p_2 = 3$ and $p_4 = 7$ we have

$$\gcd(g_n, 2) = 2,$$

$$\gcd(p_n, g_n) = \gcd(p_n - g_n, g_n) = \gcd(l_n, g_n) = \gcd(L_n, g_n) = \gcd(k_n, g_n) = 1,$$

$$3 \le k_n \le l_n,$$

$$1 \le L_n \le \frac{l_n}{3},$$

$$2 \le g_n \le k_n - 1,$$

$$2 \times g_n + 1 \le p_n.$$

Lemma 3.1. p is a prime such that p > 3 and p + 2 is also prime if and only if p has a weight equal to 3.

Proof. The primes p > 3 such that p + 2 is also prime are of the form 6n - 1, so p - 2 is of the form 6n - 3. The smallest divisor greater than 2 of a number of the form 6n - 3 is 3. If p_n has a weight equal to 3 then $p_n > 3$ and the jump g_n is equal to 2 since we know that $2 \le g_n \le k_n - 1$ and $2 \times g_n + 1 \le p_n$.

n	$p_n \underline{A000040}$	$k_n \ \underline{A117078}$	$L_n \ A117563$	$d_n \underline{A001223}$	$l_n A118534$
1	2	0	0	1	0
2	3	0	0	2	0
3	5	3	1	2	3
4	7	0	0	4	0
5	11	3	3	2	9
6	13	9	1	4	9
7	17	3	5	2	15
8	19	5	3	4	15
9	23	17	1	6	17
10	29	3	9	2	27
11	31	25	1	6	25
12	37	11	3	4	33
13	41	3	13	2	39
14	43	13	3	4	39
15	47	41	1	6	41
16	53	47	1	6	47
17	59	3	19	2	57

Table 2: The 17 first terms of the sequences of *weights*, *levels*, *jumps* and l_n in the case of the sequence of primes.

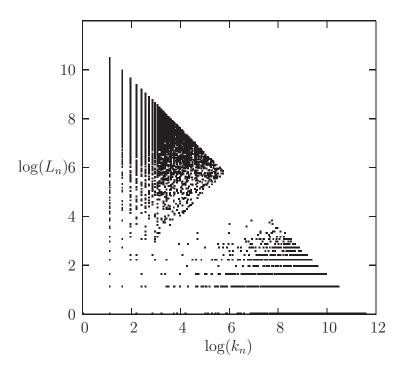


Figure 2: Plot of prime numbers in $\log(k_n)$ vs. $\log(L_n)$ coordinates (with $n \leq 10000$).

4 Classification of prime numbers

We introduce the following classification principle:

- if for p_n we have $k_n = L_n = l_n = 0$ then p_n is not classified;

- if for p_n we have $k_n > L_n$ then p_n is classified by *level*, if not p_n is classified by *weight*;

- furthermore if for p_n we have that l_n is equal to some prime p_{n-i} then p_n is of level (1; i).

For $n \leq 5.10^7$, 17, 11% of the primes p_n are classified by *level* and 82, 89% are classified by *weight*.

We have the following direct results: If p_n is classified by *weight* then

$$g_n + 1 \le k_n \le \sqrt{l_n} \le L_n \le \frac{l_n}{3}.$$

If p_n is classified by *level* then

$$L_n < \sqrt{l_n} < k_n \le l_n;$$

$$L_n + 2 \le g_n + 1 \le k_n \le l_n.$$

Pimes p_n for which $g_n > \sqrt{(l_n)}$ are : 2, 3, 5, 7, 13, 19, 23, 31, 113 for $n \le 5.10^7$.

Number of	which have	% / total of primes	% / total
primes	a <i>weight</i> equal to	classified by <i>weight</i>	
3370444	3	8.132	6.741
1123714	5	2.711	2.247
1609767	7	3.884	3.219
1483560	9	3.579	2.967
1219514	11	2.942	2.439
1275245	13	3.077	2.550
1260814	15	3.042	2.522
1048725	17	2.530	2.097
1051440	19	2.546	2.103
1402876	21	3.385	2.806
893244	23	2.155	1.786

Table 3: Distribution of primes for the 11 smallest weights (with $n \leq 5.10^7$).

Number of	which are	% / total of primes	% / total
primes	of <i>level</i>	classified by <i>level</i>	
2664810	1	31.15	5.330
2271894	3	26.56	4.544
963665	5	11.27	1.927
444506	7	5.197	0.8890
640929	9	7.493	1.282
254686	11	2.978	0.5094
155583	13	1.819	0.3112
351588	15	4.110	0.7032
115961	17	1.356	0.2319
78163	19	0.9138	0.1563
148285	21	1.734	0.297

Table 4: Distribution of primes for the 11 smallest *levels* (with $n \leq 5.10^7$).

If p_n is of *level* (1; i) then

$$L_n = 1$$
 and $l_n = k_n = p_{n-i}$,
 $p_n = p_{n-i} + g_n$ or $p_{n+1} - p_n = p_n - p_{n-i}$.

Primes of *level* (1; 1) are the so-called "balanced primes" (<u>A006562</u>). If p_n is of *level* (1; 1) then $I_n = 1 \text{ and } I_n = k_n = m$

$$L_n = 1 \text{ and } l_n = k_n = p_{n-1},$$

$$g_n = g_{n-1} \text{ or } p_{n+1} - p_n = p_n - p_{n-1},$$

$$p_n = \frac{p_{n+1} + p_{n-1}}{2}.$$

Number of	which are of <i>level</i> $(1; i)$
primes	i
1307356	1
746381	2
345506	3
153537	4
65497	5
27288	6
11313	7

Table 5: Distribution of primes of *level* (1; i) (with $n \leq 5.10^7$, $i \leq 7$).

n <=	number of primes	number of primes	% primes	% primes
	classified by level	classified by weight	classified by level	classified by weight
100	44	53	44	53
1000	324	673	32,4	$67,\!3$
10000	2766	7231	27,66	72,31
100000	22999	76998	23	77
1000000	203441	796556	20,34	79,66
10000000	1828757	8171240	18,29	81,71
50000000	8553468	41446529	17,11	82,89

Table 6: repartition between primes classified by *level* and primes classified by *weight*.

5 Conjectures on primes

From our numerical data on the decomposition of primes p_n until $n = 5.10^7$ we make the following conjectures.

Since we have shown previously that the smallest number of each twin prime pair (except 3) has a *weight* equal to 3, the well-known conjecture on the existence of an infinity of twin primes can be rewritten as

Conjecture 1. The number of primes with a weight equal to 3 is infinite.

To extend this conjecture, and by analogy with the decomposition of natural numbers for which we know that for any prime k there exist an infinity of natural numbers with a *weight* equal to k and that there exist an infinity of natural numbers of *level* 1, we make this two conjectures

Conjecture 2. The number of primes with a weight equal to k is infinite for any $k \ge 3$ which is not a multiple of 2.

Conjecture 3. The number of primes of level L is infinite for any $L \ge 1$ which is not a multiple of 2.

Now, based on our numerical data and again by analogy with the decomposition of natural numbers for which we know that the natural numbers which are classified by *level* have a l_n or a *weight* which is always prime we conjecture

Conjecture 4. Except for $p_6 = 13$, $p_{11} = 31$, $p_{30} = 113$, $p_{32} = 131$ et $p_{154} = 887$, primes which are classified by level have a weight which is itself a prime.

The conjecture on the existence of an infinity of balanced primes can be rewritten as

Conjecture 5. The number of primes of level (1; 1) is infinite.

That we can easily generalize by

Conjecture 6. The number of primes of level (1; i) is infinite for any $i \ge 1$.

We make the following conjectures, for which we have no rigorous arguments yet

Conjecture 7. If the jump g_n is not a multiple of 6 then l_n is a multiple of 3.

Conjecture 8. If l_n is not a multiple of 3 then jump the g_n is a multiple of 6.

According to the numerical data and knowing that the primes are rarefying among the natural numbers, we make the following conjecture:

Conjecture 9. The primes classified by level are rarefying among the prime numbers.

6 Decomposition of composite numbers and of 2–almost primes.

In this section we only provide the plots of the distribution of composite numbers and of $2-\text{almost primes in } \log(k_n)$ vs. $\log(L_n)$ coordinates.

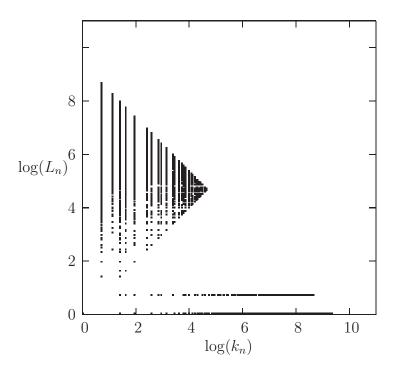


Figure 3: Plot of composite numbers (A002808) in $\log(k_n)$ vs. $\log(L_n)$ coordinates (with $n \leq 9999$).

The sequence of *weights* of composite numbers is <u>A130882</u>.

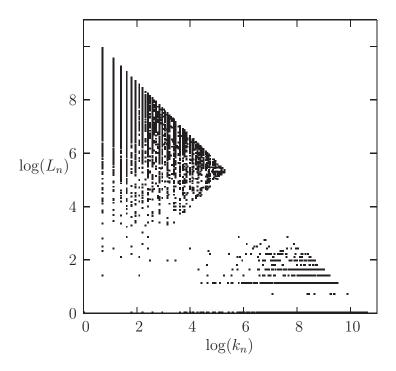


Figure 4: Plot of 2-almost primes (<u>A001358</u>) in $\log(k_n)$ vs. $\log(L_n)$ coordinates (with $n \leq 9999$).

The sequence of *weights* of 2-almost primes is <u>A130533</u>.

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2000 *Mathematics Subject Classification*: Primary 11B83; Secondary 11P99. *Keywords:* integer sequences, Eratosthenes, sieve, primes, classification, prime gaps, twin primes, balanced primes.

(Concerned with sequences <u>A000027</u>, <u>A020639</u>, <u>A032742</u>, <u>A000040</u>, <u>A117078</u>, <u>A117563</u>, <u>A001223</u>, <u>A118534</u>, <u>A001359</u>, <u>A006562</u>, <u>A125830</u>, <u>A117876</u>, <u>A074822</u>, <u>A002808</u>, <u>A130882</u>, <u>A001358</u> and <u>A130533</u>.)