THE HECKE GROUP ALGEBRA OF A COXETER GROUP AND ITS REPRESENTATION THEORY

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Abstract. Let W be a finite Coxeter group. We define its Hecke-group algebra by gluing together appropriately its group algebra and its 0-Hecke algebra. We describe in detail this algebra (dimension, several bases, conjectural presentation, combinatorial construction of simple and indecomposable projective modules, Cartan map) and give several alternative equivalent definitions (as symmetry preserving operator algebra, as poset algebra, as commutant algebra, ...).

In type A, the Hecke-group algebra can be described as the algebra generated simultaneously by the elementary transpositions and the elementary sorting operators acting on permutations. It turns out to be closely related to the monoid algebras of respectively nondecreasing functions and nondecreasing parking functions, the representation theory of which we describe as well.

This defines three towers of algebras, and we give explicitly the Grothendieck algebras and coalgebras given respectively by their induction products and their restriction coproducts. This yields some new interpretations of the classical bases of quasi-symmetric and noncommutative symmetric functions as well as some new bases.

Contents

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1. Introduction

Given an *inductive tower of algebras*, that is a sequence of algebras

$$A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n \hookrightarrow \cdots, \tag{1}$$

with embeddings $A_m \otimes A_n \hookrightarrow A_{m+n}$ satisfying an appropriate associativity condition, one can introduce two *Grothendieck rings*

$$\mathcal{G}(A) := \bigoplus_{n\geq 0} G_0(A_n) \qquad \text{and} \qquad \mathcal{K}(A) := \bigoplus_{n\geq 0} K_0(A_n)\,, \tag{2}$$

where $G_0(A)$ and $K_0(A)$ are the (complexified) Grothendieck groups of the categories of finite-dimensional A-modules and projective A-modules respectively, with the multiplication of the classes of an A_m-module M and an A_n-module N defined by the induction product

$$[M]\cdot[N] = [M\widehat{\otimes}N] = [M\otimes N\uparrow_{A_m\otimes A_n}^{A_{m+n}}]\,. \tag{3}$$

If A_{m+n} is a projective $A_m \otimes A_n$-module, one can define a coproduct on these rings by means of restriction of representations, turning these into coalgebras. Under favorable circumstances the product and the coproduct are compatible turning these into mutually dual Hopf algebras.

The basic example of this situation is the character ring of the symmetric groups (over $\mathbb{C}$), due to Frobenius. Here the $A_n := \mathbb{C}[\mathfrak{S}_n]$ are semi-simple algebras, so that

$$G_0(A_n) = K_0(A_n) = R(A_n)\,, \tag{4}$$

where $R(A_n)$ denotes the vector space spanned by isomorphism classes of indecomposable modules which, in this case, are all simple and projective. The irreducible representations $[\lambda]$ of A_n are parametrized by partitions λ of n, and the Grothendieck ring is isomorphic to the algebra Sym of symmetric functions under the correspondence $[\lambda] \leftrightarrow s_\lambda$, where s_λ denotes the Schur function associated with λ. Other known examples with towers of group algebras over the complex numbers $A_n := \mathbb{C}[G_n]$ include the cases of wreath products $G_n := \Gamma \wr \mathfrak{S}_n$ (Specht), finite linear groups $G_n := GL(n, \mathbb{F}_q)$ (Green), *etc.*, all related to symmetric functions (see [Mac95, Zel81]).

Examples involving non-semisimple specializations of Hecke algebras have also been worked out. Finite Hecke algebras of type A at roots of unity ($A_n = \mathcal{H}_n(\zeta)$, $\zeta^r = 1$) yield quotients and subalgebras of Sym [LLT96]. The Ariki-Koike algebras at roots of unity give rise to level r Fock spaces of affine Lie algebras of type A [AK94]. The 0-Hecke algebras $A_n = \mathcal{H}_n(0)$ corresponds to the pair Quasi-symmetric functions / Noncommutative symmetric functions, $\mathcal{G} =$ QSym, $\mathcal{K} =$ NCSF [KT97]. Affine Hecke algebras at roots of unity lead to $U(\widehat{sl}_r)$ and $U(\widehat{sl}_r)^*$ [Ari96], and the case of affine Hecke generic algebras can be reduced to a subcategory admitting as Grothendieck rings $U(\widehat{gl}_\infty)$ and $U(\widehat{gl}_\infty)^*$ [Ari96]. Further interesting examples are the tower of 0-Hecke-Clifford algebras [Ols92, BHT04] giving rise to the peak algebras [Ste97], and a degenerated version of the Ariki-Koike algebras [HNT06] giving rise to a colored version of QSym and NCSF.

The goal of this article is to study the representation theories of several towers of algebras which are related to the symmetric groups and their Hecke algebras $\mathcal{H}_n(q)$. We describe their representation theory and the Grothendieck algebras and coalgebras arising from them. Here is the structure of the paper together with the main results.

In Section 3, we introduce the main object of this paper, namely the *Hecke-Group algebra* $\mathcal{H}W$ of a (finite) Coxeter group W. It is constructed as the smallest algebra containing simultaneously the group algebra of W and its 0-Hecke algebra (and in fact any other q-Hecke algebra of W). It turns out that this algebra has unexpectedly nice properties. We first show that $\mathcal{H}W$ is better understood as the algebra of antisymmetry (or symmetry) preserving operators; this allows us to compute its dimension, and to give an explicit basis. We further realize it as the incidence algebra of a pre-order and derive from it its representation theory. In particular, we construct explicitly the projective and simple modules. The Cartan matrix suggests a link between $\mathcal{H}W$ and the incidence algebra of the boolean lattice. We actually show that these algebras are Morita equivalent. See also Section 7 for further comments on the links between $\mathcal{H}W$ and the affine Hecke algebras. Turning back to type A, we get a new tower of algebras $\mathcal{H}\mathfrak{S}_n$. Specifically, each $\mathcal{H}\mathfrak{S}_n$ is the algebra generated by both elementary transpositions and elementary sorting operators acting on permutations of $\{1, \dots, n\}$. We compute the restrictions and inductions of simple and projective modules. This gives rise to a new interpretation of some bases of quasi-symmetric and noncommutative symmetric functions in representation theory.

In Sections 4 and 5 we turn to the study of two other towers, namely the towers of the monoids algebras of nondecreasing functions and of nondecreasing parking functions. Finally, we prove that those two algebras are the respective quotients of $\mathcal{H}\mathfrak{S}_n$ and $\mathcal{H}_n(0)$, through their representations on exterior powers. We deduce their structure of projective and simple modules, their cartan matrices, and the induction and restrictions rules. We also show that the algebra of nondecreasing parking functions is isomorphic to the incidence algebra of some lattice.

The following diagram summarizes the relations between all the aforementioned towers of algebras:

$$\begin{tikzcd}
\mathcal{H}_n(-1) \arrow[d, two heads] \arrow[rrrr, hook, bend left] & \mathcal{H}_n(0) \arrow[d, two heads] \arrow[rrr, hook, bend left] & \mathcal{H}_n(1) = \mathbb{C}[\mathfrak{S}_n] \arrow[d, two heads] \arrow[rr, hook, bend left] & \mathcal{H}_n(q) \arrow[d, two heads] \arrow[r, hook] & H\mathfrak{S}_n \arrow[d, two heads] \\
\mathrm{TL}_n \arrow[rrrr, hook, bend right] & \mathbb{C}[\mathrm{NDPF}_n] \arrow[rrr, hook, bend right] & \mathbb{C}[\mathfrak{S}_n] \circlearrowleft \bigwedge \mathbb{C}^n \arrow[rr, hook, bend right] & \mathcal{H}_n(q) \circlearrowleft \bigwedge \mathbb{C}^n \arrow[r, hook] & \mathbb{C}[\mathrm{NDF}_n]
\end{tikzcd} \tag{5}$$

This paper mostly reports on a computation driven research using the package `MuPAD-Combinat` by the authors of the present paper [HT04]. This package is designed for the computer algebra system `MuPAD` and is freely available from `http://mupad-combinat.sf.net/`. Among other things, it allows to automatically compute the dimensions of simple and indecomposable projective modules together with the Cartan invariants matrix of a finite dimensional algebra, knowing its multiplication table.

2. Background

2.1. **Compositions and sets.** Let n be a fixed integer. Recall that each subset S of $\{1, \dots, n-1\}$ can be uniquely identified with a p-tuple $I := (i_1, \dots, i_p)$ of positive integers of sum n:

$$S = \{s_1 < s_2 < \cdots < s_p\} \longmapsto \mathrm{C}(S) := (s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_p)\,. \tag{6}$$

We say that I is a *composition of* n and we write it by $I \vDash n$. The converse bijection, sending a composition to its *descent set*, is given by:

$$I = (i_1, \dots, i_p) \longmapsto \mathrm{Des}(K) = \{i_1 + \cdots + i_j \mid j = 1, \dots, p-1\}\,. \tag{7}$$

The number p is called the *length* of I and is denoted by $\ell(I)$.

The notions of complementary of a set S^c and of inclusion of sets can be transfered to compositions, leading to the complementary of a composition K^c and to the refinement order on compositions: we say that I is *finer* than J, and write $I \succeq J$, if and only if $\mathrm{Des}(I) \supseteq \mathrm{Des}(J)$.

2.2. **Coxeter groups and Iwahory-Hecke algebras.** Let (W, S) be a Coxeter group, that is a group W with a presentation

$$W = \langle\, S \mid (ss')^{m(s,s')},\ \forall s, s' \in S\,\rangle\,, \tag{8}$$

where each $m(s, s')$ is in $\{1, 2, \dots, \infty\}$, and $m(s, s) = 1$. The elements $s \in S$ are called *simple reflections*, and the relations can be rewritten as:

$$\begin{aligned} s^2 &= \mathrm{id}\,, && \text{for all } s \in S\,,\\ \underbrace{ss'ss's\cdots}_{m(s,s')} &= \underbrace{s'ss'ss'\cdots}_{m(s,s')}\,, && \text{for all } s, s' \in S\,, \end{aligned} \tag{9}$$

Most of the time, we just write W for (W, S). In general, we follow the notations from [BB05], and we refer to this monograph for details on Coxeter groups and there Hecke algebras. Unless stated otherwise, we always assume that W is finite.

The prototypical example of Coxeter group is the n-th symmetric group $(W, S) := A_{n-1} = (\mathfrak{S}_n, \{s_1, \dots, s_{n-1}\})$, where s_i denotes the *elementary transposition* which exchanges i and $i + 1$. The relations are given by:

$$\begin{aligned} s_i^2 &= \mathrm{id}\,, && \text{for } 1 \le i \le n-1\,,\\ s_i s_j &= s_j s_i\,, && \text{for } |i - j| \ge 2\,,\\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}\,, && \text{for } 1 \le i \le n-2\,; \end{aligned} \tag{10}$$

the last two relations are called the *braids relations*. When we want to write explicitely a permutation μ in $\mathfrak{S}_n$ we will use the *one line notation*, that is the sequence $\mu_1\mu_2\cdots\mu_n := \mu(1)\mu(2)\cdots\mu(n)$.

A *reduced word* for an element μ of W is a decomposition $\mu = s_1 \cdots s_k$ of μ into a product of generators in S of minimal length $l(\mu)$. A *(right) descent* of μ is an element $s \in S$ such that $l(\mu s) < l(\mu)$. If μ is a permutation, this translates into $\mu_i > \mu_{i+1}$. A *recoil* (or *left descent*) of μ is a descent of μ^{-1}. The sets of left and right descents of μ are denoted respectively by $\mathrm{iDes}(\mu)$ and $\mathrm{Des}(\mu)$. The coxeter group W comes equipped with three natural lattice structures. Namely $\mu < \nu$, in the *Bruhat order* (resp. *(weak) left Bruhat order*, resp. *(weak) right Bruhat order*) if some reduced word for μ is a subword (resp. a right factor, resp. left factor) of some reduced word for ν. In type A, the weak Bruhat orders are the usual left and right permuthoedron.

For a subset I of S, the parabolic subgroup W_I of W is the Coxeter subgroup of W generated by I. Left and right cosets representatives for the quotient of W by

W_I are given respectively by the *recoils class*:

$$ {}^I W = \{\mu \in W \mid \mathrm{iDes}(\mu) \cap I = \emptyset\} \tag{11} $$

and the descent class:

$$ W^I = \{\mu \in W \mid \mathrm{Des}(\mu) \cap I = \emptyset\} \tag{12} $$

For q a complex number, let $\mathcal{H}(W)(q)$ be the (Iwahori-)*Hecke algebra* of W over the field $\mathbb{C}$. This algebra of dimension $|W|$ has a linear basis $\{T_\mu\}_{\mu\in W}$, and its multiplication is determined by

$$ \begin{cases} T_s T_\mu = (q-1)T_{sw} + (q-1)T_w & \text{if } s \in S \text{ and } \ell(s\mu) < \ell(\mu) \\ T_\mu T_\nu = T_{\mu\nu} & \text{if } \ell(\mu) + \ell(\nu) = \ell(\mu\nu) \end{cases} \tag{13} $$

In particular, for any element μ of W, we have $T_\mu = T_{s_1} \cdots T_{s_k}$ where $s_1, \ldots, s_k$ is any reduced word for μ. In fact, $\mathcal{H}(W)(q)$ is the algebra generated by the elements T_s, $s \in S$ subject to the same relations as the s themselves, except that the quadratic relation $s^2 = 1$ is replaced by:

$$ T_s^2 = (q-1)T_s + q\,. \tag{14} $$

Setting $q = 1$ yields back the usual group algebra $\mathbb{C}[W]$ of W. Similarly, the 0-Hecke algebra $\mathcal{H}(W)(0)$ is obtained by setting $q = 0$ in these relations. Then, the first relation becomes $T_s^2 = -T_s$ [Nor79, KT97]. In this paper, we prefer to use another set of generators $(\pi_s)_{s\in S}$ defined by $\pi_i := T_i + 1$. They also satisfy the braid-like relations together with the quadratic relations $\pi_s^2 = \pi_s$. Note that the 0-Hecke algebra is thus a monoid algebra.

2.3. **Representation theory.** In this paper, we mostly consider *right* modules over algebras. Consequently the composition of two endomorphisms f and g is denoted by $fg = g \circ f$ and their action on a vector v is written $v \cdot f$. Thus $g \circ f(v) = g(f(v))$ is denoted $v \cdot fg = (v \cdot f) \cdot g$.

It is known that $\mathcal{H}(W)(0)$ has $2^{|S|}$ simple modules, all one-dimensional, and naturally labelled by subsets I of S [Nor79]: following the notation of [KT97], let η_I be the generator of the simple $\mathcal{H}(W)(0)$-module S_I associated with I in the left regular representation. It satisfies

$$ \eta_I \cdot T_i := \begin{cases} -\eta_I & \text{if } i \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{or equivalently} \qquad \eta_I \cdot \pi_i := \begin{cases} 0 & \text{if } i \in I, \\ \eta_I & \text{otherwise.} \end{cases} \tag{15} $$

The indecomposable projective module P_I associated with S_I (that is such that $S_I = P_I/\mathrm{rad}(P_I)$) can be described as follows: it has a basis $\{b_\mu \mid \mathrm{iDes}(\mu) = I\}$ with the action

$$ b_\mu \cdot T_s = \begin{cases} -b_\mu & \text{if } s \in \mathrm{Des}(\mu), \\ b_{\mu s} & \text{if } s \notin \mathrm{Des}(\mu) \text{ and } \mathrm{iDes}(\mu s) = I \\ 0 & \text{otherwise.} \end{cases} \tag{16} $$

In type A, it is customary to index the projective and simple modules of $\mathcal{H}_n(0)$ by compositions of n. For notational convenience and when there is no ambiguity, we simply identify the subset I of $S = \{1, \ldots, n-1\}$ and the corresponding composition $C(I) = (i_1, \ldots, i_k)$ of n.

The Grothendieck rings of $\mathcal{H}_n(0)$ are naturally isomorphic to the dual pair of Hopf algebras of quasi-symmetric functions QSym of Gessel [Ges84] and of noncommutative symmetric functions NCSF [GKL$^+$95] (see [KT97]). The reader who is not familiar with those should refer to these papers, as we will only recall the required notations here.

The Hopf algebra QSym of quasi-symmetric functions has two remarkable bases, namely the *monomial basis* $(M_I)_I$ and the *fundamental basis* (also called *quasi-ribbon*) $(F_I)_I$. They are related by

$$F_I = \sum_{I \succeq J} M_J \qquad \text{or equivalently} \qquad M_I = \sum_{I \succeq J} (-1)^{\ell(I)-\ell(J)} F_J\,. \tag{17}$$

The characteristic map $S_I \mapsto F_I$ which sends the simple $\mathcal{H}_n(0)$-module S_I to its corresponding fundamental function F_I also sends the induction product to the product of QSym and the restriction coproduct to the coproduct of QSym.

The Hopf algebra NCSF of noncommutative symmetric functions [GKL$^+$95] is a noncommutative analogue of the algebra of symmetric functions [Mac95]. It has for multiplicative bases the analogues $(\Lambda^I)_I$ of the elementary symmetric functions $(e_\lambda)_\lambda$ and as well as the analogues $(S^I)_I$ of the complete symmetric functions $(h_\lambda)_\lambda$. The relevant basis in the representation theory of $\mathcal{H}_n(0)$ is the basis of so called *ribbon Schur functions* $(R_I)_I$ which is an analogue of skew Schur functions of ribbon shape. It is related to $(\Lambda_I)_I$ and $(S_I)_I$ by

$$S_I = \sum_{I \succeq J} R_J \qquad \text{and} \qquad \Lambda_I = \sum_{I \succeq J} R_{J^c}\,. \tag{18}$$

Their interpretation in representation theory goes as follows. The complete function S^n is the characteristic of the trivial module $S_n \approx P_n$, the elementary function Λ^n being the characteristic of the sign module $S_{1^n} \approx P_{1^n}$. An arbitrary indecomposable projective module P_I has R_I for characteristic. Once again the map $P_I \mapsto R_I$ is an isomorphism of Hopf algebras.

Recall that S_I is the semi-simple module associated to P_I, giving rise to the duality between $\mathcal{G}$ and $\mathcal{K}$:

$$S_I = P_I/\mathrm{rad}(P_I) \qquad \text{and} \qquad \langle P_I\ ,\ S_J\rangle = \delta_{I,J} \tag{19}$$

This translates into QSym and NCSF by setting that $(F_I)_I$ and $(R_I)_I$ are dual bases, or equivalently that $(M_I)_I$ and $(S^I)_I$ are dual bases.

3. The algebra $\mathcal{H}W$

Let (W, S) be a finite Coxeter group. Its group algebra $\mathbb{C}[W]$ and its 0-Hecke algebra $\mathcal{H}(W)(0)$ can be realized simultaneously as operator algebras by identifying the underlying vector spaces of their right regular representations. There are several ways to do that, depending on which basis elements of $\mathcal{H}(W)(0)$ we choose to identify with elements of W. It turns out that the following identification leads to interesting properties.

Namely, consider the plain *vector space* $\mathbb{C}W$. On the first hand, we identify $\mathbb{C}W$ with the right regular representation of the algebra $\mathbb{C}[W]$, i.e.: $\mathbb{C}[W]$ acts on $\mathbb{C}W$ by multiplication on the right. In type A, this is the usual action on positions, where an elementary transposition s_i acts on a permutation $\mu := (\mu_1, \dots, \mu_n)$ by exchanging μ_i and μ_{i+1}: $\mu \cdot s_i = \mu s_i$.

On the other hand, we also identify $\mathbb{C}W$ with the regular representation of the 0-Hecke algebra $\mathcal{H}(W)(0)$, i.e.: $\mathcal{H}(W)(0)$ acts on the right on $\mathbb{C}W$ by

$$\mu \cdot \pi_s = \begin{cases} \mu & \text{if } \ell(\mu s) < \ell(\mu) \\ \mu s & \text{otherwise.} \end{cases} \tag{20}$$

In type A, the $\pi_i := \pi_{s_i}$'s are the *elementary decreasing bubble sort operators*:

$$\mu \cdot \pi_i = \begin{cases} \mu & \text{if } \mu_i > \mu_{i+1}, \\ \mu s_i & \text{otherwise.} \end{cases} \tag{21}$$

The following easy lemma will be useful in the sequel.

Lemma 3.1. *Let $\sigma, \tau \in W$. Then,*

(a) *There exists τ' such that $\sigma.\pi_\tau = \sigma\tau'$ with $\ell(\sigma\tau') = \ell(\sigma) + \ell(\tau')$. Furthermore, $\tau' = 1$ if and only if $\tau \in W_{\mathrm{Des}(\sigma)}$.*
(b) *There exists σ' such that $\sigma.\pi_\tau = \sigma'\tau$ with $\ell(\sigma'\tau) = \ell(\sigma') + \ell(\tau)$. Furthermore, $\sigma' = 1$ if and only if $\sigma \in W_{\mathrm{iDes}(\tau)}$.*

Proof. Applying π_s on an element σ either leaves σ unchanged if $s \in \mathrm{Des}(\sigma)$, or extends any reduced word for σ by s otherwise. (a) follows by induction; in particular τ' is smaller than τ in the Bruhat order of W.

(b) Since $\mathbb{C}W$ is the right regular representation, the linear map

$$\Phi : \begin{cases} \mathbb{C}W & \to \mathcal{H}(W)(0) \\ \tau & \mapsto \pi_\tau \end{cases} \tag{22}$$

is a morphism of $\mathcal{H}(W)(0)$-module. Consequently, one has $\pi_{\sigma.\pi_\tau} = \pi_\sigma \pi_\tau$ which allows us to lift the computation to the 0-Hecke monoid. There σ and τ play a symmetric role, and (b) follows from (a) by reversion of the reduced words. □

Definition 3.2. The algebra $\mathcal{H}W$ is the subalgebra of $\mathrm{End}(\mathbb{C}W)$ generated by both sets of operators $\{s, \pi_s\}_{s\in S}$.

By construction, the algebra $\mathcal{H}W$ contains both $\mathbb{C}[W]$ and $\mathcal{H}(W)(0)$. In fact, it contains simultaneously all the Hecke algebras: for any values of q, $\mathcal{H}(W)(q)$ can be realized by taking the subalgebra of $\mathcal{H}W$ generated by the operators:

$$T_s := (q-1)(1-\pi_s) + qs\,, \qquad \text{for } s \in S\,. \tag{23}$$

A direct calculation show that the so-defined T_s actually verifies the Hecke relation. Reciprocally, we can recover back $\mathcal{H}W$ by choosing for each s any two generators $T_s(q_1)$ and $T_s(q_2)$ with $q_1 \neq q_2$, because for any q, q_1, q_2,

$$T_s(q) := \frac{q-q_1}{q_2-q_1} T_s(q_1) + \frac{q-q_2}{q_1-q_2} T_s(q_2)\,. \tag{24}$$

Note that setting $T_s(1) := s$ and $T_s(0) := \pi - 1$ this last equation implies the previous one when $q_1 = 1$ and $q_2 = 0$.

Let further $\overline{\pi}_s := \pi_s s$ be the operator in $\mathcal{H}W$ which removes the descent s. In type A, the $\overline{\pi}_i := \overline{\pi}_{s_i}$'s are the *elementary increasing bubble sort operators*:

$$\mu \cdot \overline{\pi}_i = \begin{cases} \mu & \text{if } \mu_i < \mu_{i+1}\,, \\ \mu s_i & \text{otherwise.} \end{cases} \tag{25}$$

Since $\pi_s + \overline{\pi}_s$ is a symmetrizing operator, we have the identity:

$$\pi_s + \overline{\pi}_s = 1 + s\,. \tag{26}$$

It follows that we can alternatively take as generators for $\mathcal{H}W$ the operators π_s's and $\overline{\pi}_s$'s.

In type A, the natural embedding of $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_m$ in $\mathbb{C}\mathfrak{S}_{n+m}$ makes $(\mathcal{H}\mathfrak{S}_n)_{n\in\mathbb{N}}$ into a tower of algebras, which contains the similar towers of algebras $(\mathbb{C}[\mathfrak{S}_n])_{n\in\mathbb{N}}$ and $(\mathcal{H}_n(()q))_{n\in\mathbb{N}}$.

3.1. **Basic properties.**

Example 3.3. Much of the structure of $\mathcal{H}W$ readily appears for $W := \mathfrak{S}_2$. Take the natural basis $(12, 21)$ of $\mathbb{C}\mathfrak{S}_2$. The matrices of the operators 1, s_1, π_1, and $\overline{\pi}_1$ are respectively:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}. \tag{27}$$

The algebra $\mathcal{H}\mathfrak{S}_2$ is of dimension 3 with basis $\{1, s_1, \pi_1\}$ and multiplication table:

(28)

	1	s_1	π_1
1	1	s_1	π_1
s_1	s_1	1	π_1
π_1	π_1	$\overline{\pi}_1 = 1 + s_1 - \pi_1$	π_1

This algebra can alternatively be described by equations. Namely, take $f \in \operatorname{End}(\mathbb{C}\mathfrak{S}_2)$ with matrix

$$\begin{pmatrix}f_{12,12} & f_{12,21}\\ f_{21,12} & f_{21,21}\end{pmatrix}; \tag{29}$$

then, the following properties are equivalent:

- f belongs to $\mathcal{H}\mathfrak{S}_2$;
- $f_{21,21} - f_{21,12} + f_{12,21} - f_{12,12} = 0$;
- $\begin{pmatrix}1\\ -1\end{pmatrix} = 12 - 21$ is an eigenvector of f;
- $(1 - s_1)f(1 + s_1) = 0$.

In general the relations in the parabolic subalgebra $\mathbb{C}[\pi_i, \overline{\pi}_i, s_i]$ of $\mathcal{H}W$ are:

$$\begin{aligned} s_i\pi_i &= \pi_i\ , & s_i\overline{\pi}_i &= \overline{\pi}_i\ ,\\ \overline{\pi}_i\pi_i &= \pi_i\ , & \pi_i\overline{\pi}_i &= \overline{\pi}_i\ ,\\ \pi_i s_i &= \overline{\pi}_i\ , & \overline{\pi}_i s_i &= \pi_i\ ,\\ & \pi_i + \overline{\pi}_i = 1 + s_i\,. \end{aligned} \tag{30}$$

In particular, any two of $\{\pi_i, \overline{\pi}_i, s_i\}$ can be taken as generators.

A computer exploration suggests that in type A the dimension of $\mathcal{H}\mathfrak{S}_n$ is given by the following sequence (sequence A000275 of the encyclopedia of integer sequences [Se03]):

$$1, 1, 3, 19, 211, 3651, 90921, 3081513, 136407699, 7642177651, 528579161353, 44237263696473, \dots$$

These are the numbers h_n of pairs (σ, τ) of permutations without common descents ($\operatorname{Des}(\sigma) \cap \operatorname{Des}(\tau) = \emptyset$). They where first considered by Carlitz [Car55, CSV76a, CSV76b] as coefficient of the doubly exponential expansion of the inverse Bessel function J_0:

$$\sum_{n>=0} \frac{h_n}{n!^2} x^n = \frac{1}{J_0(\sqrt{4x})} \,. \tag{31}$$

Together with Equation (30), this leads to state the following

Theorem 3.4. *Let W be a finite Coxeter group. A vector space basis of $\mathcal{H}W$ is given by the family of operators*

$$B := \left\{ \sigma\pi_\tau \mid (\sigma,\tau) \in W^2,\ \operatorname{Des}(\sigma) \cap \operatorname{iDes}(\tau) = \emptyset \right\} \,. \tag{32}$$

In particular the dimension of $\mathcal{H}W$ is the number h of pairs of elements of W without common descents.

Our first approach to prove this theorem was to search for a presentation of the algebra. In type A, the following relations are easily proved to hold:

$$\begin{aligned}
\pi_{i+1}s_i &= \pi_{i+1}\pi_i + s_i s_{i+1}\pi_i\pi_{i+1} - \pi_i\pi_{i+1}\pi_i \ , \\
\pi_i s_{i+1} &= \pi_i\pi_{i+1} + s_{i+1}s_i\pi_{i+1}\pi_i - \pi_i\pi_{i+1}\pi_i \ , \\
s_i\pi_{i+1}s_i &= s_{i+1}\pi_i s_{i+1} \ ,
\end{aligned} \tag{33}$$

and we conjecture that they generate all relations.

Conjecture 3.5. A presentation of $\mathcal{H}\mathfrak{S}_n$ is given by the defining relations of $\mathfrak{S}_n$ and $\mathcal{H}_n(0)$ together with the relations $s_i\pi_i = \pi_i$ and those of Equations (33).

Using those relations as rewriting rules yields a straightening algorithm which rewrites any expression in the s_i's and π_i's into a linear combination of the $\sigma\pi_\tau$. This algorithm seems, in practice *and* with an appropriate strategy, to always terminate. However we have no proof of this fact; moreover this algorithm is not efficient, due to the combinatorial explosion of the number and length of words in intermediate results.

Example 3.6. In the following computation for $W = \mathfrak{S}_8$, we multiply some element of the Hecke algebra by successive elementary transposition; we use respectively the short hand notation $\sigma_{[154]}$ and $\pi_{[154]}$ for the products $s_1s_5s_4$ and $\pi_1\pi_5\pi_4$:

$$\begin{aligned}
\pi_{[1765432]}\sigma_{[1]} &= \sigma_{[1234567]}\pi_{[675645342312]} - \sigma_{[12345]}\pi_{[7675645342312]} \\
&\quad + \sigma_{[234567]}\pi_{[675645342312]} - \sigma_{[2345]}\pi_{[7675645342312]} + \sigma_{[]}\pi_{[76543212]} \\
\pi_{[1765432]}\sigma_{[13]} &= \sigma_{[34567234561234567]}\pi_{[6756456734562345 1234]} - \sigma_{[212345]}\pi_{[7675645342312]} \\
&\quad + \sigma_{[21234567]}\pi_{[675645342312]} - \sigma_{[3452341234567]}\pi_{[67567456734562345 1234]} \\
&\quad + \sigma_{[2345671]}\pi_{[675645342312]} - \sigma_{[345234567123]}\pi_{[67567456734562345 1234]} \\
&\quad + \sigma_{[3456723456712345]}\pi_{[6756456734562345 1234]} - \sigma_{[23451]}\pi_{[7675645342312]} \\
&\quad - \sigma_{[345672345612345]}\pi_{[76756456734562345 1234]} + \sigma_{[21]}\pi_{[765432312]} \\
&\quad + \sigma_{[345234123]}\pi_{[767567456734562345 1234]} + \sigma_{[]}\pi_{[765432123]} - \sigma_{[]}\pi_{[7654323123]} \\
\pi_{[1765432]}\sigma_{[135]} &= \sigma_{[567456345672345612 34567]}\pi_{[675645673456234567123456]} + 38 \text{ shorter terms} \\
\pi_{[1765432]}\sigma_{[1357]} &= \sigma_{[7656745634567234561234567]}\pi_{[675645673456234567123456]} + 116 \text{ shorter terms}
\end{aligned} \tag{34}$$

Encountering those difficulties does not come as a surprise. The properties of such algebras often become clearer when considering their concrete representations (typically as operator algebras) rather than their abstract presentation. Here, theorem 3.4 as well as the representation theory of $\mathcal{H}W$ follow from an upcoming structural characterization of $\mathcal{H}W$ as the algebra of operators preserving certain antisymmetries.

3.1.1. *Variants.* As mentioned previously, the original goal of the definition of $\mathcal{H}W$ was to put together a Coxeter group W and its 0-Hecke algebra $\mathcal{H}(W)(0)$. Identifying their right regular representation on the canonical basis is just one possible mean. We explore quickly here some variants, and mention alternative constructions of $\mathcal{H}W$ (mostly in type A) in Sections 6 and 7.

A first variant is to still consider the right-regular actions of W and of $\mathcal{H}(W)(0)$ but this time on W itself. In other words, to consider the *monoid* $\langle s, \pi_s\rangle_{s\in S}$ generated by the operators s and π_s. In type A, the sizes of those monoids for $n = 1, 2, 3, 4$ are $1, 4, 66, 6264$, which are strictly bigger than the corresponding dimensions of $\mathcal{H}\mathfrak{S}_n$, in particular because we lose the linear relations $1 + \pi_s s = 1 + s$. Incidentally, an interesting question is to find a presentation of this monoid. If instead one takes the monoid $\langle \pi_s, \overline{\pi}_s\rangle$, the sizes are $1, 3, 23, 477$.

Another natural approach, in type A, is to start from the usual action of $\mathfrak{S}_n$ on the ring of polynomials $\mathbb{C}[x_1, \dots, x_n]$ together with the action of $\mathcal{H}_n(0)$ by *isobaric divided differences* (see [Las03]). Note that the divided differences being symmetrizing operators, this is in fact more a variant on the adjoint ${\mathcal{H}\mathfrak{S}_n}^*$ of $\mathcal{H}\mathfrak{S}_n$ (see next section). Again the obtained algebras are bigger: $1, 3, 20, ???, \dots,$ in particular because we lose the two first relations of Equation (33).

3.2. $\mathcal{H}W$ **as algebra of antisymmetry-preserving operators.** Let $\overleftarrow{s}$ be the *right operator* in $\mathrm{End}(\mathbb{C}W)$ describing the action of s_i by multiplication *on the left* (action on values in type A). Namely $\overleftarrow{s}$ is defined by

$$\sigma \cdot \overleftarrow{s} := s\sigma\,. \tag{35}$$

A vector v in $\mathbb{C}W$ is *left s-symmetric* (resp. *antisymmetric*) if $v \cdot \overleftarrow{s} = v$ (resp. $v \cdot \overleftarrow{s} = -v$). The subspace of left s-symmetric (resp. antisymmetric) vectors can be alternatively described as the image (resp. kernel) of the *quasi-idempotent* (idempotent up to a scalar) operator $1 + \overleftarrow{s}$, or as the kernel (resp. image) of the quasi-idempotent operator $1 - \overleftarrow{s}$.

Theorem 3.7. *$\mathcal{H}W$ is the subspace of* $\mathrm{End}(\mathbb{C}W)$ *defined by the* $|S|$ idempotent sandwich equations*:*

$$(1 - \overleftarrow{s})\ f\ (1 + \overleftarrow{s}) = 0\,, \qquad \textit{for } s \in S\,. \tag{36}$$

In other words, $\mathcal{H}W$ is the subalgebra of those operators in $\mathrm{End}(\mathbb{C}W)$ *which preserve left antisymmetries.*

Note that, $\overleftarrow{s}$ being self-adjoint for the canonical scalar product of $\mathbb{C}W$ (making W into an orthonormal basis), the adjoint algebra of $\mathcal{H}W$ satisfies the equations:

$$(1 + \overleftarrow{s})\ f\ (1 - \overleftarrow{s}) = 0\,, \qquad \text{for } s \in S\,; \tag{37}$$

thus, it is the subalgebra of those operators in $\mathrm{End}(\mathbb{C}W)$ which preserve left symmetries. Furthermore the group algebra $\mathbb{C}[W]$ of W can be described as the subalgebra of those operators in $\mathrm{End}(\mathbb{C}W)$ which preserve both left symmetries and antisymmetries; it is therefore the intersection of $\mathcal{H}W$ and its adjoint $\mathcal{H}W^*$.

Proof of theorems 3.4 and 3.7. We proceed as follow using three lemmas that will occupy the rest of this subsection. We first exhibit a triangularity property of the operators in B; this proves that they are linearly independent, so that $\dim \mathcal{H}W \geq h$ (lemma 3.8). Then we prove that the operators in $\mathcal{H}W$ preserve all left antisymmetries (lemma 3.9). Finally we extract from the sandwich equations $\dim \mathrm{End}(\mathbb{C}W) - h$

independent linear forms which are annihilated by all left antisymmetry preserving operators in $\mathrm{End}(\mathbb{C}W)$ (lemma 3.10). Altogether, it follows simultaneously that $\mathcal{H}W$ has dimension h with B as basis, and that $\mathcal{H}W$ is the full subspace of left antisymmetry preserving operators. $\square$

Let $<$ be any linear extension of the right Bruhat order on W. Given an endomorphism f of $\mathbb{C}W$, we order the rows and columns of its matrix $M := [f_{\mu\nu}]$ accordingly to $<$, and denote by $\mathrm{init}(f) := \min\{\mu \mid \exists \nu, f_{\mu\nu} \neq 0\}$ the index of the first non zero row of M.

Lemma 3.8. *(a) Let $f := \sigma\pi_\tau$ in B. Then, $\mathrm{init}(f) = \tau$, and*

$$f_{\tau\nu} = \begin{cases} 1 & \text{if } \nu \in W_{\mathrm{iDes}(\tau)}\sigma^{-1}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

(b) The family B is free.

Proof. (a) is a direct corollary of Lemma 3.1 (b).

(b) follows by triangularity: the operator $\sigma\pi_\tau$ has coefficient $m_{\tau\sigma^{-1}} = 1$, whereas any other operator $\sigma'\pi'_\tau$ such that $D(\sigma') \cap R(\tau) = \emptyset$, $\tau' \leq \tau$ and $\sigma' \neq \sigma$ has coefficient $m_{\tau\sigma^{-1}} = 0$. $\square$

Lemma 3.9. *The operators in $\mathcal{H}W$ preserve all left antisymmetries.*

Proof. It is sufficient to prove that the generators s and π_s of $\mathcal{H}W$ preserve any left antisymmetry. For a generator s, this is obvious since the actions of $\overleftarrow{s}$ and s commute. Let now v be an s'-antisymmetric vector; without loss of generality, we may assume that $v = (1-s')\sigma$ where σ is some permutation without recoil at position s'. We use the same linear isomorphism Φ as in lemma 3.1 to lift the computation to the 0-Hecke algebra and use its associativity:

$$\begin{aligned} v \cdot \pi_s &= \Phi^{-1}(((1-\pi_{s'})\pi_\sigma)\pi_s) \\ &= \Phi^{-1}((1-\pi_{s'})(\pi_\sigma\pi_s)) \\ &= \Phi^{-1}((1-\pi_{s'})\pi_{\sigma\ \cdot\pi_s}) \\ &= \begin{cases} 0 & \text{if } s \in \mathrm{iDes}(\sigma\cdot\pi_s)\,, \\ (1-s')(\sigma\cdot\pi_s) & \text{otherwise.} \end{cases} \end{aligned} \tag{39}$$

Therefore, $v \cdot \pi_s$ is again s'-antisymmetric. $\square$

We now turn to the explicit description of the sandwich equations. Given an endomorphism f of $\mathbb{C}W$, denote by $(f_{\mu,\nu})_{\mu,\nu}$ the coefficients of its matrix in the natural group basis. Given two elements μ, ν in W and a simple reflection $s \in S$, let $R_{\mu,\nu,i}$ be the linear form on $\mathrm{End}(\mathbb{C}W)$ which computes the (μ,ν) coefficient of the matrix of $(1-\overleftarrow{s})f(1+\overleftarrow{s})$:

$$R_{\mu,\nu,s}(f) := ((1-\overleftarrow{s})f(1+\overleftarrow{s}))_{\mu,\nu} = f_{\mu,\nu} - f_{\mu,s\nu} + f_{s\mu,\nu} - f_{s\mu,s_i\nu}\,. \tag{40}$$

By construction, $R_{\mu,\nu,i}$ annihilates any operator which preserves s-antisymmetries.

Given a pair (μ,ν) of elements of W having at least one descent in common, set $R_{\mu,\nu} = R_{\mu,\nu,s}$, where s is the smallest common descent of μ and ν (the choice of the common descent s is, in fact, irrelevant for our purposes). Finally, let

$$R := \{R_{\mu,\nu} \mid \mathrm{Des}(\mu) \cap \mathrm{Des}(\nu) \neq \emptyset\}\,. \tag{41}$$

For example in type A we have:

$$\begin{aligned}
&\text{for } n = 1\text{: } R = \{\} \\
&\text{for } n = 2\text{: } R = \{R_{21,21,1}\} = \{f_{21,21} - f_{21,12} + f_{12,21} - f_{12,12}\} \\
&\text{for } n = 3\text{: } R = \{\underbrace{R_{213,213,1}, R_{213,312,1}, R_{213,321}, \dots, R_{321,321,1}}_{17 = 6^2 - 19 \text{ items}}\}
\end{aligned} \tag{42}$$

For $n = 2$, the reader will recognize the linear relation described in example 3.3.

Lemma 3.10. *The $|W|^2 - h$ linear forms in R are linearly independent.*

Proof. Take some linear form $R_{\mu,\nu} = R_{\mu,\nu,i}$ in R, and represent it as the $|W| \times |W|$ array of its values on the elements of the canonical basis of $\mathrm{End}(W)$, with the rows and columns sorted as previously. For example, here is the array for $R_{213,312} = R_{213,312,1}$ in type A_2:

123	132	213	231	312	321	
0	−1	0	0	1	0	123
0	0	0	0	0	0	132
0	−1	0	0	1	0	213
0	0	0	0	0	0	231
0	0	0	0	0	0	312
0	0	0	0	0	0	321

This array has a coefficient 1 at position (μ, ν). Since s is both a descent of μ and ν, $s\mu < \mu$, and $s\nu < \nu$; so the three other non zero coefficients are either strictly higher or strictly to the left in the array. Furthermore, no other linear form in R has a non-zero coefficient at position (μ, ν). Hence, by triangularity the linear forms in R are linearly independent. □

3.3. **Representation theory.** Due to the particular structure of $\mathcal{H}W$ as a operator algebra, the easiest thing to start with is the study of projective module. Along the way we define a particular basis of $\mathbb{C}W$ which appear to be crucial for the representation theory.

3.3.1. *Projective modules.* Recall that $\mathcal{H}W$ is the algebra of operators preserving left antisymmetries. Thus, given $I \subset S$, it is natural to introduce the $\mathcal{H}W$-submodule

$$P_I := \bigcap_{s \in I} \ker(1 + \overleftarrow{s}\,)\,. \tag{43}$$

of the vectors in $\mathbb{C}W$ which are s-antisymmetric for all $s \in I$. For example, P_S is one dimensional, and spanned by $\sum_{\nu \in W} (-1)^{\ell(\nu)} \nu$, whereas $P_\emptyset = \mathbb{C}W$.

The goal of this section is to prove that the family of modules $(P_I)_{I \subset S}$ forms a complete set of representatives of the indecomposable projective modules of $\mathcal{H}W$. First, we need a more practical definition of P_I.

Lemma 3.11. *Let $I \subset S$. Then P_I is the $\mathcal{H}W$-submodule of $\mathbb{C}W$ generated by*

$$v_I := \sum_{\nu \in W_I} (-1)^{\ell(\nu)} \nu\,, \tag{44}$$

or equivalently the $\mathbb{C}[W]$-submodule generated by v_I.

Proof. First, it is clear that v_I belongs to P_I. Since $\mathbb{C}W$ is the right regular representation of W, we may temporarily identify $\mathbb{C}W$ and $\mathbb{C}[W]$. There, it is well known that v_I is an idempotent (up to a scalar factor). Take in general $u \in P_I$. For any s in I, one has $(1+s)u = 0$, that is $su = -u$. It follows that $v_I u = |W_I|u$, and we can conclude that $P_I = v_I \cdot \mathbb{C}[W]$. □

Actually, as we will see later, it is also an idempotent in the 0-Hecke algebra and even in the generic Hecke algebra.

For each $\sigma \in W$, define $v_\sigma := v_{S\setminus \mathrm{iDes}(\sigma)} \cdot \sigma$. Note that σ is the element of minimal length appearing in v_σ. By triangularity, it follows that the family $(v_\sigma)_{\sigma\in\mathfrak{S}_n}$ forms a vector space basis of $\mathbb{C}W$. See Figure 2 for an example.

The usefulness of this basis comes from the fact that it is compatible with the module structure.

Proposition 3.12. *For any $I \subset S$, the module P_I is of dimension $|{}^IW|$, and*

$$\{v_I \cdot \sigma \mid \sigma \in {}^IW\} \qquad \text{and} \qquad \{v_\sigma \mid \sigma \in {}^IW\} \tag{45}$$

are both vector space bases of P_I.

Proof. First note that, by the same triangularity argument, the first family is free as well. Furthermore, by the previous lemma, P_I is spanned by all the vectors $v_I \cdot \sigma$ with $\sigma \in W$. Take $\sigma \notin {}^IW$; then σ is of the form $\sigma = s\sigma'$ with $s \in I$ and $\ell(\sigma') < \ell(\sigma)$, and it follows that $v_I \cdot \sigma = -v_I \cdot \sigma'$. Applying induction on the length yields that the first family is a basis. By dimension count, the second family (each element v_σ of which is in $P_{S\setminus \mathrm{iDes}\sigma} \subset P_I$) is also a basis of P_I. □

Corollaries 3.13. *P_I is generated by v_I, either as $\mathcal{H}W$, W, or $\mathcal{H}(W)(0)$-module.*
$P_J \subset P_I$ if and only if $I \subset J$.

Proof. If a finite Coxeter group, the sets $\{w \mid \mathrm{Des}(w) = I\}$ are never empty; therefore $P_J \subsetneq P_I$ whenever $I \subsetneq J$. □

In type A, the recoil class ${}^IW$ is the set of the shuffles of the words $1\cdots a_1$, $a_1+1\cdots a_2$, …, where I is the set $\{a_1 < a_2 < \cdots\}$. As a consequence

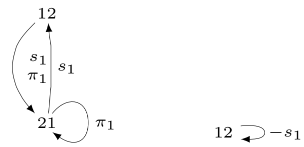
Corollary 3.14. *In type A the module P_I is of dimension $\frac{n!}{i_1!i_2!\ldots i_k!}$, where $(i_1,\ldots,i_k)$ is the composition associated to I.*

The following proposition elucidates the structure of P_I as W and $\mathcal{H}(W)(0)$-module.

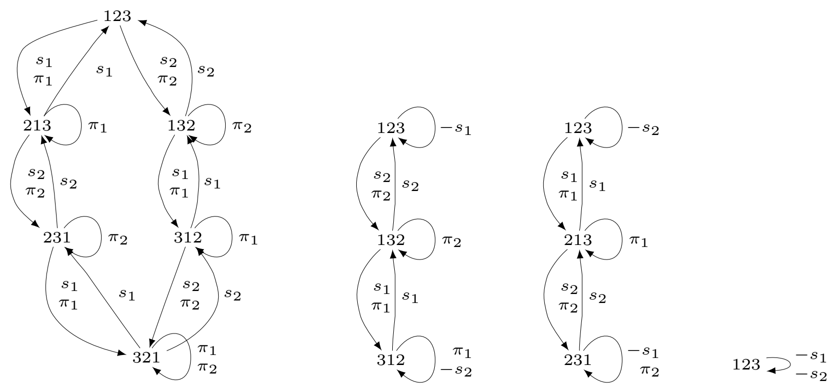
Proposition 3.15. *Let (-1) denote the sign representation of W as well as the corresponding representation of the Hecke algebra $\mathcal{H}(W)(0)$ (sending T_s to -1, or equivalently π_s to 0).*

(a) *As a W-module, $P_I \approx (-1)\uparrow_{W_I}^{W}$.*
(b) *As a $\mathcal{H}(W)(0)$-module, $P_I \approx (-1)\uparrow_{\mathcal{H}(W_I)(0)}^{\mathcal{H}(W)(0)}$; it is a projective module.*
(c) *The P_I's are non isomorphic as $\mathcal{H}(W)(0)$-modules and thus as $\mathcal{H}W$-modules.*

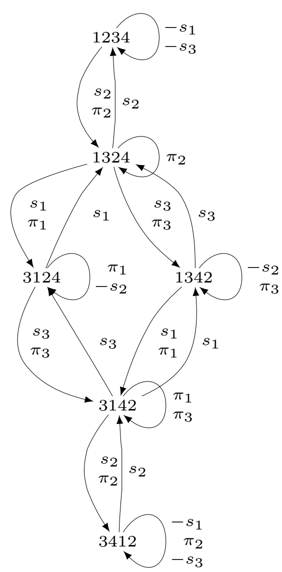
Proof. First, for s in W_I, one has $v_I \cdot s = -v_I$. Moreover since ${}^IW$ is a set of representative of left cosets for the quotient $W_I\backslash W$, the first point is proved.



(a) Projective modules $P_{\{\}}$ and $P_{\{1\}}$ of $\mathcal{H}\mathfrak{S}_2$



(b) Projective modules $P_{\{\}}$, $P_{\{1\}}$, $P_{\{2\}}$, and $P_{\{1,2\}}$ of $\mathcal{H}\mathfrak{S}_3$



(c) Projective module $P_{\{1,3\}}$ of $\mathcal{H}\mathfrak{S}_4$

FIGURE 1. Some projective modules of $\mathcal{H}\mathfrak{S}_n$, described on their $\{v_I \cdot \sigma \mid \sigma \in {}^I W\}$ bases. Note that those are not combinatorial modules, as coefficients -1 or 0 may occur; in the later case, the corresponding edges are not drawn. Those pictures have been produced automatically, using `MuPAD-Combinat`, `graphviz`, and `dot2tex`.

The second point is proved analogously, thanks to the fact that v_I interpreted in the Hecke algebra as $\Phi(v_I)$ is an idempotent:

$$\Phi(v_I)^2 = \left(\sum_{\nu\in W_I} (-1)^{\ell(\nu)}\pi_\nu\right)^2 = \sum_{\nu\in W_I} (-1)^{\ell(\nu)}\pi_\nu = \Phi(v_I)^2\,, \tag{46}$$

since $\sum_{\nu\in W_I}(-1)^{\ell(\nu)}\nu\pi_s = 0$ for any $s \in W_I$.

The P_I's being projective $\mathcal{H}(W)(0)$-modules, it is sufficient to prove that the associated semi-simple modules (obtained by factoring out by their radicals) are pairwise non-isomorphic. Namely, consider for each I

$$M_I := (-1)\uparrow_{H_I}^{H} /\mathrm{rad}\left((-1)\uparrow_{H_I}^{H}\right)\,, \tag{47}$$

writing for short $H := \mathcal{H}(W)(0)$ and $H_I := \mathcal{H}(W_I)(0)$. Since (-1) is simple as H_I-module,

$$M_I = (-1)\uparrow_{H_I/\mathrm{rad}(H_I)}^{H/\mathrm{rad}(H)}\,. \tag{48}$$

Recall [Nor79] that $H/\mathrm{rad}(H)$ (resp. $H_I/\mathrm{rad}(H_I)$) is the commutative algebra over $(T_s)_{s\in S}$ (resp. $(T_s)_{s\in I}$). Hence, a simple $H/\mathrm{rad}(H)$-module S^0_J is characterized by the set $J \in S$ of s such that $\pi_s(S^0_J) = 0$. Since there is a simple H-module for each subset,

$$M_I \approx \bigoplus_{J\supset I} S^0_J\,. \tag{49}$$

In particular, the M_I's are pairwise non isomorphic H-modules, as desired. □

As a consequence of the preceding proof, if we denote P^0_I the projective $\mathcal{H}(W)(0)$-module associated with S^0_I, then, as an $\mathcal{H}(W)(0)$-module

$$P_I \approx \bigoplus_{J\supset I} P^0_J\,. \tag{50}$$

In the following corollary, we focus on the particular type A, where a $\mathfrak{S}_n$-module (resp. $H_n(0)$-module) is characterized by its so-called characteristic, which is a symmetric function (resp. a non-commutative symmetric function). Here is the characteristic of the modules P_I:

Corollary 3.16. *In type A, the characteristic of the $\mathfrak{S}_n$-module P_I is the symmetric function $e_K := e_{k_1}\cdots e_{k_l}$, where $(k_1,\dots,k_l)$ is the composition associated to the complementary of I.*

The characteristics of the $H_n(0)$-module P_I is the noncommutative symmetric function $\Lambda^K := \Lambda_{k_1}\cdots\Lambda_{k_l}$, where $(k_1,\dots,k_l)$ is the composition associated to the complementary of I.

3.3.2. *$\mathcal{H}W$ as an incidence algebra.* Since $\{v_\sigma\}_{\sigma\in W}$ is a basis of $\mathbb{C}W$, we may consider for each (σ,τ) in W^2 the matrix element $e_{\sigma,\tau} \in \mathrm{End}(\mathbb{C}W)$ which maps $v_{\sigma'}$ to v_τ if $\sigma' = \sigma$ and to 0 otherwise.

We can now prove the main results of this section.

Theorem 3.17. *For each (σ,τ) in W^2, let $e_{\sigma,\tau} \in \mathrm{End}(\mathbb{C}W)$ defined by*

$$e_{\sigma,\tau}(\sigma') = \delta_{\sigma',\sigma}\, v_\tau\,. \tag{51}$$

Then the family $\{e_{\sigma,\tau} \mid \mathrm{iDes}(\sigma) \supset \mathrm{iDes}(\tau)\}$ is a vector space basis of $\mathcal{H}W$.

$$\begin{tikzcd}[column sep=tiny]
& v_{123} = 123 - 213 - 132 + 231 + 312 - 321 & \\
v_{2\underline{1}3} = 213 - 312 \arrow[ur, "1"] \arrow[d, leftrightarrow, "2"'] & & v_{13\underline{2}} = 132 - 231 \arrow[ul, "2"'] \arrow[d, leftrightarrow, "1"] \\
v_{23\underline{1}} = 231 - 321 & & v_{31\underline{2}} = 312 - 321 \\
& v_{3\underline{21}} = 321 \arrow[ul, "1"] \arrow[ur, "2"'] &
\end{tikzcd}$$

FIGURE 2. The basis $(v_\sigma)_\sigma$ of $\mathbb{C}\mathfrak{S}_3$, together with the graph structure which makes $\mathcal{H}\mathfrak{S}_3$ into an incidence algebra. Underlined: the recoils of the permutations.

Proof. This family is free by construction. It has the appropriate size because for any finite Coxeter group $|\{\sigma \mid \mathrm{iDes}(\sigma) = I\}| = |\{\sigma \mid \mathrm{iDes}(\sigma) = S\backslash I\}|$ (a bijection between the two sets is given by $u \mapsto \omega u$, where ω is the maximal element of W; see [BB05, Exercise 10 p. 57]).

It remains to check that any of its element $e_{\sigma,\tau}$ preserves i-left antisymmetries and therefore is indeed in $\mathcal{H}W$. Take $i \in S$, and consider the basis of $P_{\{i\}}$ of proposition 3.12: $\{v'_\sigma \mid i \notin \mathrm{iDes}(\sigma')\}$; an element of this basis is either killed by $e_{\sigma,\tau}$ or sent to another element of this basis. Therefore, $P_{\{i\}}$ is stable by $e_{\sigma,\tau}$. $\square$

Recall (see [Sta97]) that the *incidence algebra* $\mathbb{C}[P]$ of a poset $(P, \preccurlyeq)$ is the algebra whose basis elements $e_{u,v}$ are indexed by the couples $(u, v) \in P^2$ such $u \preccurlyeq v$, and whose multiplication rule is given by:

$$e_{u,v} \cdot e_{u',v'} = \delta_{v,u'}\, e_{u,v'}\,. \tag{52}$$

Here we need a slightly more general extension of this notion where P is not a partially ordered set but only a pre-order (no necessarily anti-symmetric).

Corollary 3.18. *$\mathcal{H}W$ is isomorphic to the incidence algebra of the pre-order $(W, \preccurlyeq)$ where*

$$\sigma \preccurlyeq \sigma' \quad \textit{whenever} \quad \mathrm{iDes}(\sigma) \supset \mathrm{iDes}(\sigma')\,. \tag{53}$$

Proof. By construction, the $e_{\sigma,\tau}$ satisfy the usual product rule of incidence algebras:

$$e_{\sigma,\tau} e_{\sigma',\tau'} = \delta_{\tau,\sigma'} e_{\sigma,\tau'}. \tag{54}$$

This is sufficient since $(e_{\sigma,\tau})$ is a basis. $\square$

The representation theory of $\mathcal{H}W$ (projective and simple modules and the Cartan matrix) will follow straightforwardly from this corollary. Furthermore, a good way to think about the pre-order $\preccurlyeq$ is to view it as the transitive closure of a graph G which essentially encodes the action of the generators of $\mathcal{H}W$. Namely, define G as the graph with vertex set W and where $\sigma \to \sigma'$ is an edge whenever there exists $s \in S$ such that $\sigma' = \sigma s$ and $\mathrm{iDes}(\sigma) \subset \mathrm{iDes}(\sigma')$ (cf. Figure 2).

Corollary 3.19. (a) *The ideal $e_\sigma \mathcal{H}W$ is isomorphic to $P_{S\backslash \mathrm{iDes}(\sigma)}$ as a right $\mathcal{H}W$-module;*

(b) *The idempotents $e_\sigma := e_{\sigma,\sigma}$ give a maximal decomposition of the identity into orthogonal idempotents in $\mathcal{H}W$;*

Hence we get the full description of the projective $\mathcal{H}W$-modules.

Corollary 3.20. *The family of modules $(P_I)_{I\subset S}$ forms a complete set of representatives of the indecomposable projective modules of $\mathcal{H}W$.*

Proof. Follows from (a) and (b) and Proposition 3.15 (c). □

3.3.3. *Cartan's invariants matrix and the boolean lattice.* We now turn to the description of the Cartan matrix. For any $I \subset S$, let $\alpha(I)$ be the shortest permutation such that $\mathrm{iDes}(\alpha) = S\backslash I$ (the choice of the *shortest* is in fact irrelevant). For any (I, J) such that $I \subset J \subset S$, define $e_{I,J} := e_{\alpha(I),\alpha(J)}$.

$$e_{I,J}(v_\sigma) = \begin{cases} v_J & \text{if } \sigma = \alpha(I), \\ 0 & \text{otherwise.} \end{cases} \tag{55}$$

Remark 3.21. Using the incidence algebra structure, the sandwich $e_I\mathcal{H}We_J$ is $\mathbb{C}.e_{I,J}$ if $I \subset J$ and is trivial otherwise.

By Corollary 3.19, $\mathrm{Hom}(P_I, P_J)$ is isomorphic to the sandwich $e_I\mathcal{H}We_J$. Therefore we have the following corollary:

Corollary 3.22.

$$\dim \mathrm{Hom}(P_I, P_J) = \dim e_I\mathcal{H}We_J = \begin{cases} 1 & \text{if } I \supset J, \\ 0 & \text{otherwise.} \end{cases} \tag{56}$$

In other words, the Cartan matrix of $\mathcal{H}W$ is the incidence matrix of the boolean lattice. This suggests the existence of a close relation between $\mathcal{H}W$ and the incidence algebra of the boolean lattice.

Recall that an algebra is called *elementary* (or sometimes *reduced*) if its simple modules are all one dimensional. Starting from an algebra A, it is possible to get a canonical elementary algebra by the following process. Start with a maximal decomposition of the identity $1 = \sum_i e_i$ into orthogonal idempotents. Two idempotents e_i and e_j are *conjugate* if e_i can be written as ae_jb where a and b belongs to A, or equivalently, if the projective modules e_iA and e_jA are isomorphic. Select an idempotent e_c in each conjugacy classes c and put $e := \sum e_c$. Then, it is well known [CR90] that the algebra eAe is elementary and that the functor $M \mapsto Me$ which sends a right A-module to a eAe-module is an equivalence of category. Recall finally that two algebras A and B such that the categories of A-modules and B-modules are equivalent are said *Morita equivalent.* Thus A and eAe are Morita-equivalent.

Corollary 3.23. *Let e be the idempotent defined by $e := \sum_{I\subset S} e_I$. Then the algebra $e\mathcal{H}We$ is isomorphic to the incidence algebra $\mathbb{C}[B_S]$ of the boolean lattice B_S of subsets of S. Consequently, $\mathcal{H}W$ and $\mathbb{C}[B_S]$ are Morita equivalent.*

Actually, the previous construction is fairly general when A is the incidence algebra of a pre-order $\preccurlyeq$, the construction of eAe boils down to taking the incidence algebra of the canonical order $\leq$ associated to $\preccurlyeq$ by contracting its strongly connected components, or equivalently by picking any representative. For $\mathcal{H}W$, the strongly connected components of the pre-order $(W, \preccurlyeq)$ are by construction $(\{\sigma \mid \mathrm{iDes}(\sigma) = I\})_{I\subset S}$, and the associated order is simply the boolean lattice B_S of subsets I of S.

3.3.4. *Simple modules.* The simple modules are obtained as quotients of the projective modules by their radical: $S_I := P_I / \sum_{J \subsetneq I} P_J$. This has the natural effect of making a vector v in S_I both i-antisymmetric for $i \in I$ and i-symmetric for $i \notin I$.

Theorem 3.24. *The modules $(S_I)_{I \subset S}$ form a complete set of representatives of the simple modules of $\mathcal{H}W$. Moreover, the projection of the family $\{v_\sigma \mid \mathrm{iDes}(\sigma) = I\}$ in S_I forms a vector space basis of S_I.*

Proof. This is a consequence of Proposition 3.22. Indeed, since there is no non trivial morphism from P_I to itself, the radical of P_I is the sum of the images of all morphisms from the P_J for $J \neq I$ to P_J. But there exists up to constant at most one such morphisms, that is when $I \supset J$, then $P_I \subset P_J$. Hence $\mathrm{rad}(P_I) = \sum_{J \subsetneq I} P_J$. As a consequence, together with Proposition 3.12, we get the given basis. □

The following proposition elucidates the structure of S_I as W and $\mathcal{H}(W)(0)$-module.

Proposition 3.25. *Let K be the composition associated to I.*

(a) *As a W-module, S_I is isomorphic to the subspace of $\mathbb{C}W$ of vectors which are simultaneously i-antisymmetric for i in I and i-symmetric for i in $S \backslash I$. In type A that's the usual Young's representation V_K indexed by the ribbon K, and each v_σ, with $\mathrm{Des}(\sigma) = I$ corresponds to the basis element of V_K indexed by the standard ribbon tableaux associated to σ^{-1}. Its character is the ribbon Schur symmetric function s_K.*

(b) *As a $\mathcal{H}(W)(0)$-module, S_I is an indecomposable projective module. In type A, its character is the noncommutative symmetric function R_K.*

Proof. (b) It is the projective module whose quotient by its radical is the simple module M such that $M\pi_s = 0$ if and only if $s \in I$. □

Note that S_I is also isomorphic to the simple module $S_{S \backslash I}$ of the transpose algebra $\mathcal{H}W^*$.

3.3.5. *Induction, restriction, and Grothendieck rings.* In this subsection, we concentrate on the tower of type-A algebras $(\mathcal{H}\mathfrak{S}_n)_n$.

Let $\mathcal{G} := \mathcal{G}\left((\mathcal{H}\mathfrak{S}_n)_n\right)$ and $\mathcal{K} := \mathcal{K}\left((\mathcal{H}\mathfrak{S}_n)_n\right)$ be respectively the Grothendieck rings of the characters of the simple and projective modules of the tower of algebras $(\mathcal{H}\mathfrak{S}_n)_n$. Let furthermore C be the cartan map from $\mathcal{K}$ to $\mathcal{G}$. It is the algebra and coalgebra morphism which gives the projection of a module onto the direct sum of its composition factors. It is given by

$$C(P_I) = \sum_{I \supset J} S_J \,. \tag{57}$$

Since the indecomposable projective modules are indexed by compositions, it comes out as no surprise that the structure of algebras and coalgebras of $\mathcal{G}$ and $\mathcal{K}$ are each isomorphic to QSym and NCSF. However, we do not get Hopf algebras, because the structures of algebras and coalgebras are not compatible.

Proposition 3.26. *The following diagram gives a complete description of the structures of algebras and of coalgebras on $\mathcal{G}$ and $\mathcal{K}$:*

$$
\begin{array}{ccccccc}
(\mathrm{QSym}, .) & \xleftarrow{\chi(S_I)\mapsto M_I} & (\mathcal{G}, .) & \xleftarrow{C} & (\mathcal{K}, .) & \xrightarrow{\chi(P_I)\mapsto F_I} & (\mathrm{QSym}, .) \\
(\mathrm{NCSF}, \Delta) & \xleftarrow[\chi(S_I)\mapsto R_I]{} & (\mathcal{G}, \Delta) & \xleftarrow[C]{} & (\mathcal{K}, \Delta) & \xrightarrow[\chi(P_I)\mapsto \Lambda^{I^c}]{} & (\mathrm{NCSF}, \Delta)
\end{array}
\tag{58}
$$

Proof. The bottom line is already known from Proposition 3.15 and the fact that, for all m and n, the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{H}_m(0) \otimes \mathcal{H}_n(0) & \hookrightarrow & \mathcal{H}_{m+n}(0) \\
\downarrow & & \downarrow \\
\mathcal{H}\mathfrak{S}_m \otimes \mathcal{H}\mathfrak{S}_n & \hookrightarrow & \mathcal{H}\mathfrak{S}_{m+n}
\end{array}
\tag{59}
$$

Thus the map which sends a module to the characteristic of its restriction to $\mathcal{H}_n(0)$ is a coalgebra morphism. The isomorphism from $(\mathcal{K}, .)$ to QSym is then obtained by Frobenius duality between induction of projective modules and restriction of simple modules. And the last case is obtained by applying the Cartan map C. □

It is important to note that the algebra $(\mathcal{G}, .)$ is not the dual of the coalgebra $(\mathcal{K}, \Delta)$ because the dual of the restriction of projective modules is the so-called *co-induction* of simple modules which, in general, is not the same as the induction for non self-injective algebras.

Finally, we briefly describe the Grothendieck rings for the adjoint algebra $\mathcal{H}W^*$ which preserves symmetries. The projective modules are defined by the two following equivalent formulas:

$$
P_I := \bigcap_{s\in I} \ker(1 - \overleftarrow{s}) = \left(\sum_{\nu\in W_I} \nu\right) \cdot \mathcal{H}W^* \ . \tag{60}
$$

As W-module (resp. $\mathcal{H}(W)(0)$-module), they are isomorphic to the modules induced by the trivial modules of W_I (resp. $\mathcal{H}(W_I)(0)$) whose frobenius characteristic are complete symmetric functions. The rest of our arguments can be adapted easily, yielding the following diagram:

$$
\begin{array}{ccccccc}
(\mathrm{QSym}, .) & \xleftarrow{\chi(S_I)\mapsto X_I} & (\mathcal{G}, .) & \xleftarrow{C} & (\mathcal{K}, .) & \xrightarrow{\chi(P_I)\mapsto F_{I^c}} & (\mathrm{QSym}, .) \\
(\mathrm{NCSF}, \Delta) & \xleftarrow[\chi(S_I)\mapsto R_{I^c}]{} & (\mathcal{G}, \Delta) & \xleftarrow[C]{} & (\mathcal{K}, \Delta) & \xrightarrow[\chi(P_I)\mapsto S^{I^c}]{} & (\mathrm{NCSF}, \Delta)
\end{array}
, \tag{61}
$$

where $(X_I)_I$ is the dual basis of the elementary basis $(\Lambda_I)_I$ of NCSF. Thus we have a representation theoretical interpretation of many bases of NCSF and QSym.

4. The algebra of non-decreasing functions

Definition 4.1. Let NDF_n be the set of *non-decreasing functions* from $\{1, \ldots, n\}$ to itself. Its cardinal is given by:

$$
\binom{2n-1}{n-1} = \sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}, \tag{62}
$$

where, on the right, functions are counted according to the size k of their images.

The composition and the neutral element id_n make NDF_n into a monoid and we denote by $\mathbb{C}[\mathrm{NDF}_n]$ its monoid algebra. The monoid $\mathrm{NDF}_n \times \mathrm{NDF}_m$ can be identified as the submonoid of NDF_{n+m} whose elements stabilize both $\{1, \dots, n\}$ and $\{n+1, \dots, n+m\}$. This makes $(\mathbb{C}[\mathrm{NDF}_n])_n$ into a tower of algebras.

The semi-group properties of NDF_n have been studied (see e.g. [GH92], where $\mathcal{O}_n$ coincides with NDF_n striped of the identity). In particular [GH92, Theorem 4.8], one can take as idempotent generators for NDF_n the functions π_i et $\overline{\pi}_i$ defined by:

$$\begin{aligned} &\pi_i(i+1) := i \quad \text{ and } \quad \pi_i(j) := j, \text{ for } j \neq i+1, \\ &\overline{\pi}_i(i) := i+1 \quad \text{ and } \quad \pi_i(j) := j, \text{ for } j \neq i. \end{aligned} \tag{63}$$

The functions π_i satisfy the braid relations, together with a new relation:

$$\pi_i^2 = \pi_i \qquad \text{and} \qquad \pi_{i+1}\pi_i\pi_{i+1} = \pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\,. \tag{64}$$

This readily defines a morphism $\phi : \pi_{\mathcal{H}_n(0)} \mapsto \pi_{\mathbb{C}[\mathrm{NDF}_n]}$ of $\mathcal{H}_n(0)$ into $\mathbb{C}[\mathrm{NDF}_n]$. Its image is the monoid algebra of *non-decreasing parking functions* which will be discussed in Section 5 and of which Equation (64) actually gives a presentation. The same properties hold for the operators $\overline{\pi}_i$'s. Although this is not a priori obvious, it will turn out that the two morphisms $\phi : \pi_{\mathcal{H}_n(0)} \mapsto \pi_{\mathbb{C}[\mathrm{NDF}_n]}$ and $\overline{\phi} : \overline{\pi}_{\mathcal{H}_n(0)} \mapsto \overline{\pi}_{\mathbb{C}[\mathrm{NDF}_n]}$ are compatible, making $\mathbb{C}[\mathrm{NDF}_n]$ into a quotient of $\mathcal{H}\mathfrak{S}_n$ (Proposition 4.3). This will be used in Subsection 4.2 to deduce the representation theory of $\mathbb{C}[\mathrm{NDF}_n]$.

4.1. Representation on exterior powers, and link with $\mathcal{H}\mathfrak{S}_n$.

We now want to construct a suitable faithful representation of $\mathbb{C}[\mathrm{NDF}_n]$ where the existence of the epimorphism from $\mathcal{H}\mathfrak{S}_n$ onto $\mathbb{C}[\mathrm{NDF}_n]$ becomes clear.

The *natural representation* of $\mathbb{C}[\mathrm{NDF}_n]$ is obtained by taking the vector space $\mathbb{C}^n$ with canonical basis $e_1, \dots, e_n$, and letting a function f act on it by $e_i.f = e_{f(i)}$. For $n > 2$, this representation is a faithful representation of the monoid NDF_n but not of the algebra, as $\dim \mathbb{C}[\mathrm{NDF}_n] = \binom{2n-1}{n-1} \gg n^2$. However, since NDF_n is a monoid, the diagonal action on *exterior powers*

$$(x_1 \wedge \cdots \wedge x_k) \cdot f := (x_1 \cdot f) \wedge \cdots \wedge (x_k \cdot f) \tag{65}$$

still defines an action. Taking the *exterior powers* $\bigwedge^k \mathbb{C}^n$ of the natural representation gives a new representation, whose basis $\{e_S := e_{s_1} \wedge \cdots \wedge e_{s_k}\}$ is indexed by subsets $S = \{s_1, \dots, s_k\}$ of $\{1, \dots, n\}$. The action of a function f in NDF_n is simply given by (note the absence of sign!):

$$e_S.f = \begin{cases} e_{f(S)} & \text{if } |f(S)| = |S|, \\ 0 & \text{otherwise.} \end{cases} \tag{66}$$

We call *representation of* $\mathbb{C}[\mathrm{NDF}_n]$ *on exterior powers* the representation of $\mathbb{C}[\mathrm{NDF}_n]$ on $\bigoplus_{k=1}^n \bigwedge^k \mathbb{C}^n$, which is of dimension $2^n - 1$ (it turns out that we do not need to include the component $\bigwedge^0 \mathbb{C}^n$ for our purposes).

Lemma 4.2. *For $n > 0$, the representation $\bigoplus_{k=1}^n \bigwedge^k \mathbb{C}^n$ of $\mathbb{C}[\mathrm{NDF}_n]$ is faithful.*

Proof. We exhibit a triangularity property. For a function f in NDF_n, let $\mathrm{im} f := f(\{1, \dots, n\})$ be its image set, and $R(f)$ be the preimage of $\mathrm{im} f$ defined by $R(f) := \{\min\{x \mid f(x) = y\},\ y \in S\}$. Put a partial order on $\mathbb{C}[\mathrm{NDF}_n]$ by setting $f \prec g$ if

$\mathrm{im} f = \mathrm{im} g$ and $R(f)$ is lexicographically smaller than $R(g)$. Fix a function f. If the representation matrix of a function g has coefficient 1 on row $\mathrm{im} f$ and column $R(f)$, that is if $g(R(f)) = \mathrm{im} f$, then $g \preceq f$.

Remark: for $n > 0$, the component $\bigwedge^0 \mathbb{C}^n$ is not needed because $\mathrm{im} f$ and $R(f)$ are non empty. $\square$

We now realize the representation of $\mathbb{C}[\mathrm{NDF}_n]$ on the k-th exterior power as a representation of $\mathcal{H}\mathfrak{S}_n$. Let us start from the k-th exterior product $\bigwedge^k \mathbb{C}^n$ of the natural representation of $\mathfrak{S}_n$ on $\mathbb{C}^n$. It is isomorphic to $V_{(n-k,1,\dots,1)} \bigoplus V_{(n-k+1,1,\dots,1)}$, where V_λ denotes the irreducible representation of $\mathfrak{S}_n$ indexed by the partition λ. It can be realized as a submodule of the regular representation of $\mathfrak{S}_n$ using the classical Young construction by mean of the row-symmetrizers and column-antisymmetrizers on the skew ribbon

$$
\begin{array}{|c|cccc}
\cline{1-1}
k & & & \\
\cline{1-1}
\vdots & & & \\
\cline{1-1}
1 & & & \\
\cline{1-4}
\multicolumn{1}{c|}{} & k+1 & \multicolumn{1}{|c|}{\cdots} & \multicolumn{1}{c|}{n} \\
\cline{2-4}
\end{array}\,. \tag{67}
$$

This only involves left antisymmetries on the values $1, \dots, k-1$ and symmetries on the values $k+1, \dots, n-1$. Therefore, $\bigwedge^k \mathbb{C}^n$ can alternatively be realized as the $\mathcal{H}\mathfrak{S}_n$-module

$$
P_n^k := P_{\{1,\dots,k-1\}} \quad / \quad \sum_{s=k+1}^{n-1} P_{\{1,\dots,k-1,s\}} \tag{68}
$$

(an element of $P_{\{1,\dots,k-1\}}$ is by construction left antisymmetric on the values $1, \dots, k-1$, and the quotient by each $P_{\{1,\dots,k-1,s\}}$ makes it s-left symmetric). In general, this construction turns any $\mathfrak{S}_n$-module indexed by a shape made of disconnected rows and columns into an $\mathcal{H}\mathfrak{S}_n$-module; it does not apply to shapes like $\lambda = (2, 1)$, as they involve symmetries or antisymmetries between non-consecutive values, like 1 and 3.

A basis of P_n^k indexed by subsets of size k of $\{1, \dots, n\}$ is obtained by taking for each such subset S the image in the quotient P_n^k of

$$
e_S := \sum_{\substack{\sigma(S)=\{1,\dots,k\} \\ \sigma(i)<\sigma(j),\ \text{for } i,j \notin S \text{ with } i<j}} (-1)^{\ell(\sigma)} \sigma\,. \tag{69}
$$

It is straightforward to check that the actions of π_i and $\overline{\pi}_i$ of $\mathcal{H}\mathfrak{S}_n$ on e_S of P_n^k coincide with the actions of π_i and $\overline{\pi}_i$ of $\mathbb{C}[\mathrm{NDF}_n]$ on e_S of $\bigwedge^k \mathbb{C}^n$ (justifying a posteriori the identical notations). In the sequel, we identify the modules P_n^k and $\bigwedge^k \mathbb{C}^n$ of $\mathcal{H}\mathfrak{S}_n$ and $\mathbb{C}[\mathrm{NDF}_n]$, and we call *representation on exterior powers of* $\mathcal{H}\mathfrak{S}_n$ its representation on $\bigoplus_{k=1}^n \bigwedge^k \mathbb{C}^n$.

Using Lemma 4.2 we are in position to state the following

Proposition 4.3. $\mathbb{C}[\mathrm{NDF}_n]$ *is the quotient of* $\mathcal{H}\mathfrak{S}_n$ *obtained by considering its representation on exterior powers. The restriction of this representation of* $\mathcal{H}\mathfrak{S}_n$ *to* $\mathbb{C}[\mathfrak{S}_n]$, $\mathcal{H}_n(0)$, *and* $\mathcal{H}_n(-1)$ *yield respectively the usual representation of* $\mathfrak{S}_n$ *on exterior powers, the algebra* NDPF_n *of non-decreasing parking functions (see Section 5), and the Temperley-Lieb algebra* $\mathcal{TL}_n(-1)$.

Proof. The only case which remains is $q=-1$. Recall that [Jon83] the Temperley-Lieb $\mathcal{TL}_n(q)$ algebra is the quotient of the Hecke algebra by the relations

$$\mathrm{e}_i\,\mathrm{e}_{i\pm 1}\,\mathrm{e}_i = q\,\mathrm{e}_i\,, \tag{70}$$

where $\mathrm{e}_i = T_i + q$. As NDPF_n, its dimension is the Catalan number C_n.

The algebra A is generated by the operators $\mathrm{e}_i = \pi_i - \overline{\pi}_i$, which satisfy the relations:

$$\begin{aligned} \mathrm{e}_i^2 &= 0\,, \\ \mathrm{e}_i\,\mathrm{e}_{i\pm 1}\,\mathrm{e}_i &= -\,\mathrm{e}_i\ . \end{aligned} \tag{71}$$

Therefore A is a quotient of the Temperley-Lieb algebra $\mathcal{TL}_n(-1)$. We now prove that the quotient is trivial by exhibiting C_n elements of A satisfying a triangularity property w.r.t. those of NDPF_n.

Let $\leq$ be the pointwise partial order on NDF_n such that $f \leq g$ if and only if $f(i) \leq g(i)$ for all i. The following properties are easily verified:

$$\begin{aligned} & f\pi_i \leq f \leq f\overline{\pi}_i \\ f \leq g \quad &\Longrightarrow \quad f\pi_i \leq g\pi_i \\ f \leq g \quad &\Longrightarrow \quad f\overline{\pi}_i \leq g\overline{\pi}_i \end{aligned} \tag{72}$$

Take $i_1, \dots, i_k$ such that the product $\pi_{i_1} \cdots \pi_{i_k}$ is reduced in NDPF_n, and consider a function

$$f = \pi_{i_1} \cdots \pi_{i_{k_1}} \overline{\pi}_{i_{k_1+1}} \cdots \overline{\pi}_{i_{k_2}} \pi_{i_{k_2+1}} \cdots \tag{73}$$

of NDF_n appearing in the expansion of the product

$$\mathrm{e}_{i_1} \cdots \mathrm{e}_{i_k} = (\pi_{i_1} - \overline{\pi}_{i_1}) \cdots (\pi_{i_1} - \overline{\pi}_{i_1})\,. \tag{74}$$

Clearly, $\pi_{i_1} \cdots \pi_{i_k} \leq f$. Furthermore, if equality holds then

$$\pi_{i_1} \cdots \pi_{i_k} \leq \pi_{i_1} \cdots \pi_{i_{k_1}} \pi_{i_{k_2+1}} \cdots \leq f = \pi_{i_1} \cdots \pi_{i_k}\,. \tag{75}$$

Since the product is reduced, $k_1 = k$, and therefore, $\pi_{i_1} \cdots \pi_{i_k}$ appears with coefficient 1 in $\mathrm{e}_{i_1} \cdots \mathrm{e}_{i_k}$. $\square$

4.2. **Representation theory.** In this section, we derive the representation theory of NDF_n from that of $\mathcal{H}\mathfrak{S}_n$. An alternative more combinatorial approach would be to construct by inclusion/exclusion a graded basis $(\mathrm{gr_f})_{f\in \mathrm{NDF}_n}$ of $\mathbb{C}[\mathrm{NDF}_f]$ such that:

$$\mathrm{gr_f}\,\mathrm{gr_g} = \begin{cases} \mathrm{gr_{fg}} & \text{if } |\mathrm{im} f| = |\mathrm{im} g| = |\mathrm{im} fg|, \\ 0 & \text{otherwise.} \end{cases} \tag{76}$$

Then, looking at the principal modules $\mathrm{gr_f}\,.\mathbb{C}[\mathrm{NDF}_n] = \mathbb{C}.\{\mathrm{gr_g} \mid \mathrm{fibers}(g) = \mathrm{fibers}(f)\}$ splits the regular representation into a direct summand of $\binom{n-1}{k-1}$ copies of P_n^k for each k. Combinatorial proofs for the Cartan matrix and the simple modules are then lengthy but straightforward.

4.2.1. *Projective modules, simple modules, and Cartan's matrix.* Let δ be the usual homology border map ($\delta^2 = 0$):

$$\delta : \begin{cases} P_n^k & \to P_n^{k-1} \\ S := \{s_1, \dots, s_k\} & \mapsto \sum_{i \in \{1,\dots,k\}} (-1)^{k-i} S \backslash \{s_i\} \end{cases}, \tag{77}$$

which induces the following exact sequence of morphisms of $\mathbb{C}[\mathrm{NDF}_n]$-modules:

$$0 \to P_n^n \xrightarrow{\delta} \cdots \xrightarrow{\delta} P_n^{k+1} \xrightarrow{\delta} P_n^k \xrightarrow{\delta} P_n^{k-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} P_n^1 \xrightarrow{\delta} P_n^0 \to 0 \tag{78}$$

For $k = 1, \dots, n$, set $S_n^k := P_n^k / \ker \delta$, so that we have the short exact sequence:

$$0 \to S_n^{k+1} \to P_n^k \to S_n^k \to 0\,. \tag{79}$$

In particular, $\dim S_n^k = \binom{n-1}{k-1}$ since $\dim P_n^k = \binom{n}{k}$. The following proposition states that the morphism of Equation (78) are essentially the only non trivial morphisms between the $(P_n^k)_{k=1,\dots,n}$.

Proposition 4.4. *Let k and l be two integers in $\{1, \dots, n\}$. Then,*

$$\dim \mathrm{Hom}(P_n^k, P_n^l) = \begin{cases} 1 & \textit{if } k \in \{l, l+1\}\,, \\ 0 & \textit{otherwise.} \end{cases} \tag{80}$$

Proof. From the Cartan matrix of $\mathcal{H}\mathfrak{S}_n$ (see Proposition 3.22) we can deduce the dimension of $\mathrm{Hom}(P_n^k, P_n^l)$. Indeed, any non trivial $\mathbb{C}[\mathrm{NDF}_n]$-morphism ϕ from P_n^k to P_n^l is a $\mathcal{H}\mathfrak{S}_n$-morphism, and thus can be lifted to a non trivial $\mathcal{H}\mathfrak{S}_n$-morphism ψ from $P_{\{1,\dots,k-1\}}$ to the projective module $P_{\{1,\dots,l-1\}}$.

If $k < l$, $\dim \mathrm{Hom}(P_{\{1,\dots,k-1\}}, P_{\{1,\dots,l-1\}}) = 0$, and therefore $\dim \mathrm{Hom}(P_n^k, P_n^l) = 0$. Otherwise, $\dim \mathrm{Hom}(P_{\{1,\dots,k-1\}}, P_{\{1,\dots,l-1\}}) = 1$. If $k > l+1$, then $P_{\{1,\dots,k-1\}}$ is mapped by ψ to its unique copy in $P_{\{1,\dots,l,l+1\}}$, and is therefore killed in the quotient

$$P_n^l := P_{\{1,\dots,l-1\}} \quad / \quad \sum_{s=l+1}^{n-1} P_{\{1,\dots,l-1,s\}}. \tag{81}$$

therefore $\dim \mathrm{Hom}(P_n^k, P_n^l) = 0$. In the remaining cases we can conclude that $\mathrm{Hom}(P_n^k, P_n^k) = \mathbb{C}.\mathrm{id}$, and $\mathrm{Hom}(P_n^k, P_n^{k-1}) = \mathbb{C}.\delta$. □

We are now in position to describe the projective and simple modules.

Proposition 4.5. *The modules $(P_n^k)_{k=1,\dots,n}$ form a complete set of representatives of the indecomposable projective modules of $\mathbb{C}[\mathrm{NDF}_n]$.*

The modules $(S_n^k)_{k=1,\dots,n}$ form a complete set of representatives of the simple modules of $\mathbb{C}[\mathrm{NDF}_n]$.

Proof. It follows from Proposition 4.4 that the modules $(P_n^k)_{k=1,\dots,n}$ are both indecomposable ($\dim \mathrm{Hom}(P_n^k, P_n^k) = 1$) and non isomorphic ($\dim \mathrm{Hom}(P_n^k, P_n^l) = 0$ if $k < l$). It remains to prove (i) that each of them is projective and (ii) that we obtain all the projective modules this way. It then follows from the description of the morphisms between the P_n^k that the $(S_n^k)_{k=1,\dots,n}$ form a complete set of representatives of the simple modules.

We first achieve (i) by constructing explicitly an idempotent e_n^k such that the principal ideal $e_n^k \mathbb{C}[\mathrm{NDF}_n]$ is isomorphic to P_n^k. Define the idempotent e_n^k as follows:

$$e_n^k := \pi_{n-1}\pi_{n-2} \cdots \pi_{k+1}\pi_k \; (1 - \pi_{k-1})(1 - \pi_{k-2}) \cdots (1 - \pi_2)(1 - \pi_1)\,. \tag{82}$$

To prove that e_n^k is indeed an idempotent, it is sufficient to use the presentation of $\mathbb{C}[\mathrm{NDF}_n]$ given in Equation (64) to check that

$$e_n^k \pi_i = \begin{cases} e_n^k & \text{if } i >= k \\ 0 & \text{otherwise.} \end{cases} \tag{83}$$

Alternatively, we could have shown that e_n^k is the image in $\mathbb{C}[\mathrm{NDF}_n]$ of the classical hook idempotent of the 0-Hecke algebra

$$\prod_j \pi_j \prod_j (1-\pi_j) \tag{84}$$

where the left (resp. on the right) product ranges over a reduced word of the maximal permutation of the parabolic subgroup $\mathfrak{S}_{k,\dots,n-1}$ (resp. $\mathfrak{S}_{1,\dots k-1}$).

We want now to prove that P_n^k is isomorphic to $e_n^k\mathbb{C}[\mathrm{NDF}_n]$. Let us denote B_n^k the set of functions in NDF_n which are injective on $\{1,\dots,k\}$ and such that $f(i) = f(k)$ for all $i \geq k$. It is clear that for any function f of NDF_n, the actual value of $e_n^k f$ depends only on the set $f(\{1,\dots,k\})$. Moreover, for any $f \in B_n^k$,

$$e_n^k f = f + \sum \text{functions strictly smaller than } f\,, \tag{85}$$

where the order considered on the set of functions is the product order on the tuple $(f(1),\dots,f(n))$ (ie. $f \geq g$ iff $f(i) \geq g(i)$ for all i). It follows by triangularity that $\{e_n^k f \mid f \in B_n^k\}$ is a basis for $e_n^k\mathbb{C}[\mathrm{NDF}_n]$.

Now the linear map

$$\phi_n^k : \begin{cases} e_n^k\mathbb{C}[\mathrm{NDF}_n] & \to P_n^k \\ e_n^k f & \mapsto \{f(1),\dots,f(k)\} \end{cases}, \tag{86}$$

is in fact a morphism of $\mathbb{C}[\mathrm{NDF}_n]$-module. Indeed,

$$\phi_n^k(e_n^k f \pi_i) = \phi_n^k(e_n^k(f\pi_i)) = \{f(1)\pi_i,\dots,f(k)\pi_i\} = \{f(1),\dots,f(k)\}\pi_i \tag{87}$$

and the same holds for $\overline{\pi}_i$. Since the cardinal of B_n^k is the same as the dimension of P_n^k, on gets that ϕ_n^k is in fact an isomorphism. Therefore P_n^k is projective.

To derive (ii), note that the faithful $\mathbb{C}[\mathrm{NDF}_n]$-module $\bigoplus_{k=1}^n \bigwedge^k \mathbb{C}^n = \bigoplus_{k=1}^n P_n^k$ (Lemma 4.2) is now projective. And it is a general property that any projective module of a finite dimensional algebra A occurs as a submodule of any faithful projective module M of A (the right regular representation of A is a sub-module of a certain number of copy of M, and the decomposition of any of its projective module is known to be unique up to an isomorphism).

□

Hence Proposition 4.4 actually gives the Cartan matrix of $\mathbb{C}[\mathrm{NDF}_n]$.

4.2.2. *Induction, restriction, and Grothendieck rings.*

Proposition 4.6. *The restriction and induction of indecomposable projective modules and simple modules are described by:*

$$P_{n_1+n_2}^k \downarrow_{\mathbb{C}[\mathrm{NDF}_{n_1}]\otimes\mathbb{C}[\mathrm{NDF}_{n_2}]}^{\mathbb{C}[\mathrm{NDF}_{n_1+n_2}]} \approx \bigoplus_{\substack{n_1+n_2=n \\ k_1+k_2=k \\ 1\leq k_i\leq n_i \text{ or } k_i=n_i=0}} P_{n_1}^{k_1} \otimes P_{n_2}^{k_2} \tag{88}$$

$$P_{n_1}^{k_1} \otimes P_{n_2}^{k_2} \uparrow_{\mathbb{C}[\mathrm{NDF}_{n_1}]\otimes\mathbb{C}[\mathrm{NDF}_{n_2}]}^{\mathbb{C}[\mathrm{NDF}_{n_1+n_2}]} \approx P_{n_1+n_2}^{k_1+k_2} \oplus P_{n_1+n_2}^{k_1+k_2-1} \tag{89}$$

$$S^{k}_{n_1+n_2} \downarrow^{\mathbb{C}[\mathrm{NDF}_{n_1+n_2}]}_{\mathbb{C}[\mathrm{NDF}_{n_1}]\otimes\mathbb{C}[\mathrm{NDF}_{n_2}]} = \bigoplus_{\substack{n_1+n_2=n\\ k_1+k_2\in\{k,k+1\}\\ 1\leq k_i\leq n_i \ \text{or}\ k_i=n_i=0}} S^{k_1}_{n_1}\otimes S^{k_2}_{n_2} \tag{90}$$

$$S^{k_1}_{n_1}\otimes S^{k_2}_{n_2} \uparrow^{\mathbb{C}[\mathrm{NDF}_{n_1+n_2}]}_{\mathbb{C}[\mathrm{NDF}_{n_1}]\otimes\mathbb{C}[\mathrm{NDF}_{n_2}]} \approx S^{k_1+k_2}_{n_1+n_2} \tag{91}$$

Those rules yield structures of commutative algebras and cocommutative coalgebras on the Grothendieck rings of $\mathbb{C}[\mathrm{NDF}_n]$. However, we do not get Hopf algebras, because the structures of algebras and coalgebras are not compatible (the coefficient of $\chi(P_1^1)\otimes\chi(P_1^1)$ differs in $\Delta(\chi(P_1^1)\chi(P_1^1))$ and $\Delta(\chi(P_1^1))\Delta(\chi(P_1^1)))$.

Proposition 4.7. *The Grothendieck rings $\mathcal{G}$ and $\mathcal{K}$ of $\mathbb{C}[\mathrm{NDF}_n]$ can be realized as quotients or subcoalgebras of* Sym, QSym*, and* NCSF*, as described in the following diagram:*

$$\begin{tikzcd}
(\mathrm{Sym},.) \arrow[r, two heads, "h_\lambda\mapsto\chi\left(S^{\ell(\lambda)}_{|\lambda|}\right)"] & (\mathcal{G},.) & (\mathcal{K},.) \arrow[l, hook', two heads, "C"'] \arrow[dl, dashed, leftrightarrow, "*"'] & (\mathrm{NCSF},.) \arrow[l, two heads, "R_I\mapsto\chi\left(P^{\ell(I)}_{|I|}\right)"'] \\
(\mathrm{QSym},\Delta) \arrow[r, two heads, "F_I\mapsto\chi\left(S^{\ell(I)}_{|I|}\right)"'] & (\mathcal{G},\Delta) & (\mathcal{K},\Delta) \arrow[l, hook', two heads, "C"] \arrow[r, hook, "\chi(P^k_n)\mapsto\sum_{\lambda\vdash n,\ell(\lambda)=k} m_\lambda"'] & (\mathrm{Sym},\Delta)
\end{tikzcd} \tag{92}$$

5. The algebra of non-decreasing parking functions

Definition 5.1. A *nondecreasing parking function* of size n is a nondecreasing function f from $\{1,2,\dots n\}$ to $\{1,2,\dots n\}$ such that $f(i)\leq i$, for all $i\leq n$.

The composition of maps and the neutral element id_n make the set of nondecreasing parking function of size n into a monoid denoted NDPF_n.

Parking functions where introduced in [KW66]. It is well known that the non-decreasing parking functions are counted by the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$. It is also clear that NDPF_n is the sub-monoid of NDF_n generated by the π_i's.

5.1. **Representation theory.**

5.1.1. *Simple modules.* The goal of the sequel is to study the representation theory of NDPF_n, or equivalently of its algebra $\mathbb{C}[\mathrm{NDPF}_n]$. The following remark allows us to deduce the representations of $\mathbb{C}[\mathrm{NDPF}_n]$ from the representations of $\mathcal{H}_n(0)$.

Proposition 5.2. *The kernel of the algebra epi-morphism $\phi:\mathcal{H}_n(0)\to\mathbb{C}[\mathrm{NDPF}_n]$ defined by $\phi(\pi_i)=\pi_i$ is a sub-ideal of the radical of $\mathcal{H}_n(0)$.*

Proof. It is well known (see [Nor79]) that the quotient of $\mathcal{H}_n(0)$ by its radical is a commutative algebra. Consequently, $\pi_i\pi_{i+1}\pi_i-\pi_i\pi_{i+1}=[\pi_i\pi_{i+1},\pi_i]$ belongs to the radical of $\mathcal{H}_n(0)$. □

As a consequence, taking the quotient by their respective radical shows that the projection ϕ is an isomorphism from $\mathbb{C}[\mathrm{NDPF}_n]/\mathrm{rad}(\mathbb{C}[\mathrm{NDPF}_n])$ to $\mathcal{H}_n(0)/\mathrm{rad}(\mathcal{H}_n(0))$. Moreover $\mathbb{C}[\mathrm{NDPF}_n]/\mathrm{rad}(\mathbb{C}[\mathrm{NDPF}_n])$ is isomorphic to the commutative algebra generated by the π_i such that $\pi_i^2=\pi_i$. As a consequence, $\mathcal{H}_n(0)$ and $\mathcal{H}\mathfrak{S}_n$ share, roughly speaking, the same simple modules:

Corollary 5.3. *There are 2^{n-1} simple $\mathbb{C}[\mathrm{NDPF}_n]$-modules S_I, and they are all one dimensional. The structure of the module S_I, generated by η_I, is given by*

$$\begin{cases} \eta_I \cdot \pi_i = 0 & \text{if } i \in I, \\ \eta_I \cdot \pi_i = \eta_I & \text{otherwise.} \end{cases} \tag{93}$$

5.1.2. *Projective modules.* The projective modules of NDPF_n can be deduced from the ones of NDF_n.

Theorem 5.4. *Let $I = \{s_1, \dots, s_k\} \subset \{1, \dots, n-1\}$. Then, the principal submodule*

$$P_I := (e_1 \wedge e_{s_1+1} \wedge \cdots \wedge e_{s_k+1}) \cdot \mathbb{C}[\mathrm{NDPF}_n] \quad \subset \quad \bigwedge^{k+1} \mathbb{C}^n \tag{94}$$

is an indecomposable projective module. Moreover, the set $(P_I)_{I \models n}$ is a complete set of representatives of indecomposable projective modules of $\mathbb{C}[\mathrm{NDPF}_n]$.

This suggests an alternative description of the algebra $\mathbb{C}[\mathrm{NDPF}_n]$. Let $G_{n,k}$ be the lattice of subsets of $\{1, \dots, n\}$ of size k for the *product order* defined as follows. Let $S := \{s_1 < s_2 < \cdots < s_k\}$ and $T := \{t_1 < t_2 < \cdots < t_k\}$ be two subsets. Then,

$$S \leq_G T \qquad \text{if and only if} \qquad s_i \leq t_i, \text{ for } i = 1, \dots, k. \tag{95}$$

One easily sees that $S \leq_G T$ if and only if there exists a nondecreasing parking function f such that $e_S = e_T \cdot f$. This lattice appears as the Bruhat order associated to the Grassman manifold G_k^n of k-dimensional subspaces in $\mathbb{C}^n$.

Theorem 5.5. *There is a natural algebra isomorphism*

$$\mathbb{C}[\mathrm{NDPF}_n] \;\approx\; \bigoplus_{k=0}^{n-1} \mathbb{C}[G_{n-1,k}]\,. \tag{96}$$

In particular the Cartan map $C : \mathcal{K} \to \mathcal{G}$ is given by the lattice $\leq_G$:

$$C(P_I) \;=\; \sum_{J,\ \mathrm{Des}(J) \leq_G \mathrm{Des}(I)} S_J \tag{97}$$

On the other hand, due to the commutative diagram

$$\begin{tikzcd} \mathcal{H}_m(0) \otimes \mathcal{H}_n(0) \arrow[r, hook] \arrow[d] & \mathcal{H}_{m+n}(0) \arrow[d] \\ \mathrm{NDPF}_m \otimes \mathrm{NDPF}_n \arrow[r, hook] & \mathrm{NDPF}_{m+n} \end{tikzcd} \tag{98}$$

it is clear that the restriction of simple modules and the induction of indecomposable projective modules follow the same rule as for $\mathcal{H}_n(0)$. The induction of simple modules can be deduced via the Cartan map, giving rise to a new basis G_I of NCSF. The restriction of indecomposable projective modules leads to a new operation on compositions, which seems not to be related to anything previously known. All of this is summarized by the following diagram:

$$\begin{tikzcd} (\mathrm{NCSF}, .) & (\mathcal{G}, .) \arrow[l, two heads, hook', "\chi(S_I)\mapsto G_I"'] & (\mathcal{K}, .) \arrow[l, two heads, hook', "C"'] \arrow[r, hook, two heads, "\chi(P_I)\mapsto R_I"] & (\mathrm{NCSF}, .) \\ (\mathrm{QSym}, \Delta) & (\mathcal{G}, \Delta) \arrow[l, two heads, hook', "\chi(S_I)\mapsto F_I"] & (\mathcal{K}, \Delta) \arrow[l, two heads, hook', "C"] \arrow[r, hook, two heads, "\chi(P_I)\mapsto ???"'] & ??? \end{tikzcd} \tag{99}$$

6. Alternative constructions of $\mathcal{H}\mathfrak{S}_n$ in type A

In type A, the actions of the operators s_i and π_i on permutations of $\mathfrak{S}_n$ extend straightforwardly to an action on the set A^n of words w of length n over any totally ordered alphabet A by

$$w \cdot \pi_i = \begin{cases} w & \text{if } w_i \geq w_{i+1}, \\ ws_i & \text{otherwise.} \end{cases} \tag{100}$$

We may again construct the algebra $\mathbb{C}[s_i, \pi_i]$, and wonder whether it is strictly larger than $\mathcal{H}\mathfrak{S}_n$.

Theorem 6.1. *Let A be a totally ordered alphabet of size at least n. Then the subalgebra of $\operatorname{End}(\mathbb{C}.A^n)$ generated by both sets of operators $\{s_i, \pi_i\}_{i=1,\dots,n-1}$ is isomorphic to $\mathcal{H}\mathfrak{S}_n$.*

This theorem is best restated and proved using a commuting property with the monoid of non decreasing functions. Recall that the evaluation $e(w)$ of a word w on an alphabet A is the function which counts the number of occurrences in w of each letter $a \in A$; for example, the evaluation of a permutation is the constant function $a \mapsto 1$. For a given evaluation e, write C_e the subspace of $\mathbb{C}.A^n$ spanned by the words with evaluation e, and p_e the orthogonal projection on C_e. Let finally $\operatorname{End}_e(\mathbb{C}.A^n) = \bigoplus_e \operatorname{End}(C_e)$ denote the algebra of *evaluation preserving endomorphisms* of $\mathbb{C}.A^n$.

Theorem 6.2. *Let A be a totally ordered alphabet of size at least n. Consider the monoid NDF_A of non decreasing functions from A to A, acting on values on the words of A^n. Then the commutant of NDF_A in $\operatorname{End}_e(\mathbb{C}.A^n)$ coincides with $\mathbb{C}[s_i, \pi_i]$ and is isomorphic to $\mathcal{H}\mathfrak{S}_n$.*

Alternatively, $\mathbb{C}[s_i, \pi_i]$ is the commutant of the subalgebra of $\operatorname{End}(\mathbb{C}.A^n)$ generated by both NDF_A and $(p_e)_e$.

Proof. First it is clear that both s_i and π_i preserve the evaluation and commute with non decreasing functions; so $\mathbb{C}[s_i, \pi_i]$ is a subset of the commutant.

Taking n distinct letters in A which we may call $1 < \cdots < n$ yields a component C_{std} isomorphic to $\mathbb{C}\mathfrak{S}_n$, which we call *standard component*. The restriction of $\mathbb{C}[s_i, \pi_i]$ on C_{std} is of course $\mathcal{H}\mathfrak{S}_n$.

Fix now some operator $f \in \operatorname{End}_e(\mathbb{C}.A^n)$ which commutes with NDF_A. For each i, take a function π_i in NDF_A such that for $1 \leq j < k \leq n$, $\pi_i(j) = \pi_i(k)$ if and only if $j = i$ and $k = i+1$. A vector v in the standard component is in the kernel of π_i if and only if v is i-left antisymmetric. Therefore, the restriction on C_{std} of f preserves i-antisymmetries, and thus coincides with some operator g of $\mathcal{H}\mathfrak{S}_n$. On the other hand, for any component C_e there exists a non decreasing function f_e which maps C_{std} onto C_e. Since f commutes with f_e, the action of f on C_e is determined by its action of C_{std}, that is by g. □

Note: it is in fact sufficient to consider just p_{std} instead of $(p_e)_e$ in theorem 6.2.

The argument can in fact be generalized to any subset W of words containing some words with all letters distinct, and stable simultaneously by the right action of S_n and by the left action of some monoid of non decreasing functions large enough to contain analogues of the π_i's and f_e's. The following proposition gives two typical examples of that situation (A function f from $\{1, \dots, n\}$ to itself is *initial* if there exists $k \leq n$ such that $\operatorname{im}(f) = \{1, \dots, k\}$; for parking functions see [KW66])

Proposition 6.3. *Let A be the totally ordered alphabet $\{1 < \cdots < n\}$.*

(a) *Consider the monoid* NDPF_n *of non decreasing parking functions, acting on the left on the set* PF_n *of parking functions from* $\{1, \ldots, n\}$ *to* $\{1, \ldots, n\}$. *Then the commutant of* NDPF_n *in* $\mathrm{End}_e(\mathbb{C}.\,\mathrm{PF}_n)$ *coincides with* $\mathbb{C}[s_i, \pi_i]$ *and is isomorphic to* $\mathcal{H}\mathfrak{S}_n$.

(a) *Consider the monoid* NDInit_n *of non decreasing initial functions, acting on the left on the set* Init_n *of initial functions from* $\{1, \ldots, n\}$ *to* $\{1, \ldots, n\}$. *Then the commutant of* NDInit_n *in* $\mathrm{End}_e(\mathbb{C}.\,\mathrm{Init}_n)$ *coincides with* $\mathbb{C}[s_i, \pi_i]$ *and is isomorphic to* $\mathcal{H}\mathfrak{S}_n$.

7. Research in progress

A first direction of research concerns the links between Hecke group algebras and affine Hecke algebras. It turns out that in type A, $\mathcal{H}\mathfrak{S}_n$ is in fact a natural quotient of the extended affine generic Hecke algebra $\hat{\mathrm{H}}_n(q)$, setting $m = 0$ in the definition of Ω [KN98]. We expect this to generalize one way or the other to any Coxeter group. On one hand, this may shed some new light on the representation theory of the affine Hecke algebras. On the other hand, this yields several representations of $\mathcal{H}\mathfrak{S}_n$ on the ring of polynomials $\mathbb{C}[x_1, \ldots, x_n]$, the π_i acting either by elementary sorting on monomials, or as isobaric divided differences, or ... Furthermore, the maximal commutative subring $\mathbb{C}[Y_1, \ldots, Y_n]$ of $\hat{\mathrm{H}}_n(q)$, specializes to a maximal commutative subring of dimension $n!$ of $\mathcal{H}\mathfrak{S}_n$ by evaluating the symmetric functions $\mathrm{Sym}(Y_1, \ldots, Y_n)$ on the alphabet $\frac{1-q^n}{1-q}$. This is likely to give some explicit description of the orthogonal idempotents e_σ of Theorem 3.19. More importantly, this opens the door for links with the non symmetric Macdonald polynomials (which can, among other things be defined as eigenvalues of the Y_i [KN98]).

The properties of the Hecke group algebras, and in particular their representation theory seem also to generalize nicely to infinite Coxeter groups, up to some little adaptations. First, when a parabolic subgroup W_I is infinite, it is not possible to realize the projective module P_I inside $\mathbb{C}[W]$, or at least not without an appropriate completion (because of the infinite alternating sum). Reciprocally, $P_{\mathfrak{c}I}$ is not distinct anymore from $\bigcup_{J \supset I} P_J$ (there are no element of W with descent set I; cf also proposition 3.12). This suggests that $\mathcal{H}W$ is Morita equivalent to the poset algebra of some convex subset of the boolean lattice.

A last direction of research is the generalization of sections 4 and 5. This essentially boils down to the following question: what is the natural definition of the representation on exterior powers for a general Coxeter group? One such attempt in type B gives rise to some tower of self-injective monoid of signed non decreasing parking functions whose sizes appear to be given by sequence A086618 of the encyclopedia of integer sequences [Se03].

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